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ON PRIMARY SUBMODULES

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Abstract: Let R be a commutative ring with identity and M be an R-module.M is called a multiplication module provided for each submodule N of M there exists an ideal I of R such that $N = IM$. This paper is devoted to study some properties of primary submodules of multiplication modules.

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Throughout this paper, R denotes a commutative ring with identity and all related modules are unitary R-modules. Let M be an R-module. A proper submodule N of M is called primary if whenever $rm \in N$ and $m \notin N$, for $r \in R$ and $m \in M$, then $r^n \in (N : M)$ for some positive integer n, where $(N : M) = \{r \in R : rM \subseteq N\}$, see [6], [7].

An R-module M is called a multiplication module provided for each submodule N of M there exists an ideal I of R such that $N = IM$. We say that I is a presentation ideal of N. Clearly, every submodule of M has a presentation ideal if and only if M is a multiplication module. Let N and K be submodules of a multiplication R-module with $N = I_1M$ and $K = I_2M$ for some ideals I_1 and I_2 of R. The product N and K denoted by NK is defined by $NK = I_1I_2M$. Then by [1], Theorem 3.4, the product of N and K is independent of presentation of N and K. Note that this definition is different from

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the definition of ordinary ideal multiplication. Indeed, let $R = Z$ be the ring of integers, and let $M = 2Z$ and $N = K = 4Z$. Then NK is $16Z$ by the usual definition and is $8Z$ by the our definition. Moreover, for $a, b \in M$, by ab we mean the product of Ra and Rb. Clearly, NK is a submodule of M and $NK \subseteq N \cap K$. The purpose of this paper is to explore some basic facts of primary submodules of multiplication modules.

Let I be an ideal of the ring R. Recall that nil radical of I, designated by $\sqrt{I}$, is the set $\sqrt{I} = \{r \in R : r^n \in I$ for some positive integer $n\}$. We recall the following known fact.

Proposition 1. *Let M be an R-module and N be a proper submodule of M. Then N is primary if and only if $IV \subseteq N$ and $V \nsubseteqq N$ implies that $I \subseteq \sqrt{(N : M)}$ for each ideal I of R and submodule V of M.*

Proposition 2. *Let M be an R-module. Then N is primary submodule of M if and only if $\sqrt{(N : K)} = \sqrt{(N : M)}$ for every submodule K with $N \subsetneqq K \subseteq M$.*

Proof. It is clear that $\sqrt{(N : M)} \subseteq \sqrt{(N : K)}$. Suppose that N be a primary submodule of M. Let $r \in \sqrt{(N : K)}$. Then, $r^n \in (N : K)$ and so $r^n K \subseteq N$ for some positive integer n. Therefore, $r^n k \in N$ for some $k \in K \backslash N$. Since N is primary, $(r^n)^m = r^{nm} \in (N : M)$ for some positive integer m and so $r \in \sqrt{(N : M)}$. Therefore, $\sqrt{(N : K)} = \sqrt{(N : M)}$. Conversely, let $rm \in N$ and $m \notin N$ for some $m \in M$ and $r \in R$. Then, $r \in (N : N + Rm) \subseteq \sqrt{(N : N + Rm)} = \sqrt{(N : M)}$. Therefore, N is primary submodule of M. □

Definition 1. Let N be a proper submodule of M. Then, the radical of N denoted by M-rad (N) or $r(N)$ is defined to be intersection of all prime submodules of M containing N. If N is not contained in any prime submodule of M, then M-rad $(N) = M$.

Theorem 1. (see [4], Theorem 2.12) *Let R be a commutative ring with identity, N a proper submodule of a multiplication R-module M and $A = (N : M)$. Then $M - \text{rad}\ (N) = \sqrt{A}M$.*

Definition 2. Let M be a multiplication R-module and let N be a submodule of M. Then:

(i) N is called nilpotent if $N^k = 0$ for some positive integer k, where N^k means the product of N, k times;

(ii) An element m of M is called nilpotent if $m^k = 0$ for some positive integer k.

The set of all nilpotent elements of M is denoted by N_M.

Theorem 2. (see [1], Theorem 3.13) Let N be a submodule of a multiplication R-module M. Then

$$M - \operatorname{rad}(N) = \left\{ m \in M : m^k \subseteq N \text{ for some } k > 0 \right\}.$$

Corollary 1. *Let M be a multiplication R-module. Then N_M is the intersection of all prime submodules of M.*

Theorem 3. *Let N be a proper submodule of a multiplication R-module M. If N is primary submodule of M, then $UV \subseteq N$ and $V \subsetneqq N$ implies $U \subseteq r(N)$ for each submodules U and V of M.*

Proof. Let N be a primary submodule of M. Let U and V be submodules of M such that $UV \subseteq N$ and $V \nsubseteq N$. Since M is a multiplication R-module, there exist ideals I and J such that $U = IM$ and $V = JM$. Therefore, $UV = IJM \subseteq N$ and $JM \nsubseteq N$. Then $IJ \subseteq (N : M)$ and $J \nsubseteq (N : M)$. Since N is a primary submodule of M, $(N : M)$ is a primary ideal of R. Therefore, $I \subseteq \sqrt{(N : M)}$. Consequently, $U = IM \subseteq \sqrt{(N : M)}M = r(N)$ by Theorem 1. □

Corollary 2. *Let M be a multiplication R-module. If N is primary submodule of M, then $m_1 m_2 \subseteq N$ and $m_1 \notin N$ implies $m_2 \in r(N)$ for every $m_1, m_2 \in M$.*

Theorem 4. [7, *Theorem* 3.1]*Let R be a commutative ring with identity and M a faithful (i.e. $\operatorname{Ann}(M) = 0$) multiplication R-module. Then the following statements are equivalent.*

(i) M is finitely generated.

(ii) If A and B are ideals of R such that $AM \subseteq BM$ then $A \subseteq B$.

(iii) For each submodule N of M there exists a unique ideal I of R such that $N = IM$.

(iv) $M \neq AM$ for any proper ideal A of R.

(v) $M \neq PM$ for any maximal ideal P of R.

Theorem 5. *Let M be a finitely generated faithful multiplication R-module. Then N is primary if and only if $UV \subseteq N$ and $V \nsubseteq N$ implies $U \subseteq r(N)$ for each submodule U and V of M.*

Proof. Let N be a primary submodule of M. Let U and V be submodules of M such that $UV \subseteq N$ and $V \nsubseteq N$. Then $U \subseteq r(N)$ by Theorem 3. Conversely, let $IV \subseteq N$ and $V \nsubseteq N$ for some ideal I of R and submodule V of M. Since

M is a multiplication module, $V = JM$ for some ideal J of R. Then, $IV = IMJM \subseteq N$ and $V = JM \nsubseteq N$. Therefore, $IM \subseteq r(N) = \sqrt{(N:M)}M$. Then, $I \subseteq \sqrt{(N:M)}$ by Theorem 4. □

Corollary 3. *Let M be a finitely generated faithful multiplication R-module. Let N be a proper submodule of M. Then N is primary if and only if $m_1 m_2 \subseteq N$ and $m_1 \notin N$ implies that $m_2 \in r(N)$ for every $m_1, m_2 \in M$.*

If N is a primary submodule of an R-module M, then $(N:M)$ is a primary ideal of R. However the converse is not necessarily true. For example, if M is a Z-module $Z \times Z$ and $N = (0,6)Z$, then $(N:M) = 0$ but N is not primary submodule.

Theorem 6. *Let M be a multiplication R-module and N be a proper submodule of M. Then N is primary if and only if $(N:M)$ is primary ideal of R.*

Proof. Let N be a primary submodule of M. Then $(N:M)$ is a primary ideal of R. Conversely, let $(N:M)$ is a primary ideal of R. Let $IV \subseteq N$ and $V \nsubseteq N$ for some ideal I of R and submodule V of M. Since M is a multiplication R-module, $V = JM$ for some ideal J. Therefore, $IV = IJM \subseteq N$ and $V = JM \nsubseteq N$. Then $IJ \subseteq (N:M)$ and $J \nsubseteq (N:M)$. Since $(N:M)$ is a primary ideal of $R, I \subseteq \sqrt{(N:M)}$. Consequently, N is primary submodule of M. □

Proposition 3. (see [3], Proposition 3.4) *Let M be a multiplication R-module. If N is a primary submodule of M, then $r(N)$ is a prime submodule of M.*

Proof. Assume that N is a primary submodule of M. Then $(N:M)$ is a primary ideal of R. Hence $\sqrt{(N:M)}$ is a prime ideal of R. Consequently $r(N) = \sqrt{(N:M)}M$ is a prime submodule of M, see [4], Corollary 2.11. □

Proposition 4. *Let M be a multiplication R-module. If N_1 and N_2 are two prime submodules of M such that $N_1 \nsubseteq N_2$ and $N_2 \nsubseteq N_1$, then $N_1 \cap N_2$ is not prime submodule.*

Proof. Suppose that $m_1 \in N_1 \setminus N_2$ and $m_2 \in N_2 \setminus N_1$. Then $m_1 m_2 \subseteq N_1 N_2 \subseteq N_1 \cap N_2$ but $m_1 \notin N_1 \cap N_2$ and $m_2 \notin N_1 \cap N_2$. □

Definition 3. Let M be a multiplication R-module. M is said to be a nil module if each element in M is nilpotent.

Theorem 7. *Let M be a multiplication R-module which is not a nil module and in which each non-zero submodule is primary. If N_1 and N_2 are two prime submodules in M, then either: i) $N_1 \subseteq N_2$, or $N_2 \subseteq N_1$ or ii) $N_1 \cap N_2 = 0$.*

Proof. If $N_1 \cap N_2 = N_3 \neq (0)$, then $r(N_1 \cap N_2) = r(N_1) \cap r(N_2) = r(N_3) \neq 0$. Therefore, $N_1 \cap N_2 = r(N_3)$ is a prime submodule of M by Proposition 3. Hence $N_1 \subseteq N_2$ or $N_2 \subseteq N_1$. □

Theorem 8. (see [1], Theorem 3.6) *Let P be a proper submodule of a multiplication module M. Then P is prime if and only if $UV \subseteq P \Rightarrow U \subseteq P$ or $V \subseteq P$ for each submodules of U and V of M.*

Corollary 4. *Let M be a multiplication R-module and N be a proper submodule of M. Then N is a prime if and only if $mm^* \subseteq N \Rightarrow m \in N$ or $m^* \in N$ for every $m, m^* \in M$.*

Lemma 1. *Let M be a multiplication R-module which is not a nil module and in which each non-zero submodule is primary. If N is the set of all nilpotent elements in M and if $N \neq 0$, then N is a prime submodule of M.*

Proof. Let $x \in r(N)$, then there exists an n such that $x^n \subseteq N$, i.e, x is nilpotent. Thus $N = r(N)$. This implies that N is prime submodule by Proposition 3. □

Definition 4. Let M be a multiplication R-module and N be a submodule of M. N is said to be a nil submodule of M if each element in N is nilpotent.

Theorem 9. *Let M be a multiplication R-module which is not a nil module and in which each non-zero submodule is primary. If M has nilpotents and if N_1 and N_2 are submodules in M such that $N_1 \cap N_2 = 0$, then at least one of N_1 or N_2 is a nil submodule.*

Proof. Since $r(N_1) \cap r(N_2) = r(0)$ and since $r(0)$ is a prime submodule by Lemma 1, we must have $r(N_1) \subseteq r(N_2)$ or $r(N_2) \subseteq r(N_1)$. Hence $r(N_1) = r(0)$ or $r(N_2) = r(0)$. This implies that $N_1 \subseteq r(0)$ or $N_2 \subseteq r(0)$. □

Definition 5. (i) Let M be a multiplication R-module. We say that M is a Q-module if and only if every submodule N in M such that $N \neq M$ is either primary or a product of primary submodules.

(ii) M is called primary module if 0_M is a primary submodule of M.

Theorem 10. *Let R be a Noetherian domain of dimension 1. Let M be a flat primary multiplication R-module. Let* $\operatorname{Ann}(M) = 0$. *Then M is a Q-module.*

Proof. Let N be a submodule of M. If $N = 0_M$, then N is primary submodule of M by assumption. Let $N \neq 0_M$. Since M is a multiplication module, $N = IM$ for some ideal I in R. Since R is a Noetherian domain of dimension 1, $I = Q_1 Q_2 ... Q_n$, where $Q_1, Q_2, ..., Q_n$ are primary ideals [2], Proposition 9.1. Let $\sqrt{Q_i} = P_i$ for each $i = 1, 2, ..., n$. Then $P_i M \neq M$ for each $i = 1, 2, ..., n$ by Theorem 4. Then $Q_i M$ are primary submodules of M for each $i = 1, 2, ..., n$, see [5], Theorem 3. Therefore $N = IM = Q_1 Q_2 ... Q_n M = (Q_1 M)(Q_2 M) ... (Q_n M)$ is a product of primary submodules. □

Definition 6. Let M be a multiplication R-module and N be a primary submodule of M. If $r(N) = P$ where P is a prime submodule of M, N is called P-primary submodule of M.

Theorem 11. *Let M be a Q-module. Let $N = Q_1 Q_2 ... Q_n$ be a submodule in M where Q_i is N_i-primary $(i = 1, 2, ..., n)$. Then there exist only finitely many minimal prime submodules over N.*

Proof. Let P be any prime submodule minimal over N. Since $N = Q_1 Q_2 ... Q_n \subseteq P$, for some i, $Q_i \subseteq P$. We have $Q_i \subseteq r(Q_i) = \{x \in M : x^n \subseteq Q_i\}$ and $r(Q_i) \subseteq P$. But P minimal over N implies that $r(Q_i) = P$. Thus the number of prime submodules minimal over N is less than or equal to n. □

Theorem 12. *Let M be a Q-module. If $N_1, N_2, ..., N_k$ are prime submodules in M such that $N_i \nsubseteq N_k$ for $i \neq k$, then* $\bigcap_{i=1}^{m} N_i = \prod_{i=1}^{m} N_i$.

Proof. Let $N = \bigcap_{i=1}^{m} N_i$, where $N_i \nsubseteq N_j$, $i \neq j$. Since M is a Q-module, N has a representation as a product of primary submodules, i.e. $N = \prod_{j=1}^{n} Q_j$. Further, for every i, $\prod_{j=1}^{n} Q_j = \bigcap_{i=1}^{m} N_i \subseteq N_i$. Thus for some j, $Q_j \subseteq N_i$ and $r(Q_j) \subseteq N_i$. But $\prod_{i=1}^{m} N_i \subseteq \prod_{j=1}^{n} Q_j \subseteq Q_j \subseteq r(Q_j)$. Hence for some $l, 1 \leq l \leq m, N_l \subseteq r(Q_j) \subseteq N_i$ which contradicts the hypothesis unless $l = i$. Thus $r(Q_j) = N_i$. Moreover if Q_j is N_j-primary, then Q_j is not N_k-primary, for $j \neq$

k, by distinctness of the N_i. We reindex, if necessary, so that for $i = 1, 2, ..m$, Q_i is N_i-primary. We have $N_1N_2...N_m \subseteq N = \bigcap_{i=1}^{m} N_i = Q_1Q_2...Q_mQ_{m+1}...Q_n \subseteq N_1N_2...N_mQ_{m+1}...Q_n \subseteq N_1N_2...N_m$. Thus $\bigcap_{i=1}^{m} N_i = \prod_{i=1}^{m} N_i$. □

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