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ON BANACH SPACES OF ANALYTIC FUNCTIONS AND MULTIPLICATION OPERATORS

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Abstract: We determine the general form of the operator $S : X \to X$ intertwining with the multiplication operators acting on X where X is a Banach space of analytic functions with some desired properties.

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1. Introduction

Let X be Banach space of analytic function defined on the open unit disk D of the plane such that:

a) $1 \in X, zX \subset X$.

b) For every $\lambda \in D$ the evaluation function at λ, $e_\lambda : X \to C$, given by $f \mapsto f(\lambda)$, is bounded.

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c) The set of analytic polynomials is dense in X.

d) For a fixed $n \in \mathbb{N}$, let every f in X has a unique decomposition $f = \bigoplus_{i=0}^{n-1} f_i$, where f_i belongs to X_i that is the closed linear span on the set $\{z^{nk+i} : k \geq 0\}$ in X for $i = 0, 1, \cdots, n-1$.

e) Let $h : D \to D$ be defined by $h(z) = az$ for some complex number a with $0 < |a| \leq 1$. We assume that foh is in X for each $f \in X$ and the composition operator C_h is bounded on X.

Throughout this article by a Banach space of analytic functions we mean one satisfying the above conditions. For more information on these spaces see [1]-[8]. A complex valued function φ, defined on D, for which $\varphi f \in X$ for each $f \in X$ is called a multiplier of X and the set of all multipliers is denoted by $M(X)$. Each multiplier determines a multiplication operator M_φ on X defined by $M_\varphi(f) = \varphi f$. By the Closed Graph Theorem it is easy to see that M_φ is bounded. Let $B(X)$ denotes the algebra of all bounded operators on X.

In the rest of this paper we assume that φ is a multiplier of X and the composition operator C_φ defined by $C_\varphi(f) = fo\varphi$ $(f \in X)$, is bounded on X.

2. Main Results

In the following suppose that $S \in B(X)$, where X is a Banach space of analytic functions on the open unit disc.

Lemma 1. *Assume that there exists a positive integer n such that $SM_{z^n} = M_{\phi^n}S$ and $\phi^i(z) = 0$ implies that $S(z^i) = 0$ for $i = 0, 1, 2, \cdots, n-1$. Suppose that X_i is the closed linear span of the set $\{z^{nk+i}; k \geq 0\}$ for $i = 0, 1, 2, \cdots, n-1$. If $f \in X$ having decomposition $f = f_0 + f_1 + \cdots + f_{n-1}$, where $f_i \in X_i$ for $i = 0, 1, 2, \cdots, n-1$, then*

$$S(f) = C_\varphi(f_0)\Phi_0 + C_\varphi(f_1)(\frac{\Phi_1}{\varphi} + \Phi_0) + \cdots + C_\varphi(f_{n-1})(\frac{\Phi_{n-1}}{\varphi^{n-1}} + \Phi_0),$$

when $\Phi_0 = S(1)$ and $\Phi_i = (SM_{z^i} - M_{\varphi^i}S)(1), i = 0, 1, 2, \cdots, n-1$.

Proof. Since $\Phi_0 = S(1)$ and $\Phi_i = (SM_{z^i} - M_{\varphi^i}S)(1)$ for $i = 1, 2, \cdots, n-1$, we have

$$S(z^n) = S(M_{z^n}(1)) = M_{\varphi^n}(S(1)) = \varphi^n\Phi_0,$$

so $S(z^{n_k}) = \varphi^{nk}\Phi_0$. On the other hand we have $\Phi_i = (SM_{z^i} - M_{\varphi^i}S)(1) = S(z^i) - \varphi^i\Phi_0$, hence $S(z^i) = \varphi^i\Phi_0 + \Phi_i$ for $i = 1, 2, \cdots, n-1$. Note that since the relation $\varphi^i(z) = 0$ implies that $S(z^i) = 0$, so $\frac{\Phi_i}{\varphi^i}$ makes sense for $i = 1, 2, \cdots, n-1$. We have

$$S(z^{nk+i}) = S(M_{z^{nk}} M_{z^i}(1)) = M_{\varphi^{nk}} S(M_{z^i}(1))$$
$$= M_{\varphi^{nk}}(\varphi^i \Phi_0 + \Phi_i) = \varphi^{nk+i}\Phi_0 + \varphi^{nk}\Phi_i$$

for $i = 1, 2, \cdots, n-1$. Now let $f \in X$ and p be a polynomial in X. If $p = p_0 + p_1 + \cdots + p_{n-1}$, where $p_i \in X_i$ for $i = 0, 1, 2, \cdots, n-1$, then we get

$$\begin{aligned} S(p) &= S(p_0) + S(p_1) + \cdots + S(p_{n-1}) \\ &= C_\varphi(p_0)\Phi_0 + C_\varphi(p_1)(\frac{\Phi_1}{\varphi} + \Phi_0) + \cdots + C_\varphi(p_{n-1})(\frac{\Phi_{n-1}}{\varphi^{n-1}} + \Phi_0). \end{aligned}$$

If we let p tends to f, then the boundedness of S and C_φ imply that

$$S(f) = C_\varphi(f_0)\Phi_0 + C_\varphi(f_1)(\frac{\Phi_1}{\varphi} + \Phi_0) + \cdots + C_\varphi(f_{n-1})(\frac{\Phi_{n-1}}{\varphi^{n-1}} + \Phi_0),$$

where $f_i \in X_i$ for $i = 0, 1, 2, \cdots, n-1$. Thus the proof is complete. □

Theorem 2. *Suppose that there exists a positive integer n such that $SM_{z^n} = M_{\varphi^n} S$ and $S(z^i) = 0$ whenever $\varphi^i(z) = 0$ for $i = 0, 1, 2, \cdots, n-1$. If $SM_z - M_\varphi S$ is a compact operator and $C_{\varphi^{-1}}$ is bounded on X, then $S = M_{\Phi_0} C_\varphi$ where $\Phi_0 = S(1)$.*

Proof. Let $f \in X$ has decomposition $f = f_0 + f_1 + \cdots + f_{n-1}$, where $f_i \in X_i$ for $i = 0, 1, 2, \cdots, n-1$. By Lemma 1 we have

$$\begin{aligned} (SM_z - M_\varphi S)(f) &= [C_\varphi(zf_{n-1})\Phi_0 + \cdots + C_\varphi(zf_{n-2})(\frac{\Phi_{n-1}}{\varphi^{n-1}} + \Phi_0] \\ &- M_\varphi[C_\varphi(f_0) + \cdots + C_\varphi(f_{n-1})(\frac{\Phi_{n-1}}{\varphi^{n-1}} + \Phi_0)] = C_\varphi(f_0 - f_1)\Phi_1 + \cdots \\ &+ C_\varphi(f_{n-2} - f_{n-1})(\frac{\Phi_{n-1}}{\varphi^{n-2}}) = C_\varphi(f_0)\Phi_1 + C_\varphi(f_1)(\frac{\Phi_2}{\varphi} - \Phi_1) + \cdots \\ &+ C_\varphi(f_{n-2})(\frac{\Phi_{n-1}}{\varphi^{n-2}} - \frac{\Phi_{n-2}}{\varphi^{n-3}}) - C_\varphi(f_{n-1})(\frac{\Phi_{n-1}}{\varphi^{n-2}}). \end{aligned}$$

Since $M_{\Phi_1} C_\varphi|_{X_0} = (SM_z - M_\varphi S)|_{X_0}$, so $SM_{\phi_1} C_\varphi|_{X_0}$ is compact on X_0 and it follows that

$$M_{\Phi_1 \varphi^n} C_\varphi|_{X_0} = M_{\varphi^n} M_{\Phi_1} C_\varphi|_{X_0}$$

is compact. Also we see that

$$M_{\Phi_1 \varphi^n} C_\varphi|_{X_i} = M_{\varphi^i} M_{\Phi_1} C_\varphi|_{X_0} M_{z^{n-i}}|_{X_i}$$

is compact for $i = 0, 1, 2, \cdots, n-1$. Thus the operator $M_{\Phi_1 \varphi^n} C_\varphi$ is compact on X. Now by the Fredholm Alternative Theorem, $\Phi_1 \varphi^n = 0$. Since φ is univalent, we get $\Phi_1 = 0$. Again, note that $(SM_z - M_\varphi S)|_{X_1} = M_{\frac{\Phi_2}{\varphi}} C_\varphi |_{X_i}$ is a compact operator. Hence

$$M_{\frac{\Phi_2}{\varphi}} C_\varphi (M_z | X_0) = M_{\Phi_2} C_\varphi$$

is compact on X_0. By a similar argument as above we have $\Phi_2 = 0$. Repeating this method we get $\Phi_i = 0$ for $i = 1, 2, \cdots, n-1$. Therefore $S = M_{\Phi_0} C_\varphi$. □

Corollary 3. *Let X be a Banach space of analytic functions and let $S : X \to X$ be a bounded operator such that $SM_{z^n} = a^n M_{z^n} S$ for some complex number a with $0 < |a| \leq 1$ and some positive integer n. Also suppose that $SM_z - aM_z S$ is a compact operator, then $S = M_\psi C_{az}$ for some $\psi \in M(X)$.*

Corollary 4. *Let X be a Banach space of analytic functions and let $S : X \to X$ be a bounded operator such that $SM_{z^n} = M_{z^n} S$ for some positive integer n. If $SM_z - M_z S$ is a compact operator, then $S = M_\psi$ for some $\psi \in M(X)$.*

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