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# POSITIVE SOLUTIONS OF SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS WITH PERIODIC BOUNDARY VALUE CONDITIONS

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**Abstract:** Criteria are established for existence of positive solutions to the second order periodic boundary value problem

$$-u''(t) + pu'(t) + p_1 u(t) = f(t, u), \quad t \in I = [0, T],$$
$$u(0) = u(T), \quad u'(0) = u'(T),$$

where $p \in \mathbf{R}$ and $p_1 \geq 0$. The discussion is based on the fixed point index theory in cones.

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## 1. Introduction

The existence problems of solutions for nonlinear second-order periodic boundary value problems, have attracted many authors' attention and con-

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cern. For example, see [2, 6, 9-10]. In recent years the fixed point theorems of cone mapping, have been availably applied to the two-point boundary value problems of second order differential equations, and some results of existence and multiplicity of positive solutions have been obtained (see [3, 5, 8]). Inspired and Motivated by the recent work in [1] and [7], our purpose here is to study the existence of positive solutions of nonlinear second-order periodic boundary value problem

$$-u''(t) + pu'(t) + p_1 u(t) = f(t,u), \quad t \in I = [0,T], \tag{1}$$

$$u(0) = u(T), \quad u'(0) = u'(T), \tag{2}$$

where $f : I \times \mathbf{R}^+ \to \mathbf{R}$ continuous, $p \in \mathbf{R}$, $p_1 \geq 0$.

To be convenient, we introduce the following notations

$$f_0 = \lim_{u\to 0^+}\inf \min_{t\in I} \frac{f(t,u)}{u}, \quad f^0 = \lim_{u\to 0^+}\sup \max_{t\in I} \frac{f(t,u)}{u},$$

$$f_\infty = \lim_{u\to +\infty}\inf \min_{t\in I} \frac{f(t,u)}{u}, \quad f^\infty = \lim_{u\to +\infty}\sup \max_{t\in I} \frac{f(t,u)}{u}.$$

The main result of this paper is the following theorem.

**Theorem 1.** *Assume that $f(t,u) \in C(\mathbf{R} \times \mathbf{R}^+)$. If one of the following conditions is satisfied*

($H_1$) $-\infty < f_0, \quad f^0 < p_1 < f_\infty,$

($H_2$) $-\infty < f_\infty, \quad f^\infty < p_1 < f_0.$

*Then boundary value problem (1)-(2) has at least one positive solution.*

## 2. Preliminaries and Lemmas

We denote by $C(I)$ the space of continuous functions $u : I \to I$ and by $\|\cdot\|$ its max-norm $\|u\| = \max_{t\in I} |u(t)|$.

If ($H_1$) or ($H_2$) is satisfied, it is easy to prove that $f(t,u)/u$ is lower-bounded for $t \in I$ and $u > 0$. Thus there exists $M > 0$ such that

$$f(t,u) \geq -Mu, \quad \forall t \in I, \ u \geq 0.$$

Let $f_1(t,u) = f(t,u) + Mu$, then $f_1(t,u) \geq 0$ for all $t \in I, \ u \geq 0$, and equation (1) is equivalent to

$$-u''(t) + pu'(t) + qu(t) = f_1(t,u), \quad t \in I = [0,T], \tag{3}$$

where $q = p_1 + M > 0$.

We shall consider the existence of positive solutions of BVP (3)-(2).

Given $\sigma \in C(I)$, we first consider the linear periodic boundary value problem corresponding to equation (3),

$$-u''(t) + pu'(t) + qu(t) = \sigma(t), \quad t \in I, \tag{4}$$

with the periodic boundary condition (2).

For convenience, let

$$r_1 := \frac{p + \sqrt{p^2 + 4q}}{2} > 0, \quad r_2 := \frac{p - \sqrt{p^2 + 4q}}{2} < 0. \tag{4}$$

**Lemma 2.** (see [1]) *Let $\sigma \in C(I)$. Then $u \in C^2(I)$ is a solution of (4), (2) if and only if*

$$u(t) = \int_0^T G(t,s)\sigma(s)ds, \quad t \in I, \tag{5}$$

*where*

$$G(t,s) = \frac{1}{r_1 - r_2}\begin{cases} \dfrac{e^{r_1(t-s)}}{e^{r_1 T} - 1} + \dfrac{e^{r_2(t-s)}}{1 - e^{r_2 T}}, & 0 \le s < t \le T, \\ \dfrac{e^{r_1(T+t-s)}}{e^{r_1 T} - 1} + \dfrac{e^{r_2(T+t-s)}}{1 - e^{r_2 T}}, & 0 \le t \le s \le T. \end{cases}$$

Let

$$w(t) = \frac{1}{r_1 - r_2}\left(\frac{e^{r_1 t}}{e^{r_1 T} - 1} + \frac{e^{r_2 t}}{1 - e^{r_2 T}}\right) = \frac{h(t)}{(r_1 - r_2)\left(e^{r_1 T} - 1\right)\left(1 - e^{r_2 T}\right)},$$

where $h(t) = \left(1 - e^{r_2 T}\right)e^{r_1 t} + \left(e^{r_1 T} - 1\right)e^{r_2 t}$. Then

$$G(t,s) = \begin{cases} w(t-s), & 0 \le s \le t \le T, \\ w(T+t-s), & 0 \le t \le s \le T. \end{cases}$$

Setting $h'(t) = 0$, we get

$$t = t_* = \frac{1}{r_1 - r_2}\ln\frac{-r_2\left(e^{r_1 T} - 1\right)}{r_1\left(1 - e^{r_2 T}\right)}. \tag{6}$$

By

$$e^t > 1 + t, \quad t \in \mathbf{R} \setminus \{0\},$$

we have

$$-r_2\left(e^{r_1 T} - 1\right) > -r_2 r_1 T > r_1\left(1 - e^{r_2 T}\right), \tag{7}$$

and

$$r_1\left(e^{-r_2T}-1\right) > -r_1r_2T > -r_2\left(1-e^{-r_1T}\right). \tag{8}$$

From (7) and (8), we easily know that $t_* \in (0,T)$. Noticing that

$$h''(t) = (1-e^{r_2T})r_1^2e^{r_1t} + (e^{r_1T}-1)r_2^2e^{r_2t} > 0, \quad t\in[0,T].$$

We have that $t_*$ is the minimum point of $h(t)$ in $[0,T]$, $h'(t)\le 0$ for $t\in[0,t_*]$, and $h'(t)\ge 0$ for $t\in[t_*,T]$. Hence

$$h(t_*) \le h(t) \le \max\{h(0),h(T)\} = e^{r_1T}-e^{r_2T}, \ \ t\in I. \tag{9}$$

Thus, we obtain

$$G(t,s) \le G(s,s) = \frac{e^{r_1T}-e^{r_2T}}{(r_1-r_2)\left(e^{r_1T}-1\right)\left(1-e^{r_2T}\right)} := c, \quad t,s\in I, \tag{10}$$

and

$$G(t,s) \ge \delta G(s,s) = \delta c, \quad t,s\in I, \tag{11}$$

where

$$\begin{aligned}\delta := \frac{h(t_*)}{e^{r_1T}-e^{r_2T}} &= \frac{\left(1-e^{r_2T}\right)e^{r_1t_*}+\left(e^{r_1T}-1\right)e^{r_2t_*}}{e^{r_1T}-e^{r_2T}}\\ &= \frac{1}{e^{r_1T}-e^{r_2T}}\left[\left(1-e^{r_2T}\right)\left(\frac{-r_2\left(e^{r_1T}-1\right)}{r_1\left(1-e^{r_2T}\right)}\right)^{\frac{r_1}{r_1-r_2}}\right.\\ &\qquad\left.+\left(e^{r_1T}-1\right)\left(\frac{-r_2\left(e^{r_1T}-1\right)}{r_1\left(1-e^{r_2T}\right)}\right)^{\frac{r_2}{r_1-r_2}}\right] < 1. \end{aligned}\tag{12}$$

Let $C^+(I)$ be the cone of all nonnegative function in $C(I)$. We now define a mapping $A: C^+(I)\to C^+(I)$ by

$$Au(t) = \int_0^T G(t,s)f_1(s,u(s))ds, \quad t\in I. \tag{13}$$

By Lemma 2, positive solution of BVP (1)-(2) is equivalent to nontrivial fixed point of $A$. We will find the nonzero fixed point of $A$ by using the fixed point index theory in cones. For this, choosing the sub-cone $K$ of $C^+(I)$ by

$$K = \{u\in C^+(I) | u(t)\ge \delta\|u\|, \ \forall t\in I\},$$

where $\delta>0$ as in (12), we have the following result.

**Lemma 3.** *$A(K) \subset K$, and $A : K \to K$ is completely continuous.*

*Proof.* Let $u \in K$. It is easy to see by (10) that

$$Au(t) \leq \int_0^T G(s,s) f_1(s,u(s))ds = c\int_0^T f_1(s,u(s))ds, \quad \forall t \in I,$$

and therefore

$$\|Au\| \leq c\int_0^T f_1(s,u(s))ds.$$

Using (11) and the above inequality, we have

$$Au(t) = \int_0^T G(t,s) f_1(s,u(s))ds \geq \delta c\int_0^T f_1(s,u(s))ds \geq \delta\|Au\|, \quad \forall t \in I,$$

namely $Au \in K$. Thus $A(K) \subset K$.

Obviously, $A : K \to K$ is continuous. Let $D \subset K$ be a bounded set. For every $u \in D$, since

$$(Au)'(t) = \int_0^T G_t(t,s) f_1(s,u(s))ds, \quad t \in I,$$

where $G_t(t,s)$ as in Lemma 1 in [1], a.e.,

$$G_t(t,s) = \frac{1}{r_1 - r_2}\begin{cases} \dfrac{r_1 e^{r_1(t-s)}}{e^{r_1 T} - 1} + \dfrac{r_2 e^{r_2(t-s)}}{1 - e^{r_2 T}}, & 0 \leq s < t \leq T, \\ \dfrac{r_1 e^{r_1(T+t-s)}}{e^{r_1 T} - 1} + \dfrac{r_2 e^{r_2(T+t-s)}}{1 - e^{r_2 T}}, & 0 \leq t < s \leq T, \end{cases}$$

it follows that $\{(Au)' | u \in D\}$ is a bounded set. Consequently, $A(D)$ is an equicontinuous and bounded family of functions. Thus, by Arzela-Ascoli's Theorem, $A : K \to K$ is completely continuous. $\square$

For $r > 0$, let $K_r = \{u \in K | \|u\| < r\}$, and $\partial K_r = \{u \in K | \|u\| = r\}$, which is the relative boundary of $K_r$ in $K$. The following two lemmas are needed in our argument.

**Lemma 4.** (see [4]) *Let $A : K \to K$ be completely continuous mapping. If $\lambda Au \neq u$ for any $u \in \partial K_r$ and $0 < \lambda \leq 1$, then the fixed point index $i(A, K_r, K) = 1$.*

**Lemma 5.** (see [4]) *Let $A : K \to K$ be completely continuous mapping. Suppose that the following two conditions are satisfied:*
*(i)* $\inf_{u\in\partial K_r} \|Au\| > 0$.
*(ii)* $\lambda Au \neq u$ *for any* $u \in \partial K_r$ *and* $\lambda \geq 1$.
*Then,* $i(A, K_r, K) = 0$.

## 3. Proof of the Main Result

*Proof of Theorem 1.* Assume ($H_1$) is satisfied. We show that the operator A defined by (13) has a nonzero fixed point. Since $f^0 < p_1$, by the definition of $f^0$, we may choose $\epsilon \in (0, M + p_1)$ and $r_0 > 0$ such that

$$f(t, u) \leq (p_1 - \epsilon)u, \quad \forall t \in I,\ 0 \leq u \leq r_0. \tag{14}$$

Let $r \in (0, r_0)$, we now prove that $\lambda Au \neq u$ for $u \in \partial K_r$ and $0 < \lambda \leq 1$. In fact, if there exist $u_0 \in \partial K_r$ and $0 < \lambda_0 \leq 1$ such that $\lambda_0 Au_0 = u_0$, then by the definition of $A$ and Lemma 2, we have

$$-u_0''(t) + pu_0'(t) + qu_0(t) = \lambda_0 f_1(t, u_0(t)), \quad t \in I, \tag{15}$$

along with the boundary condition (2). From (14), (15) and the definition of $f_1$, it follows that

$$\begin{aligned} -u_0''(t) + pu_0'(t) + (p_1 + M)u_0(t) &\leq \lambda_0(Mu_0(t) + (p_1 - \epsilon)u_0(t)) \\ &\leq (M + p_1 - \epsilon)u_0(t). \end{aligned}$$

Integrating the both sides of the above inequality from 0 to $T$ and using (2), we obtain

$$(p_1 + M)\int_0^T u_0(t)dt \leq (M + p_1 - \epsilon)\int_0^T u_0(t)dt.$$

Noticing that $\int_0^T u_0(t)dt > 0$ $(u_0(t) \geq \delta\|u_0\| = \delta r > 0)$, we conclude that $p_1 + M \leq p_1 + M - \epsilon$, which is a contradiction. Hence $A$ satisfies the hypothesis of Lemma 4. By Lemma 4 we have

$$i(A, K_r, K) = 1. \tag{16}$$

On the other hand, since $f_\infty > p_1$, there exist $\epsilon > 0$ and $H > 0$ such that

$$f(t, u) \geq (p_1 + \epsilon)u, \quad \forall t \in I,\ u \geq H. \tag{17}$$

Setting $C = \max_{t\in I, 0\le u\le H} |f(t,u) - (p_1+\epsilon)u| + 1$, then we have

$$f(t,u) \ge (p_1+\epsilon)u - C, \quad \forall t \in I,\ u \ge 0. \tag{18}$$

Choose $R > R_0 := \max\{H/\delta, r_0\}$. Let $u \in \partial K_R$. Since $u(t) \ge \delta\|u\| > H$ for $t \in I$, from (10), (11), and (17) we have

$$\begin{aligned}\|Au\| &= \max_{t\in I}\int_0^T G(t,s)(f(s,u(s)) + Mu(s))ds \\ &\ge c\delta\int_0^T (f(s,u(s)) + Mu(s))ds \ge (M+p_1+\epsilon)c\delta\int_0^T u(s)ds \\ &\ge (M+p_1+\epsilon)c\delta^2 T\|u\|. \end{aligned}\tag{19}$$

Therefore $\inf_{u\in\partial K_R}\|Au\| > 0$, namely the hypothesis (i) of Lemma 5 holds. Next we show that if $R$ is large enough, then $\lambda Au \ne u$ for $u \in \partial K_R$ and $\lambda \ge 1$. In fact, if there exist $u_0 \in \partial K_R$ and $\lambda_0 \ge 1$ such that $\lambda_0 Au_0 = u_0$, then $u_0(t)$ satisfies equation (15) and boundary condition (2). Thus, from (18) that

$$-u_0''(t) + pu_0'(t) + (p_1+M)u_0(t) \ge f(t,u_0(t)) + Mu_0(t) \ge (M+p_1+\epsilon)u_0(t) - C.$$

Integrating the both sides of the above inequality from 0 to $T$ and using (2), we obtain

$$(p_1+M)\int_0^T u_0(t)dt \ge (M+p_1+\epsilon)\int_0^T u_0(t)dt - CT,$$

which implies that

$$\int_0^T u_0(t)dt \le \frac{CT}{\epsilon}. \tag{20}$$

Since $u_0 \in K$, we have

$$\int_0^T u_0(t)dt \ge \int_0^T \delta\|u_0\|dt = \delta T\|u_0\|.$$

From the above inequality and (20) it follows that

$$\|u_0\| \le \frac{1}{\delta T}\int_0^T u_0(t)dt \le \frac{C}{\delta\epsilon}. \tag{21}$$

Let $R > \max\{H/\delta, r_0, C/\delta\epsilon\}$, then for any $u \in \partial K_R$ and $\lambda \ge 1$, $\lambda Au \ne u$. Hence the hypothesis (ii) of Lemma 5 also holds. By Lemma 5,

$$i(A, K_R, K) = 0. \tag{22}$$

By the additivity of fixed point index, (16) and (22) we have

$$i(A, K_R \setminus \bar{K}_r, K) = i(A, K_R, K) - i(A, K_r, K) = -1.$$

Hence $A$ has a fixed point in $K_R \setminus \bar{K}_r$, which is the positive solution of BVP (1)-(2).

Assume ($H_2$) is satisfied. The proof of this case is similar to that of above case, we omit the details here.

The proof is completed. □

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