A NOTE ON THE GENERALIZED BELTRAMI FLOW THROUGH POROUS MEDIA

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Abstract: Exact solutions to the two-dimensional, viscous fluid flow through porous media, as governed by the Darcy-Lapwood-Brinkman model, are obtained for flows where the double curl vanishes. These types of flow are known as the Beltrami flows.

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1. Introduction

Analytical solutions to the Navier-Stokes equations are rare. In fact, most of the available analytical solutions have been obtained for special cases of two dimensional flows, including parallel, laminar flows for which the Navier-Stokes equations can be linearised (cf. [8] and the references therein), or flows that are classified as generalized Beltrami flows [3], [4], [14]. These are two dimensional flows for which the curl of the cross product of the vorticity and velocity vectors vanish. Excellent reviews have been reported in the literature on special solutions of the Navier-Stokes equations [2], [3], [5], [6], [13], [14], [15].
Beltrami flows have been receiving considerable attention over the past few decades for reasons that are both practical and theoretical [11], [14]. In terms of analysis of two dimensional flows, the generalized Beltrami flows have been studied for Newtonian [4] and non-Newtonian flows [9], [12], second grade fluid flow [2], and viscoelastic fluid flow [10]. Lesser attention has been given to Beltrami flows in porous media.

The purpose of this work is to initiate discussion on aspects of the generalized Beltrami flows through porous media. This may be of interest in the analysis of heat transfer problems through porous media. We consider fluid flow governed by the Darcy-Lapwood-Brinkman model [7]. The structure of this model resembles that of the Navier-Stokes equations; they have the same source of nonlinearity. However, the Darcy-Lapwood-Brinkman equation contains a viscous damping term that is due to the effect of the porous matrix on the flow.

We parallel the work of Chandna and Oku-Ukpong [4], and obtain the Beltrami flow solution for one choice of vorticity distribution.

2. Governing Equations

The steady flow of an incompressible, viscous fluid through porous media is governed by the conservation of mass and conservation of linear momentum principles. In the absence of sources and sinks, conservation of mass takes the following form of velocity continuity:

\[ \nabla \cdot \mathbf{v} = 0, \]  

(1)

where \( \mathbf{v} \) is the macroscopic velocity vector.

Conservation of linear momentum takes the following form, referred to as the Darcy-Lapwood-Brinkman equation, [7], when viscous shear and macroscopic inertial effects are significant:

\[ \rho ( \mathbf{v} \cdot \nabla ) \mathbf{v} = -\nabla p + \mu \nabla^2 \mathbf{v} - \frac{\mu}{k} \mathbf{v}, \]  

(2)

where \( p \) is the pressure, \( k \) is the permeability, \( \mu \) is the viscosity coefficient, \( \rho \) is the fluid density, and \( \nabla^2 \) is the laplacian operator.

When the flow is in the two space dimensions, \( x \) and \( y \), the above equations can be cast in the following dimensionless streamfunction-vorticity form:

Streamfunction Equation:

\[ \nabla^2 \Psi = -\Omega. \]  

(3)
Vorticity Equation:

\[ \nabla^2 \Omega - \frac{\Omega}{K} = Re\{\Psi_y \Omega_x - \Psi_x \Omega_y\}, \]  \hspace{1cm} (4)

where \( \Psi \) is the dimensionless streamfunction, \( \Omega \) is the dimensionless vorticity, \( Re \) is the Reynolds number, \( K \) is the dimensionless permeability, and subscript notation denotes partial differentiation with respect to the dimensionless coordinates \((x, y)\). Quantities have been rendered dimensionless with respect to a characteristic length and a reference velocity.

Letting \( U \) and \( V \) be the dimensionless tangential and normal velocity components, respectively, then their relationships with the dimensionless vorticity and streamfunction are given by:

\[ U = \Psi_y, \] \hspace{1cm} (5)
\[ V = -\Psi_y, \] \hspace{1cm} (6)
\[ \Omega = V_x - U_y. \] \hspace{1cm} (7)

In this vorticity-streamfunction formulation, it is required to solve equations (3) and (4) for the unknowns \( \Psi(x, y) \) and \( \Omega(x, y) \). The dimensionless velocity components may then be found from equations (5) and (6).

3. Solution Methodology

Upon using equation (3) in (4), we obtain the following fourth order integrability condition:

\[ Re\{\Psi_x (\nabla^2 \Psi)_y - \Psi_y (\nabla^2 \Psi)_x\} = \frac{1}{K} \nabla^2 \Psi - \nabla^4 \Psi, \] \hspace{1cm} (8)

where \( \nabla^4 \equiv \partial_{xxxx} + 2\partial_{xxyy} + \partial_{yyyy} \).

Our solution methodology may proceed in accordance with the following algorithm:

1) Assume the form of the vorticity distribution, \( \Omega = -\nabla^2 \Psi \).
2) Substitute the form \( \nabla^2 \Psi = -\Omega \) in the integrability equation (8).
3) Solve the equation resulting from Step 2 for \( \Psi(x, y) \).
4) Obtain the velocity resulting from equations (5) and (6).

For the Beltrami flow at hand, we wish to satisfy the vector equation:

\[ \nabla \times (\vec{\Omega} \times \vec{V}) = \vec{0}, \] \hspace{1cm} (9)
where, for two-dimensional flow, \( \vec{\Omega} = (0, 0, \Omega) \) is the vorticity vector, \( \vec{V} = (U,V,0) \) is the velocity vector.

Equation (9) leads to the following condition (with the help of the dimensionless equation of continuity):

\[
U\Omega_x + V\Omega_y = 0,
\]

or equivalently:

\[
\Psi_x(\nabla^2\Psi)_y - \Psi_y(\nabla^2\Psi)_x = 0.
\]

We must therefore choose \( \nabla^2\Psi = -\Omega \) such that equation (10), or equivalently equation (11), holds true. Let us assume that:

\[
\Omega = -\nabla^2\Psi = -[\Psi + f(x, y)].
\]

Using equation (12) in (11) yields:

\[
\Psi_x f_y - \Psi_y f_x = 0.
\]

Equation (13) provides us with a condition that the function \( f(x, y) \) must satisfy. Equations (8) and (13) thus represent two equations in the two functions \( \Psi(x, y) \) and \( f(x, y) \). If we assume a form for \( f(x, y) \), then we can solve the compatibility equation (8) for \( \Psi(x, y) \). We can then verify that the condition (13) is satisfied.

The literature reports on a number of forms of \( f(x, y) \) that have been adopted in the treatment of the Navier-Stokes equations [4]. Popular among these is the assumption that \( f(x, y) \) is a polynomial of first or second degree. In this work we take the following form of \( f(x, y) \):

\[
f(x, y) = Cx + Dy,
\]

where \( C \) and \( D \) are constants.

Using equation (14) in (12), we get:

\[
\Omega = -\nabla^2\Psi = -[\Psi + Cx + Dy],
\]

Using equation (15) in (8), we obtain after some simplification,

\[
Re\{D\Psi_x + C\Psi_y\} + \frac{K-1}{K}\{\Psi + Cx + Dy\} = 0.
\]

In order to solve equation (16), we introduce the canonical coordinates \( (\xi, \eta) \), defined by

\[
\xi = Cx + Dy
\]
and
\[ \eta = y. \]  \hfill (18)

Partial differential operators in the two coordinate systems are related by:
\[ \frac{\partial}{\partial x} = C \frac{\partial}{\partial \xi} \]  \hfill (19)
and
\[ \frac{\partial}{\partial y} = D \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}. \]  \hfill (20)

Using (17) through (19), equation (16) is transformed into the following form:
\[ \Psi_\eta - \left( \frac{K - 1}{K} \right) \frac{1}{ReC} \Psi = \left( \frac{K - 1}{K} \right) \frac{1}{ReC} \xi. \]  \hfill (21)

Now, equation (21) admits the exponential solution
\[ \Psi = -\xi + g(\xi) \exp \left\{ \left( \frac{K - 1}{K} \right) \frac{1}{ReC} \eta \right\}, \]  \hfill (22)
where \( g(\xi) \) is an arbitrary function of \( \xi \).

In terms of the original variables \( x \) and \( y \), equation (22) takes the form
\[ \Psi = -(Cx + Dy) + g(\xi) \exp \left\{ \left( \frac{K - 1}{K} \right) \frac{1}{ReC} y \right\}. \]  \hfill (23)

In order to determine \( g(\xi) \), we substitute equation (23) in (15). The following ordinary differential equation, in the unknown function \( g(\xi) \), is obtained:
\[ \{ (C^2 + D^2) g'' + 2MDg' + (M^2 - 1)g \} \exp(My) = 0, \]  \hfill (24)
or
\[ (C^2 + D^2) g'' + 2MDg' + (M^2 - 1)g = 0, \]  \hfill (25)
where \( M = \frac{K - 1}{K} \frac{1}{ReC} \).

Equation (25) has the auxiliary equation:
\[ (C^2 + D^2) m^2 + 2MDm + (M^2 - 1) = 0 \]  \hfill (26)
whose roots are given by
\[ m_1 = \frac{-MD - \sqrt{C^2 + D^2 - C^2 M^2}}{C^2 + D^2}, \]  \hfill (27)
\[ m_2 = \frac{-MD + \sqrt{C^2 + D^2 - C^2M^2}}{C^2 + D^2}. \] (28)

It is clear that the permeability, \( K \), enters the auxiliary equation (26), and hence affects the values of the roots given in equations (27) and (28). As \( K \to \infty \), the quantity \( M = \frac{K-1}{K \Re C} \to \frac{1}{\Re C} \), and the roots \( m_1 \) and \( m_2 \) become the same as those obtained in [4] for the Navier-Stokes case.

Depending on the discriminant, the following cases arise.

(a) If \( C^2 + D^2 - C^2M^2 > 0 \), then

\[ g(\xi) = c_1 \exp(m_1 \xi) + c_2 \exp(m_2 \xi) = c_1 \exp\{m_1(Cx + Dy)\} + c_2 \exp\{m_2(Cx + Dy)\}. \] (29)

Using (29) in (23) we obtain the following expression for the streamfunction:

\[ \Psi = -(Cx + Dy) + [c_1 \exp\{m_1(Cx + Dy)\}] + c_2 \exp\{m_2(Cx + Dy)\}] \exp\{My\}. \] (30)

The velocity components are obtained from equations (5), (6), and (30). Hence we have:

\[ U = -D + \exp(My)[c_1(m_1D + M) \exp\{m_1(Cx + Dy)\}] + c_2(m_2D + M) \exp\{m_2(Cx + Dy)\}], \] (31)

and

\[ V = C - C \exp(My)[c_1m_1 \exp\{m_1(Cx + Dy)\}] + c_2m_2 \exp\{m_2(Cx + Dy)\}]. \] (32)

If \( C^2 + D^2 - C^2M^2 = 0 \), then

\[ g(\xi) = (c_3 + c_4 \xi) \exp\left(\frac{-MD}{C^2 + D^2} \xi\right) = [c_3 + c_4(Cx + Dy)] \exp\left(\frac{-MD}{C^2 + D^2}(Cx + Dy)\right). \] (33)

Using (33) in (23) we obtain the following expression for the streamfunction:

\[ \Psi = (Cx + Dy) + [c_3 + c_4(Cx + Dy)] \exp\left(\frac{-MD}{C^2 + D^2}(Cx + Dy)\right) \]
The velocity components are obtained from equations (5), (6), and (34). Hence we have:

\[
U = -D + [c_3 + c_4(Cx + Dy)] \left[ c_4D + \left\{ M - \frac{MD^2}{C^2 + D^2} \right\} \right],
\]

\[
\exp \left( \frac{-MD}{C^2 + D^2} (Cx + Dy) \right) \exp \{My\}
\]

and

\[
V = C - C \left[ [c_3 + c_4(Cx + Dy)] \left\{ \frac{MD}{C^2 + D^2} \right\} + c_4 \right]
\times \exp \left( \frac{-MD}{C^2 + D^2} (Cx + Dy) \right) \exp \{My\}.
\]

If \( C^2 + D^2 - C^2M^2 < 0 \), then

\[
g(\xi) = [c_5 \cos A\xi + c_6 \sin A\xi] \exp \left( \frac{-MD}{C^2 + D^2} \right) = [c_5 \cos A(Cx + Dy)
\]

\[
+ c_6 \sin A(Cx + Dy)] \exp \left( \frac{-MD}{C^2 + D^2} (Cx + Dy) \right),
\]

where

\[
A = \sqrt{C^2M^2 - C^2 - D^2}.
\]

Using (37) in (23) we obtain the following expression for the streamfunction:

\[
\Psi = -(Cx + Dy) + [c_5 \cos A(Cx + Dy) + c_6 \sin A(Cx + Dy)]
\times \exp \left( \frac{-MD}{C^2 + D^2} (Cx + Dy) \right) \exp \{My\}.
\]

The velocity components are obtained from equations (5), (6), and (39). Hence we have:

\[
U = -D + \left\{ c_6AD + c_5 \left\{ M - \frac{MD^2}{C^2 + D^2} \right\} \cos A(Cx + Dy) \right.
\]

\[
+ \left\{ c_6 \left\{ M - \frac{MD^2}{C^2 + D^2} \right\} - c_5AD \right\} \sin A(Cx + Dy) \left\} \exp \left( \frac{-MD}{C^2 + D^2} (Cx + Dy) \right) \exp \{My\},
\]

\[
\times \exp \{My\}.
\]
and

\[
V = C + \left\{ -c_6 A + c_5 \left( \frac{MD}{C^2 + D^2} \right) \right\} \cos A(Cx + Dy)
+ \left\{ c_6 \left( \frac{MD^2}{C^2 + D^2} \right) + c_5 A \right\} \sin A(Cx + Dy)
\times \exp \left( \frac{-MD}{C^2 + D^2} (Cx + Dy) \right) \exp \{My\}. \quad (41)
\]

While the streamfunction and velocity components are influenced by the presence of the permeability, through the term containing \(M\), the flow structure resembles that of the Navier-Stokes case, discussed in [4], and hence represents an impingement of an oblique uniform stream with an oblique rotational, divergent flow, for some choices of the parameters involved. With the knowledge of the above solutions, one can generate the flow patterns for the full range of permeability and Reynolds number.

4. Conclusion

In this work, we offered an initial discussion of Beltrami flows through porous media. Further work is needed to study the admissible forms of the function \(f(x, y)\) that is utilized in the vorticity specification. Further analysis is needed to study in deep the effect of the permeability on the locations of the stagnation points in the flow regime.

References


