ABBREVIATELY CONTINUOUS INVARIANT MEASURES
OF LINEAR INTERVAL MAPS

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Abstract: In the first part of this paper we consider a family $F$ of piecewise linear maps $\tau$ from $[-1, 1]$ into itself, where $\tau$ is expanding in one of the intervals $(-1, 0), (0, 1)$ and contracting in the other and discontinuity at $x = 0$. For a sub family $F_1 \subset F$ we give a very simple proof of Boudourides-Fotiates result [3] for the existence of absolutely continuous invariant measures (ACIM). In the second part of this paper we consider an one parameter family $S$ of tent maps $\tau_\beta$ from $[0, 1]$ onto itself, where $\beta \in (1, 2]$. It is shown that the set of parameters $\beta$ for which the $\tau_\beta$ is Markov is dense in $(1, 2]$. For certain class of $\beta \in (1, 2]$, the probability density functions $f_\beta$ of Markov $\tau_\beta$ are derived.

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1. Introduction

Existence of absolutely continuous invariant measures is one of the most important problem in the ergodic theory of dynamical systems. In particular, the existence of invariant measures which are absolutely continuous with respect to Lebesgue measure is very important from a physical point of view, because computer simulations of orbits of the system reveal only invariant measures which are absolutely continuous with respect to Lebesgue measure [6]. The Birkhoff’s
Ergodic Theorem [4] states that if \( \tau : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu) \) is ergodic and \( \mu \)-invariant and \( E \) is a measurable subset of \( X \) then the orbit of almost every point of \( X \) occurs in the set \( E \) with asymptotic frequency \( \mu(E) \). The Birkhoff’s Ergodic Theorem establishes the dynamical importance of invariant measure, but it says nothing about the existence of invariant measures.

Linear dynamical systems has significant applications in economics, in biology, in chemistry, in telecommunications [9] and in other branches of sciences and engineering. Dynamics of a simple linear dynamical system can be very chaotic. So it is important to study the long term behavior of chaotic linear dynamical systems. In the first part of this note we study a family of piecewise linear maps \( \tau \) from \([-1,1]\) into itself, where \( \tau \) is expanding in one of the intervals \((-1,0),(0,1)\) and contracting in the other and discontinuity at \( x = 0 \). Our maps are similar to those of M. Boudourides and N. Fotiades [3]. We present a simple proof of the existence of ACIM without using first return maps. Then we study a family of tent maps \( \tau_\beta, \beta \in (1,2] \) from \([0,1]\) into itself. These tent maps are differentially conjugate to Chebyshev maps. We show that the set of parameters \( \beta \) for which the \( \tau_\beta \) is Markov is dense in \((1,2]\). Moreover, for certain Markov parameter \( \beta \in (1,2] \) we explicitly find the density function \( f_\beta \) of \( \tau_\beta \) and hence the mean value of Markov Chebyshev maps.

In Section 2 we present the notation and summarize results we shall need in the sequel. In Section 3 we present a simple proof of the existence of ACIM for linear interval maps. Section 4 deals with the probability density functions of tent maps.

### 2. Notation and Review of ACIM

Let \( I \) be an interval on the real line and let it denote the state space of a dynamical system defined by the map \( \tau : I \to I \). On the measure space \((I, \mathcal{B}, \lambda)\), \( \lambda \) is an underlying measure. We assume \( \tau \) is piecewise monotonic on a partition \( \mathcal{P} = \{P_1, P_2, \ldots, P_N\} \) of \( I \). That is, \( \tau \) restricted to \( P_i \) is a monotonic function. Then the asymptotic behavior is given by a probability density function (pdf), \( f \), associated with the ACIM \( \mu \). This is stated mathematically by the following equation:

\[
\int_A f \, dx = \int_{\tau^{-1}A} f \, dx
\]

for any (measurable) set \( A \subset I \). The Frobenius-Perron operator, \( P_\tau f \), defined by
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\[
\int_A f \, dx = \int_A P_\tau f \, dx,
\]
acts on the space of integrable functions and transforms a pdf into a pdf. If \( \tau \) is piecewise smooth and piecewise differentiable on a partition of \( n \) subintervals, then we have the following representation for \( P_\tau \) [4,10]:

\[
P_\tau f(x) = \sum_{z \in \{\tau^{-1}(x)\}} \frac{f(z)}{|\tau'(z)|}, \tag{2.1}
\]

where, for any \( x \), the set \( \{\tau^{-1}(x)\} \) consists of at most \( n \) points.

**Definition 2.1.** A transformation \( \tau : I = [a, b] \to [a, b] \) is piecewise \( C^2 \) if there exists a partition \( P = \{a = a_0 < a_1 < a_2 < \ldots < a_n = b\} \) of \( I \) such that \( \tau_i = \tau|_{(a_{i-1}, a_i)}, i = 1, 2, \ldots n \) is a \( C^2 \) function which can be extended to \([a_{i-1}, a_i]\) as a \( C^2 \) function. \( \tau : I = [a, b] \to [a, b] \) is Markov with respect to the partition \( P \) if \( \tau_i, i = 1, 2, \ldots n \) is a homeomorphism from \( I_i = (a_{i-1}, a_i) \) onto some connected union of intervals of \( P \). \( \tau \) is piecewise expanding if \( |\tau'(x)| > 1 \) for all \( x \in I_i, i = 1, 2, \ldots n \). \( \tau \) is eventually expanding if \( \tau^k \) is expanding for some \( k > 1 \).

The following important result was established in [8].

**Theorem 2.2.** (see [4]) Let \( \tau \) be piecewise monotonic, piecewise \( C^2 \) map of an interval \( I \) into itself satisfying \( \inf_{x \in I} |\tau'(x)| > 1 \). Then, \( \tau \) has an ACIM whose density \( f \) is of bounded variation and satisfies \( P_\tau f = f \).

Thus, the fixed points of the operator \( P_\tau \) are the density functions of the ACIMs for the map \( \tau \).

### 3. A Family of Linear Interval Maps and their Invariant Measures

We consider a family \( \mathcal{F} \) of maps \( \tau : [-1, 1] \to [-1, 1] \) defined by

\[
\tau(x) = \begin{cases} 
\lambda_1 x + a, & x \in [-1,0), \\
0, & x = 0, \\
\lambda_2 x + b, & x \in (0,1],
\end{cases} \tag{3.1}
\]

where \( |\lambda_1| > 1 \) and \( |\lambda_2| < 1 \) and \( \lambda_1, \lambda_2, a, b \) are constants such that \( \tau([-1,1]) \subset [-1,1] \). M. Boudourides and N. Fotiades [3] studied a sub family, where \( 1 < \lambda_1 \leq 2 \) and \( -1 < \lambda_2 < 0 \) and proved the existence of absolutely continuous invariant measures (ACIM). We consider a sub family \( \mathcal{F}_1 = \{\tau \in \mathcal{F} : 1 < \lambda_1 \leq \)
2, 0 < λ₂ ≤ 1) and prove the existence of ACIM. The method of our proof is different than the method of M. Boudourides and N. Fotiades. We do not use the first return map.

3.1. Existence of Invariant Measures

Theorem 3.1. Let τ ∈ F₁ and $N = \left\lfloor 1 - \frac{\log(1 + \frac{\lambda_2 b}{\lambda_1} (\lambda_1 - 1))}{\log \lambda_1} \right\rfloor$, where $[t]$ denotes the greatest integer less than or equal to $t$. If $\lambda_1^N \lambda_2 > 1$, then $f$ is eventually expanding and there exists an absolutely continuous invariant measure with respect to Lebesgue measure.

Proof. Suppose that $x ∈ (-1, 0), τ(x) ∈ (-1, 0), τ²(x) ∈ (-1, 0), τ^{n-1}(x) ∈ (-1, 0)$. By induction we can show that $τ^n(x) = \lambda_1^n x + \frac{\lambda_1^{n-1} - 1}{\lambda_1-1} a$. For any $x ∈ [-1, 1]$, at most one of the points $x, τ(x), ..., τ^N(x)$ can be in $(0, 1)$. If not, there exists $y ∈ τ(0, 1)$, $k < N$ with $τ^k(y) ∈ (0, 1)$ and $y ∈ (-1, 0), τ(y) ∈ (-1, 0), τ²(y) ∈ (-1, 0), τ^{k-1}(y) ∈ (-1, 0)$. Since $y ∈ τ(0, 1), y ≤ λ_2 + b$. Hence

$0 < τ^k(y) ≤ \lambda_1^k (λ_2 + b) + \frac{λ_1^{k-1} - 1}{λ_1-1} a$ which implies that $k > \frac{\log(1 + \frac{λ_2 b}{λ_1} (λ_1 - 1))}{\log \lambda_1}$, and hence $k ≥ N$, a contradiction. Let $x ∈ [-1, 1]$, the slope of $τ^{N+1}$ at $x$ is equals either $λ_1^{N+1} > λ_1^N λ_2$ or $λ_1^N λ_2$. This shows that $|τ^{N+1}'| > λ_1^N λ_2 > 1$. Hence τ is eventually expanding. The result of Lasota and Yorke [8] proves the existence of the ACIM.

Example 3.2.

$τ(x) = \begin{cases} 
1.5x + .6, & x ∈ [-1, 0), \\
0, & x = 0, \\
\frac{4}{3}x - .8, & x ∈ (0, 1].
\end{cases}$ (3.2)

Here, $N = [2.512101685] = 2$ and hence, by Theorem 3.1, τ has an ACIM.
4. A Family of Interval Tent Maps and their Probability Density Functions

Consider the following one parameter family of tent maps \( \tau_\beta : [0, 1] \rightarrow [0, 1], \beta \in (1, 2] \) defined by (see Figure 1)

\[
\tau_\beta(x) = \begin{cases} 
\beta x, & 0 \leq x < \frac{1}{\beta}, \\
2 - \beta x, & \frac{1}{\beta} \leq x \leq 1.
\end{cases}
\] (4.1)

Figure 1: The graph of \( \tau_\beta, \beta = \frac{2}{3} \)

Parry [12] studied \( \beta \)-transformations \( \phi_\beta : [0, 1] \rightarrow [0, 1] \) defined by (see Figure 2) \( \phi_\beta(x) = \beta x \) (mod 1) for any \( \beta \in (1, 2] \).

Figure 2: The graph of \( \beta \) transformation

Parry’s transformations \( \phi_\beta \) and our transformations \( \tau_\beta \) are somehow similar and probability density function for transformations \( \tau_\beta \) are not known. Moreover, transformations \( \tau_\beta \) are differentially conjugate to Chebyshev maps \( T_\beta(x) = \cos(\beta \arccos(x)), -1 \leq x \leq 1, \beta > 2 \) via the differentiable homeomorphism \( h : [0, 1] \rightarrow [0, 1] \) defined by \( h(x) = \cos(\pi x)[2] \). We are interested in finding the probability density functions for tent maps \( \tau_\beta \). The probability
density function of a certain class of piecewise linear maps was studied in [1] and we follow closely [1].

![Graph of Chebyshev maps](image)

**Figure 3:** The graph of Chebyshev maps

### 4.1. Markov Tent Maps

**Theorem 4.1.** For $\beta \in (1, 2]$, $\tau_\beta$ is Markov with respect to a partition $\mathcal{P}$ if and only if any one of the following holds:

(i) $x_0 = \frac{1}{\beta}$ is periodic of period $n > 2$ and $\mathcal{P}$ is formed by the points of the periodic orbit of $\frac{1}{\beta}$.

(ii) $x_0 = \frac{1}{\beta}$ not periodic, $x_0$ is eventually periodic with period $n > 0$ and $\mathcal{P}$ is formed by the points of the eventually periodic orbit of $\frac{1}{\beta}$.

**Proof.** Assume that $x_0 = \frac{1}{\beta}$ is a periodic point of $\tau_\beta$ of period $n > 2$. Consider the orbit \{\(x_0, \tau_\beta(x_0), \tau_\beta^2(x_0), \ldots, \tau_\beta^{n-1}(x_0)\}\} and construct a partition $\mathcal{P}$ of \([0,1]\) with the points of this orbit. Let the partition be $\mathcal{P} = \{0 = a_0 < a_1 < a_2 < \ldots < a_n = 1\}$. For $i = 1, 2, \ldots, n$, let $I_i = (a_{i-1}, a_i)$ and denote the restriction of $\tau$ on $I_i$ by $\tau_i$. The image of the end points $a_{i-1}$ and $a_i$ of $I_i = (a_{i-1}, a_i)$ are in $\mathcal{P}$. Let these image points be $a_j$ and $a_k$, with $a_i < a_k$ and $j, k \in \{0, 1, \ldots, n\}$. $\tau_\beta(\frac{1}{\beta}) = 1$ and by definition of the partition $\mathcal{P}$, $\frac{1}{\beta}$ is a member of the partition points. Moreover, $\tau_\beta|_{I_i}$ is linear and hence homeomorphism and $\tau_\beta(I_i) = (a_j, a_k)$ a connected union of intervals of the partition. Hence $\tau_\beta$ is $\mathcal{P}$- Markov.

Now, assume that $x_0 = \frac{1}{\beta}$ is not periodic and it is eventually periodic of period $n \geq 1$. Then there exists an integer $k \geq 1$ and a point $x \in [0,1]$ such that $\tau^n(x) = x$ and $\tau^k(\frac{1}{\beta}) = x$. Order these $n + k + 1$ points

\[\{0, \frac{1}{\beta}, \tau_\beta(\frac{1}{\beta}), \tau_\beta^2(\frac{1}{\beta}), \ldots, \tau_\beta^{k-1}(\frac{1}{\beta}), x, \tau_\beta(x), \tau_\beta^2(x), \ldots, \tau_\beta^{n-1}(x)\}\]
and form a partition $P$ of $[0,1]$. Let the partition be $Q = \{0 = a_0 < a_1 < a_2 < \ldots < a_n = 1\}$. For $i = 1, 2, \ldots, n + k$, let $I_i = (a_{i-1}, a_i)$ and denote the restriction of $\tau$ on $I_i$ by $\tau_i$. As before it can be easily shown that $Q$ is a Markov partition for $\tau$.

Conversely, assume that for $\beta \in (1, 2], \tau_\beta$ is Markov with respect to the partition $P = \{0 = a_0 < a_1 < a_2 < \ldots < a_n = 1\}$. We want to show that (i) or (ii) holds. Since $\tau_\beta$ is Markov with respect to $P$, there exists an integer $k^*$ such that $\tau_\beta(1) = 2 - \beta = a_k^*$. Moreover, there exists a positive integer $k$ such that we have one of the following situations:

(a) $\tau_\beta^k(2 - \beta) = \frac{1}{\beta}$;
(b) $\tau_\beta^k(2 - \beta) \neq \frac{1}{\beta}$.

In case (a), $\frac{1}{\beta}$ is a member of a periodic orbit of period $k > 2$. In case (b), since $P$ is a Markov partition, there exists an integer $l \geq 1$ such that either $\tau_\beta^{k+l}(2 - \beta) = \frac{1}{\beta}$ or there exists an integer $m$ such that $\tau_\beta^k(2 - \beta) = a_m$ and $\tau_\beta^l(a_m) = a_m$.

**Definition 4.2.** Let $E_C = \left\{\frac{1}{\beta}\right\}$, $E_L = [0, \frac{1}{\beta})$, $E_R = (\frac{1}{\beta}, 1]$. Let $y_1 = 0, y_2 = \frac{1}{\beta}, y_3 = 1$. For $i = 1, 2, 3$, the kneading sequence $K$ of $y_i$ is defined by $K^i = \left\{K^i_j : j = 0, 1, 2, \ldots\right\}$, where,

$$K^i_j = \begin{cases} C & \text{if } \tau_\beta^j(y_i) \in E_C, \\ L & \text{if } \tau_\beta^j(y_i) \in E_L, \\ R & \text{if } \tau_\beta^j(y_i) \in E_R. \end{cases} \quad (4.2)$$

Let $K^2 = \left\{K^2_j : j = 0, 1, 2, \ldots\right\}$ be the kneading sequence of the point $y_2$. For $\beta^* \in (1, 2]$, a map $\tau_{\beta^*}$ is of type $\{m - p; K^2\}$ [7] if the map $\tau_{\beta^*}$ maps $y_3$ to a periodic point of period $p$ in $m$ iterations and possesses the indicated sequence $K^2$ as the kneading sequence of $y_2$.

For $\beta \in (1, 2]$, the map $\tau_\beta$ is Markov if either $\tau_\beta$ has a periodic orbit

$$\frac{1}{\beta}, \tau_\beta(\frac{1}{\beta}), \tau_\beta^2(\frac{1}{\beta}), \ldots, \tau_\beta^{p-1}(\frac{1}{\beta})$$

of period $p$ or there exists a periodic point $x$ of period $p$ such that $p_3 = \tau_\beta(\frac{1}{\beta}) = 1$ gets map to $x$ in $m$ iterations. For example, the $\{0 - 3; K^2\}$ map has period 3 orbit of the form $\left\{\frac{1}{\beta}, 1 - 2 - \beta, \ldots\right\}$. It repeats the pattern $CRL$ and the kneading sequence of $y_2 = \frac{1}{\beta}$ is $K^2 = (CRL)^\infty$. The $\{3 - 1; K^2\}$ map has orbit...
of the form
\[
\left\{ \frac{1}{\beta}, 1, \tau_\beta(1) = 2 - \beta, \tau_\beta^2(1) = \frac{2}{1 + \beta}, \tau_\beta^3(1) = \frac{2}{1 + \beta}, \frac{2}{1 + \beta}, \ldots \right\},
\]
where \( \frac{2}{1 + \beta} \) is a fixed point of \( \tau_\beta \). The kneading sequence is \( K^2 = CRLL(R)^\infty \).
In general, for a map of type \( \{ m - p; K \} \). We have the following corollary of Therom 4.1.

**Corollary 4.3.** \( \tau_\beta \) is Markov if and only if one of the following holds:

(i) \( \tau_\beta \) is of type \( \{ m - p; K \} \), \( m = 0 \), \( p > 2 \).

(ii) \( \tau_\beta \) is of type \( \{ m - p; K \} \), \( m \geq 1 \), \( p > 0 \).

### 4.2. Density Function of Markov Tent Maps

In this section we consider \( \tau_\beta \) of type \( \{ m - p; K \} \), \( m = 0 \), \( p \geq 3 \). Similar method can be applied for a general type \( \{ m - p; K \} \).

**Theorem 4.4.** (see [4]) If a transformation is \( \mathcal{Q} \)-Markov, piecewise linear and expanding, then the \( \tau \)-invariant density is constant on intervals of \( \mathcal{Q} \).

The maps \( \tau_\beta \), of type \( \{ m - p; K \} \), \( m = 0 \), \( p \geq 3 \) are Markov and have a periodic orbit of the pattern \( CRL, CRLL, CRLLL, \ldots \) and \( x = \frac{1}{\beta} \) is periodic orbit of period \( p = n + 3 \), where \( n \) represents the number of \( L \)'s after the initial \( CRL \) in the symbolic representation of the periodic orbit:

\[
\left\{ \frac{1}{\beta}, 1, 2 - \beta, \tau_\beta(2 - \beta), \tau_\beta^2(2 - \beta), \ldots \right\} = CRLLL \ldots \tag{4.3}
\]

**Lemma 4.5.** The relation between \( p = n + 3 \) and \( \beta \) in equation (4.3) is given by

\[
\beta^{n+1}(2 - \beta) = \frac{1}{\beta}. \tag{4.4}
\]

**Proof.** Denote the branch of \( \tau_\beta \) on \( E_L \) by \( \tau_L \) and the branch on \( E_R \) by \( \tau_R \). Observe that \( 2 - \beta \leq \frac{1}{\beta} \). The set of parameters \( \beta \) for which \( \frac{1}{\beta} \) has the symbolic representation (4.3) can be obtained by solving \( \tau_\beta^{n+1}(2 - \beta) = \beta^{n+1}(2 - \beta) = \frac{1}{\beta} \), where the period is \( p = n + 3 \).

**Lemma 4.6.** Equation (4.4) has a unique solution \( \beta \in (1, 2] \).

**Proof.** Equation (4.4) can be written as

\[
\beta^p - 2\beta^{p-1} + 1 = 0. \tag{4.5}
\]
Let \( f(x) = x^p - 2x^{p-1} + 1 \). Then \( f'(x) = px^{p-1} - 2(p-1)x^{p-2} = x^{p-2}(px - 2p + 2) \).
The critical points are \( x_0 = 0 \) and \( x_1 = 2 - \frac{2}{p} \). Since \( p \geq 3 \),\( x_1 = 2 - \frac{2}{p} \in (1, 2] \).
Observe that \( f(1) = 0, f(0) = 1, f(2) = 1 \) and \( 2 - p < (px - 2p + 2) < 0 \) for \( x \in (1, x_1) \) and \( 0 < (px - 2p + 2) < 2 \) for \( x \in (x_1, 2] \). Thus, \( f \) is decreasing on \((1, x_1)\) and increasing on \((x_1, 2]\). Hence the lemma is proved. \( \square \)

By Theorem 4.1, the map \( \tau_\beta \) is Markov with respect to the partition
\( Q = \{0, \frac{1}{p}, 1, 2 - \beta, \tau_\beta(2 - \beta), \ldots, \tau^{p-2}_\beta(2 - \beta), \tau^{p-2}_\beta(2 - \beta) = \frac{1}{p}, 1\} \) of \([0, 1]\), where
\( \beta \) is the solution of the equation (4.4). By Theorem 4.4, the invariant density
of \( \tau_\beta \) is piecewise constant on the elements of \( Q \). Let the density function \( f_\beta \) of
\( \tau_\beta \) with respect to the partition \( Q \) be

\[
f_\beta(x) = \begin{cases}
0 & \text{if } x < 0, \\
f_1 & \text{if } 0 \leq x \leq 2 - \beta, \\
f_2 & \text{if } 2 - \beta \leq x \leq \tau(2 - \beta), \\
f_3 & \text{if } \tau(2 - \beta) \leq x \leq \tau^2(2 - \beta), \\
\vdots & \\
f_{p-1} & \text{if } \tau^{p-3}(2 - \beta) \leq x \leq \tau^{p-2}(2 - \beta), \\
f_p & \text{if } \frac{1}{p} = \tau^{p-2}(2 - \beta) \leq x \leq 1.
\end{cases}
\] (4.6)

Then the action of the Perron-Frobenius operator is

\[
P_\beta f(x) = \frac{f(\tau^{i-1}_L(x))}{|\tau^{i-1}_L(x)|} + \frac{f(\tau^{i-1}_R(x))}{|\tau^{i-1}_R(x)|} \chi_{[2-\beta, 1]}(x) = \frac{1}{p} [f(\tau^{i-1}_L(x)) + f(\tau^{i-1}_R(x))] \chi_{[2-\beta, 1]}(x).
\] (4.7)

The matrix representation of the Perron-Frobenius operator is a \( p \) by \( p \) matrix

\[
M = \begin{bmatrix}
\frac{1}{p} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{p} \\
0 & \frac{1}{p} & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{p} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & \frac{1}{p} & 0 & \cdots & 0 & \frac{1}{p} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{p} & \frac{1}{p}
\end{bmatrix}.
\] (4.8)

We consider the equation \( fM = f \) with the normalizing condition \( \int_0^1 f(x)dx = 1 \), i.e.,

\[
(2 - \beta)f_1 + (2 - \beta) \sum_{i=1}^{p-2} (\beta^i - \beta^{i-1})f_{i+1} + (1 - \frac{1}{p})f_p = 1.
\] (4.9)
Let \( r = 2 - \beta \). Using (4.7), we have the system of linear equations \( Af = f^* \), where \( A \) is a \( p + 1 \) by \( p \) matrix,
\[
A = \begin{bmatrix}
\frac{1}{\beta} - 1 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
\frac{1}{\beta} - 1 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & \frac{1}{\beta} - 1 & 0 & \ldots & 0 & \frac{1}{\beta} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & \frac{1}{\beta} - 1 \\
r & r(\beta - 1) & r(\beta^2 - \beta) & \ldots & \ldots & \ldots & \ldots & r(\beta^{p-2} - \beta^{p-3}) & 1 - \frac{1}{\beta}
\end{bmatrix}
\]
(4.10)
and \( f^* = (f_1, f_2, \ldots, f_p, 1)^t \).

Theorem 4.7. Let \( p = n + 3 \). Suppose that \( \beta \) satisfies equation (4.4) for some \( n \). Then the probability density function of \( \tau_\beta \) is
\[
f_1 = 0,
\]
\[
f_{i+1} = \frac{\beta}{1 - \beta(2 - \beta) - (p - 2)(\beta - 1)(2 - \beta) + (2 - \beta)^2(1 - \frac{1}{\beta})},
\]
\[
\vdots
\]
\[
f_p = \frac{\beta(\beta - 1)}{1 - \beta(2 - \beta) - (p - 2)(\beta - 1)(2 - \beta) + (2 - \beta)^2(1 - \frac{1}{\beta})},
\]
i = 1, 2, \ldots, p - 2.

Proof. It can be easily shown that the system of equations \( Af = f \) can be written as
\[
f_1 = 0;
\]
\[
f_{i+1} = f_p \frac{1}{\beta} \sum_{j=0}^{i-1} \left( \frac{1}{\beta} \right)^j, \quad i = 1, 2, \ldots, p - 2;
\]
\[
(1 - \frac{1}{\beta}) f_p = \frac{1}{\beta} f_{p-1};
\]
\[
1 = (2 - \beta) \sum_{i=1}^{p-2} (\beta^i - \beta^{i-1}) f_{i+1} + (1 - \frac{1}{\beta}) f_p.
\]
Now,
\[
f_{i+1} = f_p \frac{1}{\beta} \frac{1 - \frac{1}{\beta}}{1 - \frac{1}{\beta}}, \quad i = 1, 2, 3, \ldots, p - 2.
Thus,
\[ (2 - \beta) \sum_{i=1}^{p-2} (\beta^i - \beta^{i-1}) f_p \frac{1 - \frac{1}{p}}{1 - \frac{1}{\beta}} + (1 - \frac{1}{\beta}) f_p = 1, \]
or
\[ f_p \left\{ \frac{(2 - \beta)}{\beta(1 - \frac{1}{\beta})} \left( \sum_{i=1}^{p-2} (\beta^i - \beta^{i-1}) - \sum_{i=1}^{p-2} (1 - \frac{1}{\beta}) \right) + (1 - \frac{1}{\beta}) \right\} = 1, \]
or
\[ f_p \left\{ \frac{(2 - \beta)}{\beta(1 - \frac{1}{\beta})} \left( \frac{\beta^{p-2} - 1}{\beta - 1} - \frac{\beta^{p-2} - 1}{\beta - 1} - (p - 2)(1 - \frac{1}{\beta}) \right) + (1 - \frac{1}{\beta}) \right\} = 1, \]
or
\[ f_p = \frac{\beta(\beta - 1)}{1 - \beta(2 - \beta) - (p - 2)(\beta - 1)(2 - \beta) + (\beta - 1)^2}. \]

Therefore,
\[ f_{i+1} = \frac{\beta}{1 - \beta(2 - \beta) - (2 - \beta)(p - 2)(\beta - 1) + (\beta - 1)^2 (1 - \frac{1}{\beta^i})}, \]
where \( i = 1, 2, \ldots, p - 2. \)

**Corollary 4.8.** Suppose \( \beta \in (1, 2] \) such that \( \beta \) satisfies equation (4.4). Then the unique invariant probability density function \( F_\beta(x) = (F_1, F_2, \ldots, F_p) \) of the Chebyshev map \( \phi_\beta \) is
\[
F_\beta(x) = \begin{cases} 
0, & 0 \leq \frac{1}{\pi} \arccos(x) \leq 2 - \beta, \\
\frac{f_2}{\pi \sqrt{(1 - x^2)}}, & 2 - \beta \leq \frac{1}{\pi} \arccos(x) \leq \tau_\beta (2 - \beta), \\
\frac{f_3}{\pi \sqrt{(1 - x^2)}}, & \tau_\beta (2 - \beta) \leq \frac{1}{\pi} \arccos(x) \leq \tau_\beta^2 (2 - \beta), \\
\vdots \\
\frac{f_{p-1}}{\pi \sqrt{(1 - x^2)}}, & \tau_\beta^{p-3} (2 - \beta) \leq \frac{1}{\pi} \arccos(x) \leq \tau_\beta^{p-2} (2 - \beta), \\
\frac{f_p}{\pi \sqrt{(1 - x^2)}}, & \frac{1}{\beta} = \tau_\beta^{p-2} (2 - \beta) \leq \frac{1}{\pi} \arccos(x) \leq 1.
\end{cases}
\]

**Proof.** The Chebyshev map \( \phi_\beta \) is conjugated to \( \tau_\beta \) via the differentiable conjugacy \( h(x) = \cos(\pi x) \). Thus, by Proposition 1 of [5], the \( \phi_\beta \)-invariant density is
\[ F_\beta(x) = \phi_\beta(h^{-1}(x))|(h^{-1})'| = f_\beta \left( \frac{\arccos(x)}{\pi} \right) \frac{1}{\pi \sqrt{(1 - x^2)}}. \]
Hence, the theorem is proved. \( \square \)
**Theorem 4.9.** Suppose $\beta \in (1, 2]$ satisfies equation (4.4). Then the mean value $M(\beta)$ of the Chebyshev map $\phi_\beta$ is

$$M(\beta) = -\frac{C}{\pi} \{(\beta - 1) \sin \frac{\pi}{\beta} + \{\sin(\pi(2 - \beta)) - \sin \frac{\pi}{\beta} \}
- \sum_{i=1}^{p-2} \frac{1}{\beta^i}(\sin(\pi\beta^{i-1}(2 - \beta)) - \sin(\pi\beta^i(2 - \beta)))\},$$

(4.12)

where

$$C = \frac{\beta}{1 - \beta(2 - \beta) - (2 - \beta)(p - 2)(\beta - 1) + (\beta - 1)^2}.$$  

**Proof.** The density function of the Chebyshev map $\phi_\beta$ in equation (4.11) can be rewritten as

$$F_\beta(x) = \begin{cases} 0, & \cos(\pi(2 - \beta)) \leq x \leq 1, \\ \frac{f_{i+1}}{\pi \sqrt{(1 - x^2)}}, & \cos(\pi\beta^i(2 - \beta)) \leq x \leq \cos(\pi\beta^{i+1}(2 - \beta)), \\ \frac{f_p}{\pi \sqrt{(1 - x^2)}}, & -1 \leq x \leq \cos \frac{\pi}{\beta}. \end{cases}$$

(4.13)

The mean value of the Chebyshev map $\phi_\beta$ is $M(\beta) = \int_{-1}^{1} xF_\beta(x)$, where $F_\beta(x)$ is the invariant density function of the Chebyshev map $T_\beta$. Now,

$$M(\beta) = \int_{-1}^{1} xF_\beta(x)$$

$$= \int_{-1}^{\cos \frac{\pi}{\beta}} \frac{xf_p}{\pi \sqrt{(1 - x^2)}} + \sum_{i=1}^{p-2} \int_{\cos(\pi\beta^i(2 - \beta))}^{\cos(\pi\beta^{i+1}(2 - \beta))} \frac{x f_{i+1}}{\pi \sqrt{(1 - x^2)}}$$

$$= -\frac{C}{\pi} \{(\beta - 1) \sin \frac{\pi}{\beta} + \{\sin(\pi(2 - \beta)) - \sin \frac{\pi}{\beta} \}
- \sum_{i=1}^{p-2} \frac{1}{\beta^i}(\sin(\pi\beta^{i-1}(2 - \beta)) - \sin(\pi\beta^i(2 - \beta)))\}. \quad \square$$

### 4.3. Denseness of Parameters

Let

$$D = \left\{ \beta : \beta > 1, \frac{2}{\beta} \geq 1 \right\}.$$ 

Let $\mathcal{M}$ denotes the set of all kneading sequences $K(\beta)$ of $\tau_\beta$. 
Theorem 4.10. (see [11]) If $\beta_1 \in D$ and $\beta_2 \in D$ such that $\beta_1 > \beta_2$. Then $K(\beta_1) > K(\beta_2)$.

Theorem 4.11. (see [11]) If $\beta \in D$, then $K(\beta) \in \mathcal{M}$.

Theorem 4.12. (see [10]) If a one parameter family $g_t$ of a continuous unimodal maps depends continuously on $t$ and $K(g_{t_0}) < K < K(g_{t_1})$, $K \in \mathcal{M}$, then there exists $t \in (t_0, t_1)$ such that $K(g_t) = K$.

Theorem 4.13. The set $S$ of all $\beta \in (1, 2]$ such that $\tau_\beta$ is Markov is dense in $(1, 2]$.

Proof. Let $\beta \in (1, 2]$ such that $\tau_\beta$ is non-Markov. Given small $\epsilon > 0$, let $\beta^* = \beta + \epsilon \in D$. Clearly $\beta < \beta^*$ and by the Theorem 5.1, $K(\beta) < K(\beta^*)$. Moreover, by Theorem 5.2, $K(\beta), K(\beta^*) \in \mathcal{M}$. Since periodic or eventually periodic kneading sequences are dense in $\sigma_2$, there exists periodic or eventually periodic $A \in \mathcal{M}$ such that $K(\beta) < A < K(\beta^*)$. Observe that $\tau_\beta$ varies continuously over the parameter $\beta$. Hence, by Theorem 5.3, there exists $\beta_1$ such that $\beta < \beta_1 < \beta^*$ such that $K(\beta_1) = A$. \qed

Example 4.14. Let $p = 4$, i.e., $n = 1$. Then $\beta$ satisfies equation (4.4), $\beta^2 (2 - \beta) = \frac{1}{\beta}$. The positive solution of this equation is $\beta = 1.839286755$. The partition is

$$Q = \left\{ 0, 2 - \beta, \beta (2 - \beta), \beta (2 - \beta) = \frac{1}{\beta}, 1 \right\}.$$ 

The probability density function of $\tau_\beta$ is

$$f_1 = 0;$$

$$f_2 = \frac{\beta (\beta - 1)}{1 - \beta (2 - \beta) - (2 - \beta)(p - 2)(\beta - 1) + (\beta - 1)^2 (1 - \frac{1}{\beta})} = 0.73684;$$

$$f_3 = \frac{\beta (\beta - 1)}{1 - \beta (2 - \beta) - (2 - \beta)(p - 2)(\beta - 1) + (\beta - 1)^2 (1 - \frac{1}{\beta^2})} = 1.13745;$$

$$f_4 = \frac{\beta}{1 - \beta (2 - \beta) - (2 - \beta)(p - 2)(\beta - 1) + (\beta - 1)^2} = 1.35526.$$ 

The mean value of Chebyshev map $T_\beta$ is

$$M(\beta) = K \{ (\beta - 1) \sin \frac{\pi}{\beta} + \{ \sin(\pi (2 - \beta)) - \sin \frac{\pi}{\beta} \}ight.$$

$$- \sum_{i=1}^{p-2} \frac{1}{\beta^i} \{ \sin(\pi \beta^{i-1}(2 - \beta)) - \sin(\pi \beta^i (2 - \beta))) \} \} = -0.284249,$$
where $K = -\frac{\beta}{\pi(1-\beta)(2-\beta)-(2-\beta)(p-2)(\beta-1)+(\beta-1)^2)}$.

This value of the mean value of $T_\beta$ agree with the numerical result found in [9].

**Example 4.15.** Let $p = 3$, i.e., $n = 0$. Then $\beta$ satisfies equation (4.4), $\beta(2 - \beta) = \frac{1}{p}$. The positive solution of this equation is $\beta = 1.618$. The partition is $Q = \{0, 2 - \beta, \beta(2 - \beta) = \frac{1}{p}, 1\}$. The probability density function of $\tau_\beta$ is

\begin{align*}
  f_1 &= 0; \\
  f_2 &= \frac{\beta(\beta - 1)}{1 - \beta(2 - \beta) - (2 - \beta)(p - 2)(\beta - 1) + (\beta - 1)^2}(1 - \frac{1}{\beta}) = 1.171 , \\
  f_3 &= \frac{\beta}{1 - \beta(2 - \beta) - (2 - \beta)(p - 2)(\beta - 1) + (\beta - 1)^2} = 1.894 .
\end{align*}

The mean value of Chebyshev map $T_\beta$ is $M(\beta) = -0.562076$.

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**References**


