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STABLE COHERENT SYSTEMS ON INTEGRAL PROJECTIVE CURVES: AN ASYMPTOTIC EXISTENCE THEOREM

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Abstract: Fix integers $k > n \geq 2$, an integral projective curve X, a rank n vector bundle E on X and an ample line bundle H on X. Here we prove the existence of an integer t_0 (depending only on k, n, X, H, E) such that for all integers $t \geq t_0$ a general k-dimensional linear subspace V of $E \otimes H^{\otimes t}$ spans E and the coherent system (E, V) is α-stable for every $\alpha \gg 0$. To prove this result we prove the existence of an integer t_0 (depending only on k, n, X, H, E) such that for all integers $t \geq t_0$ a general k-dimensional linear subspace V of $E \otimes H^{\otimes t}$ spans E and the natural map $\bigwedge^n(V) \to H^0(C, \det(E))$ is injective.

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1. The Statements

Let X be an integral projective curve. A coherent system on X is a pair (E, V) such that E is a torsion free bundle on X and $V \subseteq H^0(X, E)$ is a linear subspace. The pair (E, V) is of type (n, d, k) if $\mathrm{rank}(E) = n$, $\deg(E) = d$ and $\dim(V) = k$. Hence (E, V) is uniquely determined by E and by a map of $\mathcal{O}_X$-sheaves ${\mathcal{O}_X}^{\oplus k} \to E$ which induces an injection of global sections. Fix

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$\alpha \in \mathbb{R}$. Let $\mu(E) := d/n$ denote the slope of E. Set $\mu_\alpha(E, V) := \mu(E) + \alpha k/n$. The real number μ_α is called the α-slope of the pair (E, V). A subcoherent system$(F, W) \subseteq (E, V)$ is a coherent system such that $F \subseteq E$ and $W \subseteq V \cap H^0(X, F)$. The pair (E, V) is said to be α-stable (resp. α-semistable) if $\mu_\alpha(F, W) < \mu_\alpha(E, V)$ (resp. $\mu_\alpha(F, W) \leq \mu_\alpha(E, V)$) for all proper coherent subsystems (F, W) of (E, V). See [3] for the general theory in the case X smooth and when the algebraically closed base field has characteristic zero. See [5] for the general theory for non-integral curves. See [8] for the general theory for higher dimensional projective varieties. See [4], [6] and [7] for further results on coherent systems. Here we prove the following result.

Theorem 1. *Fix integers $k > n \geq 2$, an integral projective curve X, a rank n vector bundle E on X and an ample line bundle H on X. There is an integer t_0 (depending only on k, n, X, H, E) such that for all integers $t \geq t_0$ a general k-dimensional linear subspace V of $E \otimes H^{\otimes t}$ spans E and the coherent system (E, V) is α-stable for every $\alpha \gg 0$.*

According to [3] the α stability of a coherent system for all $\alpha \gg 0$, say for $\alpha \geq \alpha_0$ is an important concept. In the set-up of Theorem 1 we may take α_0 depending only on k, n, X and E, but not from t and H (see Remark 1 and part (iii) of the proof of Theorem 2). We will show that Theorem 1 is an easy consequence of the following result.

Theorem 2. *Fix integers $k > n \geq 2$, an integral projective curve X, a rank n vector bundle E on X and an ample line bundle H on X. There is an integer t_0 (depending only on k, n, X, H, E) such that for all integers $t \geq t_0$ a general k-dimensional linear subspace V of $E \otimes H^{\otimes t}$ spans E and the natural map $u_{(E,V)} : \bigwedge^n(V) \to H^0(C, \det(E))$ is injective.*

We may take as the integer t_0 appearing in the statement of Theorem 1 the same integer t_0 appearing in the statement of Theorem 2 (see Remark 1).

We work over an algebraically closed field $\mathbb{K}$.

2. The Proofs

For all integers $a > b > 0$ let $G(b, a)$ denote the Grassmannian of all $(a - b)$-dimensional linear subspaces of $\mathbb{K}^{\oplus a}$. Hence $\dim(G(b, a)) = b(a - b)$ and $G(b, a)$ has a universal rank b quotient vector bundle $Q_{G(b,a)}$ and a universal rank $(a - b)$ subbundle $S_{G(b,a)}$ such that $S^*_{G(b,a)}$ is spanned. Furthermore, $h^0(G(b, a), Q_{G(b,a)}) = h^0(G(b, a), S^*_{G(b,a)}) = a$. Any finite-dimensional vector space is isomorphic (not in a canonical way) to its dual. Any such isomorphism induces an isomorphism of $G(b,a)$ onto $G(a-b,a)$. Any such isomorphism maps $Q_{G(a-b,a)}$ isomorphically onto $S^*_{G(b,a)}$ and $S_{G(a-b,a)}$ isomorphically onto $Q^*_{G(b,a)}$. We have $\mathrm{Aut}^0(G(b,a)) \cong \mathrm{Aut}(\mathbf{P}^{a-1}) = \mathrm{PGL}(a)$. If $a \ne 2b$, then $\mathrm{Aut}^0(G(b,a)) = \mathrm{Aut}(G(b,a))$. If $b = 2a$, then $\mathrm{Aut}^0(G(b,a))$ is an index two normal subgroup of $\mathrm{Aut}(G(b,a))$. If $a = 2b$ a representative of the order two group $\mathrm{Aut}^0(G(b,a))/\mathrm{Aut}(G(b,a))$ is induced by any isomorphism between $G(b,a)$ and $G(a-b,a)$ induced by any isomorphism between an a-dimensional vector space and its dual. Set $N_{b,a} := \binom{a}{b} - 1$. Let $j_{b,a} : G(b,a) \to \mathbf{P}^{N_{a,b}}$ denote the Plücker embedding, i.e. the embedding induced by the complete linear system associated to the positive generator $\mathcal{O}_{G(b,a)}(1)$ of $\mathrm{Pic}(G(b,a)) \cong \mathbb{Z}$. We have $\det(Q_{G(b,a)}) \cong \det(S^*_{G(b,a)}) \cong \mathcal{O}_{G(b,a)}(1)$. Fix integers $a > b > 0$, $x > 0$, an x-dimensional linear subspace A of $\mathbb{K}^{\oplus(a+x)}$ and an a-dimensional linear subspace B of $\mathbb{K}^{\oplus(a+x)}$ such that $A + B = \mathbb{K}^{\oplus(a+x)}$, i.e. such that $A \cap B = \{0\}$. Set $V(A;b) := \{D \in G(b,a+x) : D \cap A \ne \{0\}\}$. For any $D \in G(b,a+x)$ set $\psi_{A,B,b}(D) := (D + A) \cap B$. If $D \notin V(A;b)$, then $\dim(D + A) = b + x$ and hence $\dim((D + A) \cap B) = b$. In this way we obtain a smooth surjective morphism $\psi_{A,B,b} : G(b,a+x)\backslash V(A;b) \to G(b,a)$, which is usually called the linear projection from A. The morphism $\psi_{A,B,b}$ does not depend on the choice of B, up to an element of $\mathrm{Aut}^0(G(b,a))$. The corresponding dominant rational map from $G(b,a+x)$ onto $G(b,a)$ does not depend on the choice of A, up to an element of $\mathrm{Aut}^0(G(b,a+x))$. For any reduced projective curve Y, any vector bundle G on Y and any linear subspace W of $H^0(Y,G)$ spanning G let $\phi_{G,W} : Y \to G(n,k)$, $n := \mathrm{rank}(G)$, $k := \dim(W)$, denote the morphism associated to the pair (G,W) by the universal property of the Grassmannian $G(n,k)$.

Remark 1. Here, following [9] and [1], we explain the geometric meaning of Theorem 2 and why it implies Theorem 1. Fix an integer $t \ge t_0$ and a general k-dimensional linear subspace V spanning $E(tH)$. The natural map $\bigwedge^n(V) \to H^0(C,\det(E))$ is injective if and only if the curve $j_{n,k}\circ\phi_{E(tH),V}(X)$ spans $\mathbf{P}^{N_{n,k}}$. Hence for all linear subspaces M of V such that $\dim(M) \le n$ the evaluation map $\mathcal{O}_X \otimes M \to E(tH)$ is an inclusion of sheaves. Hence $\dim(V \cap H^0(X,A)) \le \mathrm{rank}(A)$ for all subsheaves A of E such that $1 \le \mathrm{rank}(A) \le n-1$. There is $\gamma \in \mathbb{R}$ (depending only on E, but not from t) such that $\mu(A) \le \mu(E(tH)) + \gamma$ for all subsheaves A of E such that $1 \le \mathrm{rank}(A) \le n-1$. Hence the inequality $k > n$ implies that the coherent system $(E(tH),V)$ is α-stable for all $\alpha \gg 0$. An exact lower bound for the admissible α may be given in terms of Lange's stability degrees of E and it is independent on the integer $t \ge t_0$.

Notation 1. Let X be an integral projective curve, $P \in X_{reg}$ and E a rank n vector bundle on X. Let $\mathbb{K}_P$ denote the skyscraper sheaf supported by P and with $h^0(X, \mathbb{K}_P) = 1$. Let $E|\{P\}$ denote the fiber of E over P. Fix any surjection $m : E|\{P\} \to \mathbb{K}$ of vector spaces. The surjection m corresponds to a unique surjection $\mu : E \to \mathbb{K}_P$ of $\mathcal{O}_X$-sheaves. The coherent sheaf $\mathrm{Ker}(\mu)$ is locally free, because it is locally free (as E) on $X \backslash \{P\}$ and it is locally free at P because it is torsion free and X is smooth at P. We will say that $\mathrm{Ker}(\mu)$ is obtained from E making a negative elementary transformation supported by P. $\mathrm{Ker}(\mu)$ has rank n and degree $\deg(E) - 1$. A sheaf F will be said to be obtained from E making a positive elementary transformation supported by P if F^* is obtained from E^* making a negative elementary transformation.

Remark 2. Let X be an integral projective curve, $P \in X_{reg}$ and E a rank n vector bundle on X. A torsion free sheaf F on X is obtained from E making a positive elementary transformation supported by P if and only if it fits in an exact sequence

$$0 \to E \to F \to \mathbb{K}_P \to 0\,. \tag{1}$$

Fix any such F. F is locally free, $\mathrm{rank}(F) = n$, $\det(F) \cong \det(E)(P)$ and $\deg(F) = \deg(E) + 1$. From the exact sequence (1) we obtain $h^0(X, E) \leq h^0(X, F) \leq h^0(X, E) + 1$, $h^1(X, F) \leq h^1(X, E)$ and $h^0(X, F) = h^0(X, E) + 1$ if $h^1(X, E) = 0$. Notice that $E(P)$ is obtained from E making n positive elementary transformations supported by P. From now on we assume $h^1(X, E) = 0$ and that E is spanned by its global sections. The exact sequence 1 implies $h^1(X, F) = 0$. From the exact sequence (1) we obtain that F is spanned by its global sections at each point of $X \backslash \{P\}$. From the exact sequence (1) and the assumption $h^1(X, E) = 0$ we also obtain that $H^0(X, F)$ spans $F|\{P\}$. Hence F is spanned by its global sections. For any linear subspace $V \subseteq H^0(X, E)$ spanning E at P, there is a linear subspace $V' \subseteq H^0(X, F)$ such that V' spans F at P, $\dim(V') = \dim(V) + 1$ and $V' \cap H^0(X, E) = V$, where we identify E to a subsheaf of F and $H^0(X, E)$ to a linear subspace of $H^0(X, F)$ using the exact sequence (1). If V spans E, then V' spans F. If $\phi_{E,V}$ is an embedding, then $\phi_{F,V'}$ is an embedding.

Remark 3. Let X be an integral projective curve, $P \in X_{reg}$ and E a rank n vector bundle on X. Let Y be the reducible curve which is the union of X and of a smooth rational curve D such that $D \cap X = \{P\}$ and Y has an ordinary node at P. Hence $p_a(Y) = p_a(X)$. An elementary property of the nodal singularities gives the existence of a vector bundle G on Y such that $G|X \cong E$ and $G|D$ is the direct sum E_1 of one line bundle of degree one and $n - 1$ line bundles of degree zero. More precisely, the set of all isomorphism classes of such bundles G is parametrized by the projective space $\mathbf{P}(E|\{P\}) \cong \mathbf{P}^{n-1}$: the point of the parameter space corresponding to G is the point of $\mathbf{P}(G|\{P\})$ corresponding to the degree one factor of E_1 (after the identification of the projective spaces $\mathbf{P}(E|\{P\})$ and $\mathbf{P}(E_1|\{P\})$ with $\mathbf{P}(G|\{P\})$ obtained by requiring $E = G|X$ and $E_1 = G|D$). Hence there is a natural isomorphism between this parameter space and the set of all positive elementary transformations of E supported by P. We do not claim that different points of this parameter space give non-isomorphic vector bundles; for instance, if $E \cong \mathcal{O}_X{}^{\oplus n}$, then all such bundles G are isomorphic. Fix any such bundle G. From now on we assume $h^1(X, E) = 0$ and that E is spanned by its global sections. The Mayer-Vietoris exact sequence

$$0 \to G \to G|X \oplus G|D \to G|\{P\} \to 0 \tag{2}$$

gives $h^1(Y, G) = 0$ and that G is spanned by its global sections. Fix an integer $k > 0$ and a linear subspace $V \subseteq H^0(X, E)$ such that $\dim(V) = k$ and V spans E. Identify $G|X$ with E and hence see V as a linear subspace of $H^0(X, G|X)$. By the exact sequence (2) we obtain the existence of a linear subspace W of $H^0(Y, G)$ such that $\dim(W) = k + 1$, $W|X = V$ and $W|D = H^0(D, G|D)$. Furthermore, if $\phi_{E,V} : X \to G(n, k)$ is an embedding, then $\phi_{G,W} : Y \to G(n, k + 1)$ is an embedding and (up to the fixed identification of E with $G|X$ and V with a linear subspace of $H^0(X, G|X)$) the curve $\phi_{G,W}(Y)$ is the union of $\phi_{E,V}(X)$ and of a line D' such that $\phi_{E,V}(X) \cap D' = \{P\}$. Conversely, fix $\phi_{E,V}(X) \subset G(n, k)$ and a line $T \subset G(n, k + 1)$ such that $G(n, k) \cap T = \{P\}$. Any such line gives a positive elementary transformation of the restricted universal quotient bundle $Q_{n,k}|\phi_{E,V}(X)$ supported by $\phi_{E,V}(P)$, i.e. a positive elementary transformation of E supported by P. Notice that all such lines (for a fixed positive elementary transformation) are projectively equivalent because the subgroup of $\mathrm{Aut}^0(G(n, k + 1))$ fixing pointwise the subvariety $G(n, k)$ acts transitively on the set of all such lines. Furthermore, every positive elementary transformation of E supported by P arises from some line and $\phi_{E,V}(X) \cap T = \{P\}$ and $\phi_{E,V}(X) \cup T$ has an ordinary node at P for any such line T. Hence any such reducible curve $\phi_{E,V}(X) \cup T$ arises from the construction given in the first part of this remark.

Lemma 1. *Let X be an integral projective curve and E a rank n spanned vector bundle on X such that* $\deg(E) = 1$. *Then $X \cong \mathbf{P}^1$ and E is isomorphic to the direct sum of one line bundle of degree one and $n - 1$ line bundles of degree zero. Furthermore, $h^0(X, E) = n + 1$ and no proper linear subspace of $H^0(X, E)$ spans E.*

Proof. Set $k := h^0(X, E)$. Since E is spanned, it induces a morphism $\phi : X \to G(n, k) \subseteq \mathbf{P}^{N_{n,k}}$ such that $\deg(\phi) \cdot \deg(\phi(X)) = \deg(E) = 1$. Hence ϕ

is not constant, $\deg(\phi) = 1$ (i.e. ϕ is birational onto its image) and $\phi(X)$ is a line. Since ϕ is finite and birational and $\phi(X)$ is smooth, ϕ is an isomorphism. Apply the classification of all vector bundles on $\mathbf{P}^1$. □

Remark 4. Let X be an integral projective curve, $P \in X_{reg}$ and E a rank n vector bundle on X. Assume $h^1(X, E) = 0$ and that E is spanned by its global section. Let Y be the union of X and of n smooth rational curves $D_1 \cup \cdots \cup D_n$. Assume $X \cap (D_1 \cup \cdots \cup D_n) = \{P\}$, $p_a(Y) = p_a(X)$, that Y is connected and that Y has only seminormal points outside $\mathrm{Sing}(X)$. Since X is smooth at P and each D_i is smooth, the latter condition is equivalent to require that at each point $Q \in \{P\} \cup \mathrm{Sing}(D_1 \cup \cdots \cup D_n)$ the Zariski tangent space to Y at Q has dimension equal to the number of branches of Y at Q. Elementary properties of seminormal singularities give the existence of a rank n vector bundle on Y such that $G|X \cong E$ and for every $i \in \{1, \dots, n\}$ the vector bundle $G|D_i$ is the direct sum of one line bundle of degree one and $n-1$ line bundles of degree zero. For a parametrization (in which however isomorphic vector bundles may correspond to different point of the irreducible parameter space) use Remark 3 and induction on the number of the components; the property of ordinary nodes used in Remark 3 is true for seminormal singularities. Using n Mayer-Vietoris exact sequences we easily obtain $h^1(Y, G) = 0$ and that G is spanned by its global sections.

We will not use the family $\{Y_\lambda\}_{\lambda \in \Delta}$ described in the next lemma, but we believe that its existence will be useful to obtain more explicit results (roughly speaking, to obtain results for coherent systems of type $(n, d, k+1)$ instead of (n, d, k)).

Lemma 2. *Fix integers $k > n > 0$. Let X be an integral projective curve, $P \in X_{reg}$, and E a rank n vector bundle on X such that $h^1(X, E) = 0$ and there is a k-dimensional linear subspace $V \subseteq H^0(X, E)$ spanning E and such that the associated morphism $\phi_{E,V} : X \to G(n, k)$ is an embedding. See $G(n, k)$ as a Schubert cycle of $G(n, k+1)$. Let Y be the connected projective curve such that Y has two irreducible components X, D, $D \cong \mathbf{P}^1$ and $X \cap D = \{P\}$ and Y has an ordinary node at P. Hence $p_a(Y) = p_a(X)$. Fix a vector bundle F obtained from E making a positive elementary transformation supported by P. Let $V' \subset H^0(X, F)$ be the linear subspace such that $V' \cap H^0(X, E) = V$ (Remark 2). Let G be the unique (up to isomorphisms) rank n vector bundle on Y such that $G|X \cong E$, $G|D$ is isomorphic to a direct sum of one line bundle of degree one and $n-1$ line bundles of degree zero and G is associated to the positive elementary transformation giving F (Remark 3). Let $W \subseteq H^0(Y, G)$ be any linear subspace of $H^0(Y, G)$ such that $\dim(W) = k+1$, $W|X = V$ and $W|D = H^0(D, G|D)$. By*

Remark 3 if $\phi_{E,V} : X \to G(n,k)$ is an embedding, then $\phi_{G,W} : Y \to G(n,k+1)$ is an embedding and $\phi_{G,W}(D)$ is a line intersecting the curve $\phi_{E,V}(X)$ only at P and quasi-transversally. There is an open neighborhood Δ of 0 in $\mathbb{K}$ and flat families $\{C_\lambda\}_{\lambda\in\Delta}$ (resp. $\{Y_\lambda\}_{\lambda\in\Delta}$) of curves in $G(n,k)$ (resp. $G(n,k+1)$) with the following properties:

(a) for all $\lambda \in \Delta\backslash\{0\}$ the curve C_λ is isomorphic to X and it is an isomorphic linear projection (as a curve in $G(n,k)$) of $\phi_{F,V'}(X)$;

(b) $C_0 = \phi_{E,V}(X) \cup D$ with D a line;

(c) $D \cap \phi_{E,V}(X) = \{\phi_{E,V}(P)\}$ and C_0 has an ordinary node;

(d) each Y_λ, $\lambda \in \Delta\backslash\{0\}$, is projectively equivalent to $\phi_{F,V'}(X)$; $A_0 = \phi_{E,V}(X) \cup D'$ with D' a line associated to the positive elementary transformation supported by P used to obtain F from E; hence $D' \nsubseteq G(n,k)$; Y_0 is an isomorphic linear projection of A_0 into $G(n,k)$;

(e) The family $\{C_\lambda\}_{\lambda\in\Delta}$ is obtained from the family $\{A_\lambda\}_{\lambda\in\Delta}$ taking a linear projection.

Proof. Fix a symmetric and non-degenerate bilinear form $(-,-)$ on V'. For any linear subspace $U \subseteq V'$ set $U^\perp := \{a \in V' : (a,u) = 0$ for every $u \in U\}$. If U has codimension one in V' the decomposition $U \oplus U^\perp$ defines a linear projection from $G(n,k+1)$ to $G(n,k)$. The curve $\phi_{E,V}(X) \subset G(n,k)$ is the closure in $G(n,k+1)$ of the linear projection of $\phi_{F,V'}(X)\backslash\phi_{F,V'}(P)$ from the point P. In this way we only have a linear projection $\mathbf{P}^{N_{n,k+1}}\backslash\{P\} \to \mathbf{P}^{N_{n,k+1}-1}$; see [2], Proposition 1.1 and Figure 1, for this case. To obtain a map $\phi_{F,V'}(X)\backslash\phi_{F,V'}(P) \to G(n,k)$ we need to specify the right Grassmannian $G(n,k)$ inside $G(n,k+1)$ passing through P. We choose the $G(n,k)$ passing through P and associated with V. The linear subspace V identifies E as a subsheaf of F, because E is the subsheaf of F spanned by V. Hence for this choice of $G(n,k)$ the curve $\phi_{F,V'}(X)\backslash\phi_{F,V'}(P)$ is mapped isomorphically by the linear projection onto $\phi_{E,V}(X)\backslash\phi_{E,V}(P)$ and hence the closure the image of $\phi_{F,V'}(X)\backslash\phi_{F,V'}(P)$ is isomorphic to X. Let $\{V_\lambda\}_{\lambda\in\mathbb{K}}$ be any family of codimension one linear subspaces of V' such that $V_0 = V$, while V_λ spans G and induces an embedding of X in $G(n,k)$ for general $\lambda \in \mathbb{K}$, say for every $\lambda \in \Delta\backslash\{0\}$. For any $\lambda \in \Delta\backslash\{0\}$ the curve C_λ is obtained from $\phi_{F,V'}$ using the linear projection associated to the splitting $V' = V_\lambda \oplus {V_\lambda}^\perp$. For degree reasons the one-dimensional part of the flat limit of this family is the union of $\phi_{E,V}(X)$ and a line $R \subset G(n,k+1)$ (Lemma 1). Notice that $Q_{n,k+1}|C$ is isomorphic to the direct sum of one line bundle of degree one and $n-1$ line bundles of degree zero for any line $C \subset G(n,k+1)$. Hence the limit line is obtained as a projection of a line associated to the positive elementary transformation which gives F from E. To obtain the family $\{Y_\lambda\}_{\lambda\in\Delta}$ one take two copies $G(n,k+1)'$ and $G(n,k+1)''$ of $G(n,k+1)$ inside $G(n,k+2)$; for any $Q \in G(n,k+2)\backslash G(n,k+1)'$ the corresponding linear projection maps isomorphically $G(n,k+1)'$ onto $G(n,k+1)''$. When Q goes to a point $\phi_{F,V'}(P) \in G(n,k+1)'$, the image of $G(n,k+1)'$ is contained in a $G(n,k)'' \subset G(n,k+1)''$ and the flat limit of the corresponding curves has as a component a line L of $G(n,k+1)''$ which intersects $G(n,k)''$ only in one point and quasi-transversally. □

Recall that $E(P)$ is obtained from E making n positive elementary transformations supported by P. Hence iterating n times Lemma 2 and using Remark 4 we obtain the following result.

Lemma 3. *Fix integers $k > n > 0$. Let X be an integral projective curve, $P \in X_{reg}$, and E a rank n vector bundle on X such that $h^1(X,E) = 0$ and there is a k-dimensional linear subspace $V \subseteq H^0(X,E)$ spanning E and such that the associated morphism $\phi_{E,V} : X \to G(n,k)$ is an embedding. See $G(n,k)$ as a Schubert cycle of $G(n,k+n)$. Since $h^0(X,E(P)) = h^0(X,E)+n$, there is a unique linear subspace $V' \subseteq H^0(X,E(P))$ such that $V' \cap H^0(X,E) = V$ (seeing E as a subsheaf of $E(P)$ and hence $H^0(X,E)$ as a linear subspace of $H^0(X,E(P))$) and $\dim(V') = k+n$; the latter condition is equivalent to require that V' spans $E(P)$ at P; furthermore, $\phi_{E(P),V'} : X \to G(n,k+n)$ is an embedding. There exists a connected projective curve Y such that Y has $n+1$ irreducible components $X, D_1,\dots,D_n$, $D_i \cong \mathbf{P}^1$ for all i, Y with only seminormal singularities outside* Sing(X), *$p_a(Y) = p_a(X)$ and such that there is an open neighborhood Δ of 0 in $\mathbb{K}$ and a flat family $\{C_\lambda\}_{\lambda\in\Delta}$ of curves in $G(n,k)$ with the following properties:*

- *(a) for all $\lambda \in \Delta\backslash\{0\}$ the curve C_λ is isomorphic to X and it is an isomorphic linear projection (as a curve in $G(n,k+n)$) of $\phi_{E(P),V'}(X)$;*
- *(b) $C_0 \cong Y$ and in this embedding of Y into $G(n,k)$ each D_i is mapped onto a line, while $\phi_{E,V}(X)$ is the component isomorphic to X.*

Proof of Theorem 2. We divide the proof into 3 parts.

(i) Here we assume $H \cong \mathcal{O}_X(P)$ for some $P \in X_{reg}$. Since $\mathcal{O}_X(P)$ is ample, there is an integer β such that $h^1(X,E(tP)) = 0$, $h^0(X,E(tP)) \geq k$ and $E(tP)$ is spanned. For any integer $t \geq \beta$ let γ_t be the dimension of the image in $H^0(X,\det(E(tP)))$ of $\bigwedge^n(V_t)$, where V_t is a general k-dimensional linear subspace of $H^0(X, E(tP))$. Obviously, $\gamma_t \le \binom{k}{n}$ for all t. By Lemma 3 we have $\gamma_{t+1} \ge \min\{\binom{k}{n}, 1 + \gamma_t\}$. Hence $\gamma_t = \binom{k}{n}$ for all $t \gg 0$, say for all $t \ge \beta + \binom{k}{n}$, proving Theorem 2 in this case.

(ii) Here we assume $H \cong \mathcal{O}_X(P_1 + \cdots + P_s)$ for some $P_1, \ldots, P_s \in X_{reg}$ such that $P_i \ne P_j$ for all $i \ne j$. By part (i) we may assume $s \ge 2$. Apply part (i) with respect to the point P_s to the bundle E and get an integer t_0. We claim that the same integer t_0 works for H. Indeed, for any integer $t \ge t_0$ there is an inclusion $H^0(X, E(tP_s)) \subseteq H^0(X, E(tH))$ induced by the multiplication with the equation of the effective divisor $tP_1 + \cdots + tP_s$ and hence the same linear subspace of $H^0(X, E(tP_s))$ works for $H^0(X, E(tH))$ (although it never spans $E(tH)$).

(iii) Here we consider the general case. Since H is ample, there is an integer $m > 0$ (depending only on $p_a(X)$ because $\deg(H) > 0$) such that $H^{\otimes m}$ is very ample. Hence $H^{\otimes m} \cong \mathcal{O}_X(P_1 + \cdots + P_s)$ for some $P_1, \ldots, P_s \in X_{reg}$ such that $P_i \ne P_j$ for all $i \ne j$. Apply part (ii) to the finite set of vector bundles $E \otimes H^{\otimes i}$, $0 \le i \le m - 1$. □

Proof of Theorem 1. Apply Theorem 2 and Remark 1. □

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