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NUMERICAL GENERALIZED SOLUTION OF FUZZY DIFFERENTIAL EQUATION

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Abstract: Generalized solution of fuzzy differential equations is defined and discussed. Furthermore, we proposed numerical algorithms of generalized solution of fuzzy differential equations.

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1. Introduction

The topic of fuzzy differential equations (FED) and fuzzy integral equations (FIE) which attracted growing interest for sometime. Until now, there are many works on fuzzy differential equations. A thorough theoretical research of fuzzy Cauchy problem was given by Kaleva [6], Seikkala [9], Wu and Song [12], etc. In 2002, Xue and Fu [13] obtained the existence of Caratheodory solution of fuzzy differential equations. In 2002, Gong Zeng-Tai first introduced generalized solution of discontinuous fuzzy system [3].

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Numerical algorithms for calculating approximates to solutions of fuzzy differential and integral equations were designed. In 1999, Ma and Friedman [8], [2] studied numerical solutions of fuzzy differential and integral equations.

In this paper we replace the fuzzy differential equations by fuzzy integral equations and discussed numerical procedures for the new integral equations.

In Section 2 we briefly present the basic notation of fuzzy number, fuzzy valued function, fuzzy derivative, and fuzzy integral. In Section 3 generalized solution of fuzzy differential equation is defined and the existence of generalized solution is studied. In Section 4 we replace the fuzzy differential equations by fuzzy integral equations and numerical algorithm for solving fuzzy integral equation is proposed.

2. Notations and Preliminaries

Let R^1 be the real number field. Denote $E^1 = \{u|u : R^1 \to [0,1]\}$, which has the following properties (i)-(iv):

(i) u is normal, i.e., there exists an $x_0 \in R^1$with $u(x_0) = 1$;

(ii) u is fuzzy convex, i.e., $u(rx + (1-r)y) \geq \min(u(x), u(y))$ whenever $x, y \in R^1$ and $r \in [0,1]$;

(iii) $u(x)$ is upper semi-continuous; and

(iv) $[u]^0 = cl\{x \in R^1 : u(x) > 0\}$ is a compact set.

For any $u \in E^1$, u is called a fuzzy number and E^1 is called the fuzzy number space, see [6], [9], [12], [13], [3].

Obviously, $u(\lambda)$ is nonempty bounded closed intervals for any $u \in E^1$ and $\lambda \in [0,1]$, where $u(\lambda) = \{x \in R^1 : u(x) \geq \lambda\}$ when $\lambda \in [0,1]$.

For $u, v \in E^1$, define

$$D(u,v) = \sup_{\lambda \in [0,1]} d(u_(\lambda), v(\lambda)) = \max(|u^-(\lambda) - \leq v^-(\lambda)|, |u^+(\lambda) - v^+(\lambda)|)\,.$$

Using the results of WU et al [11], we see that:

(1) (E^1, D) is a complete metric space;

(2) $D(\tilde{A} + \tilde{C}, \tilde{B} + \tilde{C}) = D(\tilde{A}, \tilde{B})$;

(3) $D(k\tilde{A}, k\tilde{B}) = |k| D(\tilde{A}, \tilde{B}), k \in R$;

(4) $D(\tilde{A} + \tilde{B}, \tilde{0}) \leq D(\tilde{A}, \tilde{0}) + D(\tilde{B}, \tilde{0})$;

(5) $D(\tilde{A} + \tilde{C}, \tilde{B} + \tilde{D}) \leq D(\tilde{A}, \tilde{B}) + D(\tilde{C}, \tilde{D})$.

Where $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D} \in E^1, \tilde{0} = \mathcal{X}_{\{0\}}$.

Lemma 1. (see [12], [13], [3], [2]) *Let $\tilde{A} \in E^1$, then:*

(1) $A_\lambda$is a bounded interval, $\lambda \in [0,1]$;

(2) $A_{\lambda_1} \supset A_{\lambda_2}$ *when* $0 \le \lambda_1 \le \lambda_2 \le 1$*;*

(3) $\bigcap_{n=1}^{\infty} A_{\lambda_n} = A_\lambda$, *for any* $\lambda_n \nearrow \lambda \in (0,1]$.

Conversely, if $\{B_\lambda | \lambda \in [0,1]\}$ *fulfills (1)-(3), then there exists a unique* $\tilde{M} \in E^1$, *such that* $M_\lambda = B_\lambda, \lambda \in [0,1]$, *for* $\lambda \in [0,1]$ *and* $[M]^0 \subset B_0$.

Let $\{u_n\} \subset E^1, u \in E^1$, we say that $\{u_n\}$ is convergent to u (denoted by $u_n \to u (n \to \infty)$) if $D(u_n, u) \to 0 (n \to \infty)$.

Let $x, y \in E^1$. If there exists a $z \in E^1$ such that $x = y + z$, then we call z the (H) difference of x and y, denoted by $x - y$ (see [6], [9], [13], [8], [2], [11]).

Definition 2.3. (see [11]) Let $\tilde{f} : [a,b] \to E^1$, the (FH) integral of $\tilde{f}(t)$ over $[a,b]$, denoted (FH) $\int_a^b \tilde{f}(t)dt$ is defined levelwise by the equation

$$[\int_a^b \tilde{f}(t)dt](\lambda) = \{(H)\int_a^b F(t)dt : F : [a,b] \to R \text{ is a (H) integrable selection for } (f(t))(\lambda)\}.$$

Lemma. (see [11]) *If* $\tilde{f} : [a,b] \to E^1$, *then* $\tilde{f}(x)$ *is* (FH) *integrable on* $[a,b]$, *if and only if* $f_\lambda^-(x), f_\lambda^+(x)$ *is* (FH) *integrable on* $[a,b]$, *for any* $\lambda \in [0,1]$, *and*

$$\big[(FH)\int_a^b \tilde{f}(x)dx\big]_\lambda = \big[(H)\int_a^b f_\lambda^-(x)dx,\ (H)\int_a^b f_\lambda^+(x)dx\big].$$

Definition 2.4. (see [6], [13], [8], [2]) A fuzzy-valued function $\tilde{f} : [a,b] \to E^1$ is said to be differentiable at $x \in [a,b]$ if there exists a $\tilde{f}'(x) \in E^1$ such that

$$\lim_{h \to 0^+} \frac{\tilde{f}(x+h) - \tilde{f}(x)}{h}, \quad \lim_{h \to 0^-} \frac{\tilde{f}(x+h) - \tilde{f}(x)}{h}$$

exist and equal to $\tilde{f}'(x)$.

Definition 2.4. (see [7]) Let $\delta(x) > 0$ be a function on $[a,b]$, a division $T = \{[x_{i-1}, x_i]; \xi_i\}$ is said to be δ-fine, if the following conditions are satisfied:

(1) $a = x_0 < x_1 < \cdots < x_n = b$;

(2) $\xi_i \in [x_{i-1}, x_i] \subset (\xi_i - \delta(\xi), \xi_i + \delta(\xi))$.

Definition 2.3. (see [5]) Let $\tilde{f} : [a,b] \to E^1$, if there exists an additive fuzzy-valued function $\tilde{F}$ on $[a,b]$ such that for every $\varepsilon > 0$, there is a function $\delta(x) > 0$ and for any δ-fine division $T = \{[x_{i-1}, x_i]; \xi_i\}$ of $[a,b]$, we have

$$\sum D(\tilde{f}(\xi)(x_i - x_{i-1}), \tilde{F}([x_{i-1}, x_i])) < \varepsilon.$$

We write (SFH)$\int_a^b \tilde{f}(x)dx = \tilde{F}([a,b])$, where $\tilde{F}([s,t]) = \tilde{F}(s) - \tilde{F}(t)$.

Remark 1. If $\tilde{f}$ is (SFH) integrable on $[a,b]$, then $\tilde{f}$ is (FH) integrable on $[a,b]$, and $\int_a^x \tilde{f}(t)dt = \tilde{F}(x)$.

3. Generalized Solution of Fuzzy Differential Equations

Without loss of generality, we next discuss the generalized solutions for initial problem of the following ordinary differential equation:

$$\tilde{x}' = \tilde{f}(t, x(t)), \quad t \in [a,b]; \quad \tilde{x}(a) = \tilde{x_0}, \tag{1}$$

where x is a fuzzy function of t, $\tilde{f}(t,x)$ is a fuzzy function of the crisp variable t and the fuzzy variable $\tilde{x}$, and $\tilde{x}'$ is the fuzzy derivative of $\tilde{x}$ and an initial value $\tilde{x}(t_0) = \tilde{x_0}$ is given.

It is well known that the initial problem when $f(t,x)$ is continuous equal a fuzzy integral equation

$$\tilde{x}(t) = \tilde{x_0} + (K)\int_a^t \tilde{f}(s, x(s))ds\,,$$

similarly, we have the following definition.

Definition 3.1. $\tilde{x}(t) : [a,b] \to E^1$ is called a generalized solution of (1), for simplicity written as G-solution, if $\tilde{x}(t)$ satisfies the integral equation

$$\tilde{x}(t) = \tilde{x_0} + (SFH)\int_a^t \tilde{f}(s, x(s))ds.$$

This definition is well. By Gong [5], Theorem 4.1, we have $x(t)$ is differentiable almost everywhere and $\tilde{x}'(t) = \tilde{f}(t, x(t))$ almost everywhere in $[a,b]$, $\tilde{x}(a) = x_0$.

Remark 2. Generalized solution of fuzzy initial valued problem (1) is defined by (SFH) integrable, not by (K) integrable. Since there exists a fuzzy valued function which is (K) integrable, but its primitive is not differentiable almost everywhere Gong [5].

Example 1.

$$\begin{cases} \tilde{x}'(t) = \tilde{f}(t, x(t)) = g(t)\tilde{A}, \quad t \in [0,1], \\ x(0) = \tilde{0}, \end{cases}$$

where

$$g(t) = \begin{cases} 2t\sin\frac{1}{t^2} - \frac{2}{t}\cos\frac{1}{t^2}, & t \neq 0\,, \\ 0, & t = 0\,, \end{cases} \tag{2}$$

$$\tilde{A}(s) = \begin{cases} 1-s, & s \in [0,1]\,, \\ 0, & s \notin [0,1]\,. \end{cases}$$

By [1], Corollary 15, we have

$$\tilde{F}(t) = \begin{cases} \tilde{A}t^2 \sin \dfrac{1}{t^2}, & t \notin (0,1]\,, \\ 0, & t = 0 \end{cases}$$

is differentiable on $[0,1]$ and

$$\tilde{f}(t, x(t)) = \tilde{F}'(t) = \begin{cases} \tilde{A}(2t \sin \frac{1}{t^2} - \frac{2}{t} \cos \frac{1}{t^2}), & t \in (0,1]\,, \\ \tilde{0}, & t = 0\,. \end{cases}$$

By Gong [5], Theorem 5.1, $\tilde{f}(t, x(t))$ is (SFH) integrable on $[0,1]$. So $\tilde{F}(t)$ is G-solution of fuzzy initial valued problem (2).

Definition 3.2. A fuzzy-number-valued function F defined on $[a,b]$ is said to be $ACG^*[a,b]$ if F is continuous on $[a,b]$ and $[a,b] = \bigcup_{i=1}^{\infty} \mathbf{X_i}$ such that F is $AC^*(\mathbf{X_i})$. F is $AC^*(\mathbf{X})$, if for every $\varepsilon > 0$, there exists $\eta > 0$ such that for every finite or infinite sequence of non-overlapping intervals $[a_i, b_i]$ satisfying $\sum_i |b_i - a_i| < \eta$, where $a_i, b_i \in X$ for all i, we have

$$\sum_i \omega(F; [b_i - a_i]) < \varepsilon\,,$$

where ω denotes the oscillation of F over $[a_i, b_i]$, i.e.,

$$\omega(F; [b_i - a_i]) = \sup\{D(F(x) - F(y)); x, y \in [a_i, b_i]\}.$$

Theorem 1. *Let $\tilde{x} : [a,b] \to E^1$, if $\tilde{x}$ is the G-solution of (1), Then $\tilde{x}$ is ACG^* on $[a,b]$.*

Proof. For every $\varepsilon > 0$, there is a function $\delta(t) > 0$ such that for any δ-fine partial division $T = \{[u,v]; \xi\}$ in $[a,b]$, we have

$$(T) \sum D(x(u,v)), f(\xi)(v-u) < \varepsilon.$$

We may assume that $0 < \delta(t) \leq 1$. Let

$\mathbf{X}_{ni} = \{t \in [a,b] :$

$$D(\tilde{f}(t),\tilde{0}) \leq n; \frac{1}{n} < \delta(t) \leq \frac{1}{n-1} \text{ and } x \in [a+\frac{i-1}{n}, a+\frac{i}{n}]\},$$

for $n = 2, 3, \cdots; i = 1, 2, \cdots$, we have $\bigcup_{n=2}^{\infty}\bigcup_{i=1}^{\infty} \mathbf{X}_{ni} = [a, b]$.

Fix $\mathbf{X}_{ni}$ and let $\{[a_k, b_k]\}$ be any finite sequence of non-over lapping intervals with $a_k, b_k \in \mathbf{X}_{ni}$ for all k. The $\{([a_k, b_k], a_k)\}$ is a δ-fine partial division of $[a, b]$. Furthermore, if $a_k \leq u_k \leq v_k \leq b_k$, the $\{([a_k, u_k], a_k)\}$, $\{([v_k, b_k], b_k)\}$ are δ-fine partial division of $[a, b]$. Thus

$$\begin{aligned}\sum_k D(\tilde{x}(u_k, v_k), \tilde{0}) &\leq \sum_k D(\tilde{x}(a_k, u_k), 0) + \sum_k D(x(u_k, b_k), \tilde{0}) \\ &+ \sum_k D(\tilde{x}(a_k, b_k), \tilde{0}) \leq 3\varepsilon + \sum_k D(\tilde{f}(a_k)(u_k - a_k), \tilde{0}) + \sum_k D(\tilde{f}(b_k - v_k), \tilde{0}) \\ &+ \sum_k D(f(b_k - v_k), \tilde{0}) \leq 3\varepsilon + 3n\sum_k (b_k, a_k).\end{aligned}$$

Choose $n \leq \frac{\varepsilon}{3n}$ and $\sum_k (b_k - a_k) < \eta$, then

$$\sum_k w(x; [a_k, b_k]) \leq 3\varepsilon + \varepsilon = 4\varepsilon.$$

Therefore, x is AC$^*(\mathbf{X})$. Consequently, x is ACG* on $[a, b]$. □

4. Numerical Method for *G*-Solution of Fuzzy Differential Equation

4.1. The Fuzzy Euler Method

To calculate the (SFH) integral of $\tilde{f}(t, x(t))$ we apply the trapezoidal rule. The interval $[a, b]$ is partitioned by equally spaced points:

$$a = t_0 < t_1 < \cdots < t_n = b; \quad t_i - t_{i-1} = \frac{b-a}{n} = h, \quad 1 \leq i \leq n\,.$$

On $[t_n, t_{n+1}]$, we have

$$\int_{t_n}^{t_{n+1}} x'(t)dt = \int_{t_n}^{t_{n+1}} f(t, x(t))dt\,, \tag{3}$$

this is $x(t_{n+1}) = x(t_n) + \int_{t_n}^{t_{n+1}} f(t, x(t))dt$.

We apply the left rectangle rule, let $\int_{t_n}^{t_{n+1}} f(t,x(t))dt \approx f(t_n, x(t_n))(t_{n+n} - t_n)$, then

$$x_{n+1} = x_n + hf(t_n, x_n) \tag{4}$$

are the Euler approximates to $x(t_{n+1})$.

For any $r \in [0,1]$, it is sufficient to show that:

$$\underline{x}_{n+1}(r) = \underline{x}_n(r) + h\overline{f}(t_n, x_n)(r), \quad \overline{x}_{n+1}(r) = \overline{x}_n(r) + h\overline{f}(t_n, x_n)(r)\,.$$

This is Euler method. We apply the trapezoidal rule. Let

$$\begin{aligned} &\int_{t_n}^{t_{n+1}} f(t,x(t))dt \approx \frac{h}{2}[f(t_n, x(t_n)) + f(t_{n+1}, x(t_{n+1}))], \\ &\tilde{x}_{n+1} = \tilde{x}_n + \frac{h}{2}[\tilde{f}(t_n, x_n) + \tilde{f}(t_{n+1}, x_{n+1})] \end{aligned} \tag{5}$$

are the trapezoidal rule approximates to $x(t_{n+1})$.

In this paper, we only discuss Euler method.

Theorem 2. *Let $\tilde{f}(t,\tilde{x})$ is satisfied Lipschitz condition of x.*

$$D(\tilde{f}(t,x_1), \tilde{f}(t,x_2)) \le LD(x_1,x_2), \quad t \in [a,b]$$

and (SFH) integrable on $[a,b]$, then Euler rule (4) converge to the exact solutions $x(t_n)$.

Proof. By (3) and Lemma 1 we have

$$\tilde{x}_n = \tilde{x}_{n-1} + hf(t_{n-1}, x(t_{n-1}))\,,$$

$$\begin{aligned} D(\tilde{x}(t_n), \tilde{x}_n) &= D(\tilde{x}_{n-1} + h\tilde{f}(t_{n-1}, x_{n-1}), \tilde{x}(t_{n-1}) + \int_{t_{n-1}}^{t^n} \tilde{f}(t,x(t))dt) \\ &\le D(\tilde{x}_{n-1}, \tilde{x}(t_{n-1})) + D(h\tilde{f}(t_{n-1}, x_{n-1}), \int_{t_{n-1}}^{t_n} \tilde{f}(t,x(t))dt) \\ &\le D(\tilde{x}_{n-1}, \tilde{x}(t_{n-1})) + D(h\tilde{f}(t_{n-1}, x_{n-1}), hf(t_{n-1}, \tilde{x}_{n-1})) \\ &+ D(hf(t_{n-1}, x(t_{n-1}), \int_{t_{n-1}}^{t_n} \tilde{f}(t,x(t))dt) \le D(\tilde{x}_{n-1}, \tilde{x}(t_{n-1})) \\ &+ hLD(\tilde{x}_{n-1}, \tilde{x}(t_{n-1}) + D(h\tilde{f}(t_{n-1}, \tilde{x}(t_{n-1}), \int_{t_{n-1}}^{t_n} \tilde{f}(t,x(t))dt) \\ &\le (1+hL)^n D(\tilde{x_0}, \tilde{x}(t_0)) + D(h\tilde{f}(t_{n-1}, \tilde{x}(t_{n-1})), \int_{t_{n-1}}^{t_n} \tilde{f}(t,\tilde{x}(t))dt)\,. \end{aligned}$$

When $\tilde{x_0} = \tilde{x}(t_0)$ and $h \to 0$, we have

$$\lim_{h\to 0} D(\tilde{x}(t_n), \tilde{x}_n) = 0.$$

That is, Euler rule (3) converges to the exact solutions $x(t_n)$. □

Example 2. For Example 1, we have:

$$0 = t_0 < t_1 < \cdots < t_n = 1; \quad t_i - t_{i-1} = \frac{1}{n} = h, 1 \le i \le n\,,$$
$$\tilde{x}_1 = \tilde{0} + h\tilde{f}(0, \tilde{x}_0) = \tilde{0} + h\tilde{f}(0, \tilde{0}) = \tilde{0},$$
$$\tilde{x}_2 = \tilde{x}_1 + h\tilde{f}(\frac{1}{n}, \tilde{x}_1) = \tilde{0} + h\tilde{f}(\frac{1}{n}, \tilde{0}) = (\frac{2}{n}\sin n^2 - 2n\cos n^2)\tilde{A},$$
$$\vdots$$
$$\tilde{x}_n = \tilde{x}_{n-1} + h\tilde{f}(1 - \frac{1}{n}, \tilde{x}_{n-1}) = [2(1 - \frac{2}{n})\sin(\frac{n}{n-2})^2$$
$$- (\frac{2n}{n-2})\cos(\frac{n}{n-2})^2]\tilde{A} + [2(1 - \frac{1}{n})\sin(\frac{n}{n-1})^2 - (\frac{2n}{n-1})\cos(\frac{n}{n-1})^2]\tilde{A}.$$

4.2. The Fuzzy Computational Method

If $\tilde{f}(t, \tilde{x})$ is (SFH) integrable on $[a, b]$, then $\tilde{f}$ is (FH) integrable on $[a, b]$. Let

$$L(\lambda) = (H)\int_a^b f_\lambda^-(t, \tilde{x})dt, \; R(\lambda) = (H)\int_a^b f_\lambda^+(t, \tilde{x})dt.$$

Suppose that $A_\lambda = [L(\lambda), R(\lambda)]$. The membership $\int_a^b \tilde{f}(t, \tilde{x})dt$ is

$$\int_a^b \tilde{f}(t, \tilde{x})dt(s) = \sup_{0 \le \lambda \le 1} \lambda \mathcal{X}_{A_\lambda}(s) = \sup\{\lambda | \; 0 \le x \le 1, s \in A_\lambda\}.$$

That is the relaxed nonlinear program

$$\int_a^b \tilde{f}(t, \tilde{x})dt(s) = \max \lambda, \qquad \text{s.t.} \quad \begin{cases} L(\lambda) \le s, \\ R(\lambda) \ge s, \\ 0 \le \lambda \le 1. \end{cases}$$

(1) If $s \notin [L(0), R(0)]$, then $\int_a^b \tilde{f}(t, \tilde{x})dt(s) = 0$.

(2) If $s \in [L(1), R(1)]$, then $\int_a^b \tilde{f}(t, \tilde{x})dt(s) = 1$.

(3) If $s < L(1)$, then $s < R(1)$. So $R(\lambda)$ is a bounded monotonic decreasing that $R(\lambda) \ge s$. By (5.1), we have

$$\int_a^b \tilde{f}(t, \tilde{x})dt(s) = \max \lambda, \qquad \text{s.t.} \quad \begin{cases} L(\lambda) \le s, \\ 0 \le \lambda \le 1. \end{cases}$$

(4) If $s > R(1)$, then $L(1) < s$, so $L(\lambda)$ is a bounded monotonic increasing that $L(\lambda) \le s$. By (5.1) we have

s	$\int_0^1 \tilde{f}(t,\tilde{x})dt(s)$
-1	0
0	1
$\frac{1}{9}\sin 1$	0.889
$\frac{1}{6}\sin 1$	0.833
$\frac{1}{3}\sin 1$	0.667
$\frac{1}{2}\sin 1$	0.500
$\frac{2}{3}\sin 1$	0.333
$\frac{3}{4}\sin 1$	0.250
$\sin 1$	0
1	0

Table 1:

$$\int_a^b \tilde{f}(t,\tilde{x})dt(s) = \max\lambda\,, \quad \text{s.t.} \quad \begin{cases} R(\lambda) \geq s, \\ 0 \leq \lambda \leq 1. \end{cases}$$

Example 3. For Example 1, we have

$$f_\lambda^-(t,\tilde{x}) = 0\,, \quad f_\lambda^+(t,\tilde{x}) = [2t\sin\frac{1}{t^2} - \frac{2}{t}\cos\frac{1}{t^2}](1-\lambda),$$

$\int_0^1 f_\lambda^-(t,\tilde{x})dt = 0 \equiv L(\lambda)$; $\int_0^1 f_\lambda^+(t,\tilde{x})dt = [\int_0^1(2t\sin\frac{1}{t^2} - \frac{2}{t}\cos\frac{1}{t^2})dt](1-\lambda) \equiv R(\lambda)$. Let $A_\lambda = [L(\lambda), R(\lambda)]$, the membership $\int_0^1 \tilde{f}(t,\tilde{x})dt$ is

$$\int_0^1 \tilde{f}(t,\tilde{x})dt(s) = \sup_{0\leq\lambda\leq1} \lambda\mathcal{X}_{A_\lambda}(s).$$

(1) $L(0) = 0, R(0) = \sin 1$.
(2) $L(1) = 0, R(1) = 0$.
So the membership $\int_0^1 \tilde{f}(t,\tilde{x})dt$ is given in Table 1.

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References

[1] B. Bede, S. Gal, Genealizations of fuzzy-number-valued functions with applications to fuzzy differential equations, *Fuzzy Sets and Systems*, **151** (2005), 581-599.

[2] M. Friedman, M. Ma, A. Kandel, Numerical solutions of fuzzy differential and integral equations, *Fuzzy Sets and Systems*, **106** (1999), 35-48.

[3] Z.T. Gong, Discontinuous fuzzy systems and fuzzy Henstock integral, *Fuzzy Systems and Mathematics*, **16** (2002), 211-214, In Chinese.

[4] Z.T. Gong, Discontinuous fuzzy systems and fuzzy Henstock integral, *Fuzzy Systems and Mathematics*, **16** (2002), 211-214.

[5] Z.T. Gong, On the problem of characterizing derivative for the fuzzy-valued functions (II): almost everywhere differentiability and strong Henstock integral, *Fuzzy Set and Systems*, **145** (2004), 381-393.

[6] O. Kaleva, Fuzzy differential equations, *Fuzzy Sets and Systems*, **24** (1987), 301-317.

[7] P.Y. Lee, *Lanzhou Lectures on Henstock Integration*, Singapore: World Scientific (1989).

[8] M. Ma, M. Friedman, A. Kandel, Numerical solutions of fuzzy differential equations, *Fuzzy Sets and Systems*, **105** (1999), 133-138.

[9] S. Seikkala, On the fuzzy initial value problem, *Fuzzy Set and Systems*, **24** (1987),319-330.

[10] H. Wu, The fuzzy Riemann integral and its numerical integration, *Fuzzy Sets and Systems*, **110** (2000), 1-25.

[11] C.X. Wu, Z.T. Gong, On Henstock integrals of fuzzy-valued functions, *Fuzzy Sets and Systems*, **120** (2001), 523-532.

[12] C.X. Wu, S.E. Song, S. Lee, Approximate solutios, existenec and uniqueness of the Cauchy problem of fuzzy differential equations, *J. Math. Appl.*, **202** (1996), 629-644.

[13] X.P. Xue, Y. Fu, Caratheodory solutions of fuzzy differential equations, *Fuzzy Sets and Systems*, **125** (2002), 239-243.