INVARIANCES AND CONVERGENCE PROPERTIES
OF BAIRSTOW’S METHOD

Wolfgang Gabler
Oberer Kümmelbergsweg 1
Neuwied, D-56567, GERMANY
e-mail: wolfgang.gabler@email.com

Abstract: Impressed with the simplicity of Bairstow’s iterative algorithm for calculating a real quadratic factor of a polynomial $f(x)$ with real coefficients, has prompted me to use the method on the digital computer. Dissatisfaction in practice, because it is fallible, has led me to extensive research into the effects of the observations made from computer runs. Its goal was to understand the reasons and from there to derive expansions for the algorithm in order to avoid divergence. As it is shown, Bairstow’s method has two types of invariance properties which are stated in Theorem I and Theorem II (Section 2). These two invariance theorems become the key for explaining phenomena around convergence in general, convergence rate like quadratic and linear and divergence of Bairstow’s iterative technique. It is invariant to certain straight lines which define subsets of all possible points in a two dimensional plane. These “invariant lines” may contain at least one or more problem solutions, i.e. quadratic factors of $f(x)$ or so called fixed points in the mapping under Bairstow’s method. This situation signifies the case of convergence and thus these invariant lines are considered to be “entry areas for pairs of roots” or synonymous with entry areas into fixed points. The divergence is a trivial outcome of the invariance theorems, an entry area for pairs of roots not containing any fixed point. Conclusively, it is shown that Theorem I implies divergence for all polynomials with odd degrees and Theorem II the equivalent for all even polynomials. In case of divergence, one would want to know how to proceed in an expanded Bairstow-algorithm. The remedies suggested by the author are presented in the form of rules to be observed by the programmer in order to use Bairstow’s technique successfully. Also presented is a theorem valid for cubics which has the surprising property of finding the problem solution in one step. Bairstow’s iterative
algorithm provides in this case a closed-form solution, given in Theorem III (Section 4). Another outcome of the invariance theorems are closed-form formulas for Bairstow’s method, i.e. an algebraic expression, that allows, at first glance, to judge the rate of convergence such as quadratic, linear or divergence for all invariant lines. These are shown in Theorem IV in four parts for the different root combinations of all cubics. Same is done for all quartics in Theorem V in five parts for the root combinations of special interest (Section 5). Again, rules for the programmer are derived from multiple root cases of Theorem IV and V, and a corresponding algorithm for practical use is recommended. In the context of polynomial reduction by forcing the polynomial coefficient representing the sum of the roots (VIETA) to zero, the famous Wilkinson polynomial of degree 20 with integer roots from -1 to -20 is addressed and it is shown that the “ill-conditioned polynomial” is transformed to a “well-conditioned” one (Section 3). This is to support the author’s recommendation of introducing polynomial reduction before applying Bairstow’s method. As a byproduct of the investigations, the fourths degree polynomials factorization is derived in closed-form by the resolvent cubic in possibly unknown derivation shortness (Section 7). As far as software implementation as well as verification & validation are concerned, Matlab of The MathWorks, Inc. on PC has been used to produce numerical results and their corresponding graphical illustrations. If appropriate, influences of finite mantissa length arithmetic are taken into account.

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1. Introduction: Bairstow’s Method

Consider the following polynomial with real coefficients

\[ f(x) = a_{n+1}x^n + a_n x^{n-1} + \cdots + a_2x + a_1, \ a_1, a_{n+1} \neq 0, \ a_{n+1} = 1 \] (1)

(normalization).

Bairstow’s method calculates the real and complex zeros of the polynomial (1) using only two simultaneous equations in an iterative process. By correcting the coefficients of a trial quadratic factor, it avoids calculations with complex numbers.

The polynomial (1) of degree \( n \) will be divided by the real quadratic polynomial

\[ h(x) = x^2 - px - q, \ p, q \ real, \] (2)
the quotient of degree \((n - 2)\) being \(g(x; p, q)\) with a linear remainder \(A(p, q)x + B(p, q)\). Thus

\[
f(x) = (x^2 - px - q)g(x; p, q) + A(p, q)x + B(p, q).
\]

(3)

If, for a given pair \((p^*, q^*)\),

\[
f(x) = (x^2 - p^*x - q^*)g(x; p^*, q^*)
\]

is valid, the zeros of the quadratic polynomial \(x^2 - p^*x - q^*\) are also zeros of \(f(x)\). This corresponds to the solution of the simultaneous equations

\[
A(p^*, q^*) = 0, \quad B(p^*, q^*) = 0.
\]

(5)

If real numbers \(x_1\) and \(x_2\) satisfy \(f(x_1) = 0\) and \(f(x_2) = 0\), i.e. they are real zeros of (1), it is further valid (VIETA)

\[
p^* = x_1 + x_2 \quad \text{and} \quad q^* = -x_1x_2,
\]

(6)

or just as well for pairs of complex conjugate roots \(x_{1,2} = u \pm iv\) which leads to (7) instead of (6):

\[
p^* = 2u \quad \text{and} \quad q^* = -(u^2 + v^2).
\]

(7)

Starting from values \((p_0, q_0)\) the corresponding Newton type method may be derived from the two dimensional Taylor series, from Lagrange’s Mean Value Theorem for functions of two variables, or by linearization. Selecting the Taylor series approach, the Taylor series for functions of two variables and Lagrange remainder reads

\[
f(x+h, y+k) = f(x, y) + \left( \frac{\partial}{\partial x} h + \frac{\partial}{\partial y} k \right) f(x, y)
\]

\[
+ \frac{1}{2!} \left( \frac{\partial}{\partial x} h + \frac{\partial}{\partial y} k \right)^2 f(x, y) + \cdots + \frac{1}{n!} \left( \frac{\partial}{\partial x} h + \frac{\partial}{\partial y} k \right)^n f(x + \vartheta h, y + \vartheta k),
\]

where \(0 < \vartheta < 1\). (8)

Substituting \(x = p_\nu, \ y = q_\nu, \ h = p_{\nu+1} - p_\nu, \ k = q_{\nu+1} - q_\nu, \ f = \{ A, B, \) and breaking the series off after the first derivative yields

\[
A(p_{\nu+1}, q_{\nu+1}) = A(p_\nu, q_\nu) + \frac{\partial A(p_\nu, q_\nu)}{\partial p}(p_{\nu+1} - p_\nu) + \frac{\partial A(p_\nu, q_\nu)}{\partial q}(q_{\nu+1} - q_\nu) = 0,\]

(9)
\[ B(p_{\nu+1}, q_{\nu+1}) = B(p_{\nu}, q_{\nu}) + \frac{\partial B(p_{\nu}, q_{\nu})}{\partial p}(p_{\nu+1} - p_{\nu}) + \frac{\partial B(p_{\nu}, q_{\nu})}{\partial q}(q_{\nu+1} - q_{\nu}) = 0. \] (9)

Applying matrix notation and abbreviating: \[ A_p = \frac{\partial A(p_{\nu}, q_{\nu})}{\partial p}, \ldots, A = A(p_{\nu}, q_{\nu}), \ldots \] leads to

\[ \begin{pmatrix} A_p & A_q \\ B_p & B_q \end{pmatrix} \begin{pmatrix} p_{\nu+1} - p_{\nu} \\ q_{\nu+1} - q_{\nu} \end{pmatrix} = -\begin{pmatrix} A \\ B \end{pmatrix}, \] (10)

\[ \begin{pmatrix} \Delta p \\ \Delta q \end{pmatrix} = \begin{pmatrix} p_{\nu+1} - p_{\nu} \\ q_{\nu+1} - q_{\nu} \end{pmatrix} = \begin{pmatrix} A_p & A_q \\ B_p & B_q \end{pmatrix}^{-1} \begin{pmatrix} -A \\ -B \end{pmatrix} = \frac{1}{J} \begin{pmatrix} B_{Aq} - AB_q \\ AB_p - BA_p \end{pmatrix}, \] (11)

\[ \begin{pmatrix} p_{\nu+1} \\ q_{\nu+1} \end{pmatrix} = \begin{pmatrix} p_{\nu} \\ q_{\nu} \end{pmatrix} + \frac{1}{J} \begin{pmatrix} B_{Aq} - AB_q \\ AB_p - BA_p \end{pmatrix} = \begin{pmatrix} p_{\nu} \\ q_{\nu} \end{pmatrix} + \begin{pmatrix} D_1/J \\ D_2/J \end{pmatrix}, \] (12)

where \( D_1 = -\left| \begin{array}{cc} A & A_q \\ B & B_q \end{array} \right| = BA_q - AB_q, \) \( D_2 = 1/ \left( \begin{array}{c} A_p \\ B_p \end{array} \right) \left| \begin{array}{c} A \\ B \end{array} \right| = AB_p - BA_p, \)

Jacobian determinant \( J = +\left| \begin{array}{cc} A_p & A_q \\ B_p & B_q \end{array} \right| = A_pB_q - A_qB_p. \)

Summarizing, the appropriate Newton iterative algorithm may be expressed as follows

\[ \begin{pmatrix} p_{\nu+1} \\ q_{\nu+1} \end{pmatrix} = \begin{pmatrix} \varphi_1(p_{\nu}, q_{\nu}) \\ \varphi_2(p_{\nu}, q_{\nu}) \end{pmatrix} = \begin{pmatrix} p_{\nu} + \frac{B_{A_p} - AB_q}{A_pB_q - A_qB_p} \\ q_{\nu} + \frac{AB_p - BA_p}{A_pB_q - A_qB_p} \end{pmatrix} \] (\( \nu = 0, 1, 2, \ldots \)), (13)

that, illustratively speaking, maps in the \( p,q - \text{plane} \) the point \( (p_{\nu}, q_{\nu}) \) into the new point \( (p_{\nu+1}, q_{\nu+1}) \). Consequently this mapping will have for every point \((p^*, q^*)\) according to (4)-(7) the property that it is mapped to itself and this for any repeated iteration as well:

\[ \begin{pmatrix} p^* \\ q^* \end{pmatrix} = \begin{pmatrix} \varphi_1(p^*, q^*) \\ \varphi_2(p^*, q^*) \end{pmatrix} \] for each quadratic factor of \( f(x) = 0. \) (14)

Each point \( (p^*, q^*) \) is therefore named also \textit{fixed point} of the mapping, or \textit{fixed point} of the iterative algorithm (13) of Bairstow’s method. As to the question on how many of such \textit{fixed points} exist in the \( p,q - \text{plane} \) for a polynomial
of degree \(n\), the answer is simply between \(\binom{n}{2} = \frac{n(n-1)}{2}\) and 1 depending on how many roots are real and distinct leading for the number \(n\) to the maximal number as given. The minimal number 1 is obtained for a real root of multiplicity \(n\) or single multiple complex conjugate root pair and possibly one real root for the case of odd degree polynomials. All other root distributions lead to numbers in-between those extreme values.

Analogously to the well known convergence conditions of the conventional Newton method for functions of a single variable, giving \(\varphi(x) = x - f/f'\) and its derivative \(\varphi'(x) = f f''/f'^2\), convergence rate could be shown for Bairstow’s method as well, using the corresponding Newton method for functions of two variables, in order to show quadratic convergence in case where \(f\) has only simple roots. This is deferred for the reason that the issue of rate of convergence is best handled, after having learned about invariance properties of the mapping given by (13).

The determination of the values of \(A\) and \(B\) is best done using the double row Horner Scheme, the equivalent to the conventional Horner Scheme, which is referred to as well as synthetic division or nested multiplication for treating linear factors of polynomials. This Horner Scheme in two rows is derived from the following coefficient comparison:

\[
a_{n+1}x^n + a_nx^{n-1} + \cdots + a_2x + a_1 = (c_{n+1}x^{n-2} + c_nx^{n-3} + \cdots + c_5x^2 + c_4x + c_3) \\
\times (x^2 - px - q) + c_2x + c_1, \quad (15)
\]

where the quotient is denoted:

\[
g_{(n-2)}(x; p, q) = c_{n+1}x^{n-2} + c_nx^{n-3} + \cdots + c_5x^2 + c_4x + c_3,
\]
The comparison of coefficients of equal power in the three rows above yield:
\[
x^n : \quad a_{n+1} = c_{n+1} \quad \Rightarrow \quad c_{n+1} = a_{n+1},
\]
\[
x^{n-1} : \quad a_n = c_n - pc_{n+1} \quad \Rightarrow \quad c_n = a_n + pc_{n+1},
\]
\[
x^{n-2} : \quad a_{n-1} = c_{n-1} - pc_n - qc_{n+1} \quad \Rightarrow \quad c_{n-1} = a_{n-1} + qc_{n+1} + pc_n,
\]
\[
x^{n-3} : \quad a_{n-2} = c_{n-2} - pc_{n-1} - qc_n \quad \Rightarrow \quad c_{n-2} = a_{n-2} + pc_{n-1},
\]
\[\vdots\]
\[
x^2 : \quad a_3 = c_3 - pc_4 - qc_5 \quad \Rightarrow \quad c_3 = a_3 + qc_5 + pc_4,
\]
\[
x^1 : \quad a_2 = c_2 - pc_3 - qc_4 \quad \Rightarrow \quad c_2 = a_2 + qc_4 + pc_3,
\]
\[
x^0 : \quad a_1 = c_1 - qc_3 \quad \Rightarrow \quad c_1 = a_1 + qc_3,
\]

with the obvious equalities \(c_2 = A\) and \(c_1 = B\).

Therefore the coefficients \(c_1\) up to \(c_{n+1}\) will be best determined by the double row Horner scheme

\[
\begin{array}{cccccccc}
\text{coeff. of } f(x) & a_{n+1} & a_n & a_{n-1} & a_{n-2} & \ldots & a_3 & a_2 & a_1 \\
\text{const. factor } q & - & - & q_{c_{n+1}} & q_{c_n} & \ldots & q_c & q & q_p \\
\text{const. factor } p & - & pc_{n+1} & pc_n & pc_{n-1} & \ldots & pc & pc & - \\
\hline
\text{column } \Sigma & c_{n+1} & c_n & c_{n-1} & c_{n-2} & \ldots & c_3 & c_2 & A \quad c_1 = B
\end{array}
\quad (16)
\]

In a continued Double Row Horner Scheme the partial derivatives \(A_p, A_q, B_p, B_q\) are similarly determined (derivation omitted):

\[
\begin{array}{cccccccc}
\text{quotient } g_{n-2} & c_{n+1} & c_n & c_{n-1} & c_{n-2} & \ldots & c_4 & c_3 & 0 \\
\text{const. factor } q & - & - & q_{d_{n+1}} & q_{d_n} & \ldots & q_{d_4} & q_{d_5} & q_{d_4} \\
\text{const. factor } p & - & pd_{n+1} & pd_n & pd_{n-1} & \ldots & pd_{d_5} & pd_{d_4} & - \\
\hline
\text{column } \Sigma & d_{n+1} & d_n & d_{n-1} & d_{n-2} & \ldots & d_4 & A_q & d_3 = A_p \quad d_2 = B_p
\end{array}
\quad (17)
\]

where \(A_p = d_3, B_p = d_2, A_q = d_4, B_q = d_3 - pd_4\).

The simplicity of Bairstow method is best demonstrated by the 19 lines of actual code representing a Matlab-function to calculate from \((p_\nu, q_\nu)\) the corrections \((\Delta p, \Delta q)\) in order to find improved values \((p_{\nu+1}, q_{\nu+1}) = (p_\nu + \Delta p, q_\nu + \Delta q)\):

\[
\text{function } [dp, dq, Pnr, BCV] = \text{bairstow}(p, q, Pn)
\]

\[
% \text{For a polynomial } Pn \text{ of degree } n \text{ \textgreater } 2 \text{ and the values } p \text{ and } q \\
% \text{relating to the quadratic factor } x^2 - px - q \text{ the values } \\
% dp=delta p \text{ and } dq=delta q \text{ for improving } p=p+dp, \text{ } q=q+dq \\
% \text{are calculated using Bairstow's method} \\
% \text{Input: } Pn \text{ polynomial } a_{n+1}x^n + a_nx^{n-1} + \ldots + a_1x + a_0, \text{ where } n \text{ \textgreater } 2 \\
% p, q \text{ initial values of the trial quadratic factor } x^2 - px - q \\
% \text{Output: dp delta p to calculate an improved value } p+dp \\
% \text{ dq delta q to calculate an improved value } q+dq
\]
\% BCV(1:9) characteristic values related to the calculation of Bairstow’s method: BCV=[D1 D2 J A B Ap Aq Bp Bq]
\% Pnr remainder polynomial when dividing Pn by x^2-px-q
\%************************************************************************************qe=[1. -p -q];
\%the trial quadratic equation

\[\text{[Pnr,r]}=\text{deconv}(\text{Pn},\text{qe}); \ \%\text{polynomial division (deconvolution)}\]
\[\text{n=}\text{length(r)};\]
\[\text{A}=\text{r(n-1)};\]
\[\text{B}=\text{r(n)};\]
\[\text{pr1}=[\text{Pnr} \ 0];\]
\[\text{[prr,rr]}=\text{deconv}(\text{pr1},\text{qe}); \ \%\text{continued poly. division (deconvol.)}\]
\[\text{nr=}\text{length(rr)};\]
\[\text{Ap}=\text{rr(nr-1)};\]
\[\text{Bp}=\text{rr(nr)};\]
\[\text{np=}\text{length(prr)};\]
\[\text{Aq}=\text{prr(np)};\]
\[\text{Bq}=\text{rr(nr-1)}-\text{p*prr(np)};\]
\[\text{D}=\text{Ap*Bq-Aq*Bp}; \ \%\text{JACOBIAN determinant}\]
\[\text{D1}= \text{B*Aq-A*Bq}; \ \%\text{D=JACOBIAN determinant}\]
\[\text{D2}= \text{A*Bp-B*Ap};\]
\[\text{dp}\text{=D1/D}; \ \%\text{dp=(B*Aq-A*Bq)/JACOBIAN determinant}\]
\[\text{dq}\text{=D2/D}; \ \%\text{dq=(A*Bp-B*Ap)/JACOBIAN determinant}\]
\[\text{BCV}=[\text{D1} \ \text{D} \ \text{A} \ \text{B} \ \text{Ap} \ \text{Aq} \ \text{Bp} \ \text{Bq}];\]

An example shall follow that pursues more than one purpose. Besides demonstrating the method and its provided tools it shall show that the computer is calculating with finite mantissa length causing round-off errors with sometimes effects that mislead the scientist’s observations made from computer runs. For ease of hand calculations a cubic with only two out of four coefficients being nonzero has been chosen for this numerical example:

\[f(x) = x^3 + 0x^2 + 0x + 0.064\]

using the initial value pair \((p_0 = -0.4, q_0 = 0)\) (18)

<table>
<thead>
<tr>
<th>coef. of (f(x))</th>
<th>1</th>
<th>0</th>
<th>0</th>
<th>.064</th>
</tr>
</thead>
<tbody>
<tr>
<td>(q_0 = 0)</td>
<td>-</td>
<td>-</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
| \(p_0 = -0.4\) | - | - | 0 | 0

\[\text{column } \Sigma \]
| 1 | - | .16 | A | .064 = B |

| \(g(1)\) | 0 | 1 | - | 0 |
| \(q_0 = 0\) | - | - | 0 |
| \(p_0 = -0.4\) | - | - | 0 |

\[\text{column } \Sigma \]
| 1 = Aq | -.8 = Ap | 0 = Bp |
\[ B_q = d_3 - pd_4 = -0.4, \]

\[
\begin{pmatrix} p_1 \\ q_1 \end{pmatrix} = \begin{pmatrix} p_0 + \frac{BAB_q - AB_p}{A_p B_q - A_q B_p} \\ q_0 + \frac{AB_p - BA_p}{A_p B_q - A_q B_p} \end{pmatrix} = \begin{pmatrix} -0.4 + \frac{0.064 \times 1 - 0.16 \times (-0.4)}{-0.8 \times (-0.4) - 1} \\ 0 + \frac{0.16 + 0.064 \times (-0.8)}{0.8 + 0.4} \end{pmatrix} = \begin{pmatrix} 0 \\ 0.16 \end{pmatrix}.
\]

Repetition of the process yields:

\[
\begin{array}{c|cccc}
\text{coef. of } f(x) & 1 & 0 & 0 & .064 \\
\hline
q_1 = .16 & - & - & .16 & 0 \\
p_1 = 0 & - & 0 & 0 & - \\
\hline
\text{column } \Sigma & 1 & 0 & .16 = A & .064 = B \\
\end{array}
\]

\[ B_q = d_3 - pd_4 = 0, \]

\[
\begin{pmatrix} p_2 \\ q_2 \end{pmatrix} = \begin{pmatrix} p_1 + \frac{BAB_q - AB_p}{A_p B_q - A_q B_p} \\ q_1 + \frac{AB_p - BA_p}{A_p B_q - A_q B_p} \end{pmatrix} = \begin{pmatrix} 0 + \frac{0.064 \times 1 - 0.16 \times 0}{0.8 \times 1 - 0.16} \\ .16 + \frac{0.16 + 0.064 \times 0}{-0.16} \end{pmatrix} = \begin{pmatrix} -A \\ 0 \end{pmatrix} = \begin{pmatrix} p_0 \\ q_0 \end{pmatrix}.
\]

Consequently

\[
\begin{array}{c|cccc}
\nu & -0.4 & 0 & \nu = 0, 2, 4, 6, \ldots \text{(even)}, \\
\hline
q_\nu & 0 & 0.16 & \nu = 1, 3, 5, \ldots \text{(odd)}. \\
\end{array}
\]

(19)

In this numerical example, Bairstow’s method demonstrates what could be best characterized by the word “cage-effect” or “pendulum-effect” as well.

This example applied to IEEE standard double precision for binary arithmetic with 1 sign bit, 52 bits for mantissa, 11 bits for exponent will not calculate the same result, as is shown from the following Matlab results (note: Matlab uses IEEE double precision):

\[ \nu p_\nu q_\nu - B (p_\nu, q_\nu) / A (p_\nu, q_\nu) \quad \{\leftarrow \text{Rule 1, Section 2}\} \]
The floating point calculations in its first iteration $p_1 = -0.4 + (0.064 + 0.16 \times 0.4)/(0.8 + 0.4)$ yields

\[
p_1 = \begin{cases} 
-2^{-53} & \text{personal computer: IEEE binary arithmetic (52 bit mantissa length)}, \\
0 & \text{hand calculation: mantissa length \text{"\rightarrow\} inf}}. 
\end{cases}
\]

The simple finite mantissa length calculation spoils the “cage-effect” and after completing 1356 iterations the only fixed point $(p^* = 0.4, q^* = -0.16)$ of the cubic polynomial is computed. As we will learn later from “Invariance Theorem I”, this should absolutely not have been possible when using calculations of infinite mantissa length. We must conclude that round-off errors are again (2-nd occurrence (!)) responsible for the reason of ending up in a fixed point. Certainly the “unexplained” high number of iterations should indicate something abnormal, although the wanted fixed point had been finally computed.
This numerical example of the “missed cage-effect” and the “nonconvergence case that converges(!)” has brought about the following: Whenever the computer is used to do software validation and verification and the computer runs are viewed, we must acknowledge the fact that we are looking at results being a mixture of mathematical law and order accompanied by rounding errors that might have falsifying effects. Since we do not know whether least significant bit round-off errors have become significant, it got to be thoroughly investigated, like done in this paper, in order to avoid false interpretations of observations made. This could mean comparing hand calculations without rounding errors with computer runs that may explain otherwise undiscovered phenomena. The difficulties of finding the invariance properties of Bairstow’s method are probably due to the fact that a strong hint had to come from computer runs, but that these hints did not come through purely but mixed with rounding error effects.

2. Properties of Bairstow’s Method: Two Invariance Theorems

Bairstow’s iterative algorithm (13) has the following property:

**Theorem I.** When \( f(x) \) has a single real root \( x = k \) and for the index \( \nu \),

\[
k^2 - p_\nu k - q_\nu = 0
\]  

(20)

is true, it is also true for the following index (and therefore for all further indices):

\[
k^2 - p_{\nu+1}k - q_{\nu+1} = 0.
\]  

(21)

For the proof, form the partial derivatives of (3) with respect to \( p \) and \( q \):

\[
0 = -xg(x; p, q) + (x^2 - px - q) \frac{\partial g}{\partial p}(x; p, q) + \frac{\partial A}{\partial p}(p, q)x + \frac{\partial B}{\partial p}(p, q),
\]  

(22)

\[
0 = -g(x; p, q) + (x^2 - px - q) \frac{\partial g}{\partial q}(x; p, q) + \frac{\partial A}{\partial q}(p, q)x + \frac{\partial B}{\partial q}(p, q),
\]  

(23)

and in addition form the equations which follow from (3), (22) and (23) for \( x = k, p = p_\nu \) and \( q = q_\nu \) with respect to (20) and abbreviating \((A)_\nu = A(p_\nu, q_\nu), \ldots\):

\[
0 = (A)_\nu k + (B)_\nu,
\]  

(24)

\[
0 = -kg(k; p_\nu, q_\nu) + (A_p)_\nu k + (B_p)_\nu,
\]  

(25)
\[
0 = -g(k; p_\nu, q_\nu) + (A_q)_\nu k + (B_q)_\nu. \tag{26}
\]

By eliminating \(g(k; p_\nu, q_\nu)\) from (25) and (26) we get
\[
(A_q)_\nu k^2 + (B_q - A_p)_\nu k - (B_p)_\nu = 0. \tag{27}
\]

Thus it follows from elementary calculations using (27) and applying (20), (24), (11) to (27)
\[
-k = \frac{(B_p + kA_p)_\nu}{(-kA_q - B_q)_\nu} = \left(\frac{\frac{4}{\nu}B_p + \frac{4k}{\nu}A_p}{\frac{4}{\nu}A_q - \frac{4}{\nu}B_q}\right)_\nu = \left(\frac{(AB_p - BA_p) / J}{(BA_q - AB_q) / J}\right)_\nu
= \frac{\Delta q}{\Delta p} = \frac{q_{\nu+1} - q_\nu}{p_{\nu+1} - p_\nu},
\]

that
\[
(p_{\nu+1} - p_\nu)k + (q_{\nu+1} - q_\nu) = 0. \tag{28}
\]

But this is equivalent to the proposition (21) by reason of (20).

Observe that Theorem I has a geometrical interpretation in the \(p, q - plane\), refer to Figure 1: Bairstow’s iterations (13) are invariant on the straight lines
\[
q(p) = -kp + k^2 \text{ with } k \text{ each real root of } f(x). \tag{29}
\]

Thus, if \(f(x)\) has \(r\) distinct real roots \(x = k_i\) \((i = 1, 2, ..., r, \ r \leq n)\), then there exist \(r\) different straight lines (29) with the property of being an invariant under the transformation of Bairstow’s iterative technique. It should further be noted that the envelope of these \(r\) invariant lines is the discriminant of the quadratic equation (2) set to zero for the loci of double roots being \(q = -p^2/4\). Consequently the invariant lines given by Theorem I cannot reach the area \(q < -p^2/4\), this must be kept in mind for the divergence considerations to come shortly.

The invariant lines are at the same time the locus of all points in the \(p, q - plane\) for which one real root of (1), \(x = k\), is kept constant. For the proof consider the quadratic equation and VIETA’s equalities: \(p = k + x_2; q = -kx_2; \Rightarrow q = -kp + k^2\).

Convergence considerations of Theorem I: As long as an invariant line stated by Theorem I includes at least one fixed point \((p^*, q^*)\), this invariant line is considered to be an entry area into the single fixed point or, if appropriate, the various distinct fixed points located on the invariant line. Instead of the term fixed point one may use also problem solution or pairs of roots. This is to explain the already used nomenclature in the abstract.
Considering the contrary of convergence, divergence and speaking of Theorem I and its invariant lines, the failure transforms into an entry area for pairs of roots not containing any pair of roots. The difficult problem of proving divergence for Bairstow’s method has been reduced to a trivial conclusion from Theorem I. As a matter of fact, it is not difficult to state the nonconvergence for all odd degree polynomials in case where the polynomial consists of only complex conjugate roots up to one real root. The real root assures the existence of an invariant straight line, but the single real root cannot be combined with all other complex or conjugate complex roots to form a fixed point that belongs to the set of points contained within the invariant line. Or, as already said above, the invariant line(s) cannot reach into the area $q < -p^2/4$, where complex conjugates root pairs are located.

For the two cases, the one where the infinite number of iterations cannot be terminated successfully, and the one where it converges, as well as multiple root combinations with the special interest in convergence rate, will be treated in Section “4. Convergence rate . . . ”. The invariant lines that follow from Theorem I allow the derivation of algebraic closed-form solutions of Bairstow’s iterative technique valid for just these invariant straight lines. That is of great benefit for the research on especially the convergence rate of this iterative procedure for the multiple root cases.

Although Theorem I uses a real factor of the wanted solution, it is possible
to expand the algorithm for the digital computer in a way that the failure is recognized and compensated for as well. From the proof of Theorem I, equation (24), it is learned that the ratio $-(B)_{\nu}/(A)_{\nu}$ remains constant and equal to a real root of $f(x)$, if and only if Bairstow’s method operates on an invariant line. It is certainly helpful that $(A)_{\nu}$ and $(B)_{\nu}$ are characteristic values computed anyway in each Bairstow iteration. Thus the following rule is proposed for the expansion of Bairstow’s algorithm to avoid divergence of odd degree polynomials according to Theorem I.

**Rule 1.** In addition to the usual method, the ratio $-(B)_{\nu}/(A)_{\nu}$ should be calculated and the values stored for all consecutive iterations. In the event, where the counter for the number of allowable iterations is exceeded (because no fixed point found), check if the successive values of these ratios differ by less than a small tolerance. If yes, a single root has already been found. If the accuracy required demands it, the value $-(B)_{\nu}/(A)_{\nu}$ may be used as an excellent approximation for a reiteration with the conventional Newton method. This way Bairstow’s method is terminated after having found a linear factor and not, as in the regular case, with extraction of a quadratic factor.

It should be noted, that for invariant lines

$$\frac{-\Delta q}{\Delta p} = \frac{q_{\nu+1} - q_{\nu}}{p_{\nu+1} - p_{\nu}},$$

would compute the same value as $-(B)_{\nu}/(A)_{\nu}$ does from values that are also determined in any Bairstow iteration. The only difference is given by the number of floating point operations while computing these numbers, which has about doubled here, because $(A)_{\nu}$ and $(B)_{\nu}$ require only one time the double row Horner Scheme, where the $\Delta p$ and $\Delta q$ calculation does twice. If rounding errors are intended to be studied, this should be kept in mind and can provide insight.

It is well known, that Bairstow’s method [1] has been first published in 1914, and due to difficult access to the isolated literature, the method has been later independently reinvented by Hitchcock [5] and again later by others as well. Some similarity is to be observed with the contents of Theorem I. The author published in 1969 this very theorem [4], Boyd rediscovered it in 1977 [2]. But there is still one more thing to be learned from Theorem I and that is based on general mathematical experience and expectation: Knowing Theorem I and its consequences as stated above, there must exist another type of invariance, i.e. a Theorem II, explaining divergence of even degree polynomials. This conjecture has initiated research that culminated in the following: Bairstow’s iterative algorithm (13) has also this property.
Theorem II. When the polynomial \( f(x) \) is of even degree, \( n = n_e \), \( f(x) = a_{n_e+1}x^{n_e}+a_{n_e}x^{n_e-1}+\cdots+a_2x+a_1 \), where \( n_e = 4, 6, 8, \ldots \) and if all coefficients \( a_i \) with even index are equal to zero,

\[
a_{2i} = 0, \quad i = 1, 2, \ldots, n_e/2
\]

and is for the index \( \nu \)

\[
p_\nu = 0,
\]

true, it is also true for the following index (and therefore for all further indices):

\[
p_\nu+1 = 0.
\]

Remark to Theorem II. The validity of Theorem II does not depend on the odd coefficients

\[
a_{2i-1}, \quad i = 1, 2, \ldots, (n_e/2 + 1), \quad n_e = 4, 6, 8, \ldots
\]

whether they are zero or unequal to zero!

For the proof, start from (13) and use the modified notation

\[
p_{\nu+1} = p_\nu + \left( \frac{B \frac{\partial}{\partial q} - B_q}{A_pB_q - A_qB_p} \right)^{\nu},
\]

where \( A \) in the numerator has been isolated. Assume that the explicit representation of \( A(p, q) \) dependent on the polynomial degree \( n \) has been determined to be

\[
A(n)(p, q) = \sum_{\nu=0}^{n-1} a_{\nu+2}p^{\nu} + \sum_{\nu=1}^{[(n-1)/2]} a_{2\nu+2}q^{\nu} + \sum_{\mu=1}^{[(n-2)/2]} q^{\mu} \sum_{i=1}^{n-(2\mu+1)} \binom{i + \mu}{i} a_{i+2\mu+2}p^i,
\]

\( n \geq 3 \) or for even degree \( n = n_e \), where a combination of the 3 summation terms into one is possible,

\[
A(n_e)(p, q) = \sum_{\mu=0}^{n_e/2-1} q^{\mu} \sum_{i=0}^{n_e-(2\mu+1)} \binom{i + \mu}{i} a_{i+2\mu+2}p^i;
\]
and the formulas have successfully passed the induction proof (see further below). Applying the condition of the polynomial coefficients in Theorem II to $A_{(n_e)}(p_\nu, q_\nu)$, equation (34) will produce a multiplier $p_\nu$ which can be put outside the summation expressions:

$$A_{(n_e)}(p_\nu, q_\nu; a_2 = a_4 = \cdots = a_{n_e} = 0) = p_\nu \left( \sum_{\nu=0}^{n_e/2-1} (q^\nu)^{\nu} \sum_{i=1}^{n_e-(2\nu+1)} \binom{i + \mu}{i} a_{i+2\nu+2} (p^{i-1})^\nu \right) ,$$  \hspace{1cm} (35)

where $n_e = 4, 6, 8, \ldots$ and $\nu$-th iteration

$$A_{(n_e)}(p_\nu = 0, q_\nu; a_2 = a_4 = \cdots = a_{n_e} = 0) = 0 .$$  \hspace{1cm} (36)

From equations (32) and (36) the result

$$p_\nu = p_{\nu+1} = 0$$

can be deduced, which is equivalent to the proposition (31) by reason of (30).

We are postponing the induction proof of (33) to the section’s end, in order to continue reasoning, here.

Observe that Theorem II, again, has a geometrical interpretation in the $p, q - plane$. Bairstow’s iterations (13) are invariant on the straight line $p = 0$, which is, illustratively speaking, the $q$-axis of that plane. All mappings under Bairstow’s method follow ($p_\nu = 0$, $-\infty < q_\nu < +\infty$) if iterations have been started from the beginning with ($p_0 = 0$, $-\infty < q_0 < +\infty$) or entered the $q$-axis by the $\nu$-th iteration.

Convergence considerations of Theorem II are the same as given for the sister theorem, Theorem I. This means that the invariant line stated by Theorem II is an entry area into the single fixed point ($p^* = 0, q^*$) or into several fixed points situated on the $q$-axis, respectively. Convergence requires at least one fixed point within the set of points on the $q$-axis.

The contrary of convergence, divergence, again, concerns the lack of any fixed point on the $q$-axis for Theorem II’s invariant straight line. This leads to the question whether there are polynomials of even degree meeting the coefficient condition of Theorem II and lacking any fixed point on the $q$-axis. Indeed these conditions can easily be met:

a) Consider all polynomials

$$x^{n_e} + a_1 = 0 , \text{ where } n_e = 4, 6, 8, \ldots ,$$  \hspace{1cm} (37)
which meet and exceed the condition for the polynomial coefficients of Theorem II. For the roots which lie here on a circle in the complex plane, the following angle condition is valid for the location of the roots:

\[ \varphi(n) = \frac{\pi + 2k\pi}{n}, \quad \text{where} \quad k = 0, 1, 2, \ldots, (n-1). \]

To avoid any root pair \((p^*, q^*)\) at \(p = 0,\)

\[ \varphi(n) \neq \frac{\pi}{2} \]

is necessary, which is met for

\[ n_e = 2(1 + 2k) + 2, \quad \text{where} \quad k = 0, 1, 2, \ldots, \quad (38) \]

consequently for all polynomials \((37)\) of degree \(n_e = 4, 8, 12, 16, \ldots.\) Bairstow’s algorithm will for these polynomials not converge if for some \(\nu, \rho_\nu = 0\) is met.

b) For quartic polynomials one may avoid the existence of a fixed point \((p^*, q^*)\) on the invariant line \(p = 0\) by the following requirements:

\[
\begin{align*}
    p_1^* &= -p_2^* \neq 0, \\
    q_1^* &= q_2^* = q^*,
\end{align*}
\]

under the condition \(q^* < -\left(p_1^*\right)^2/4. \quad (39)\)

In this case the corresponding polynomial coefficients are

\[
\begin{align*}
    a_1 &= (q^*)^2; \quad a_2 = 0; \quad a_3 = -(p_1^*)^2 - 2q^*; \quad a_4 = 0; \quad a_5 = 1. \quad (40)
\end{align*}
\]

Remark. Refer to Theorem Ve (Section 4) with the corresponding requirement \(a_3^2 < 4a_1, \quad a_1 > 0\) or \((p_1^*)^2 = 2q^*, \) i.e. \(a_3 = 0.\) Here it will be required only that \(a_1 > 0\) in order not to have any fixed point \((p^*, q^*)\) on invariant line \(p = 0.\) The latter case is certainly contained in a) for \(n_e = 4.\)

Regarding b) one could think of creating comparable polynomials of degree \(n_e > 4.\) For convenience this will not be pursued here any further. Instead, it will be discussed how an infinite number of new polynomials can be generated starting from divergence polynomials according to a) and b). Applying the transformation of axis shifting \(x = y - c\) with \(-\infty < c < +\infty\) by selecting any finite value of \(c\) to already above mentioned polynomials a) or b), will create a new polynomial \(f(y) = b_{n+1}y^n + b_ny^{n-1} + \cdots + b_2y + b_1\) having now a shifted invariant line, i.e. shifted from \(p = 0\) to the new invariant line \(p = \text{const.} \neq 0.\) where Bairstow’s method, now will diverge. The transformed polynomial will no more have any special zero coefficient condition. In order to recognize from such
a polynomial \( f(y) \) the existence of an invariant line, a reverse transformation \( y = x - b_n/(b_{n+1}n) \) is required. Section 3 provides details about this subject.

The significance of Theorem II is not covered in full, if it had not been addressed that also invariant lines \( p = \text{const.} \neq 0 \) without fixed points are possible. Theorem II is still sufficient, since a well known transformation for polynomials can force any polynomial to a reduced form which then allows its application.

Theorem II’s coefficient condition is fulfilled for all members of the class of all polynomials \( f(x, a_1, \ldots, a_n, a_{n+1}) \) with equidistant roots. If \( a_n \neq 0 \) and \( n \) even, the transformation \( x = y - a_n/(a_{n+1}n) \) will lead just to the zero coefficient condition demanded by Theorem II. But in these cases the existing invariant line \( p = 0 \) will include more than one fixed point with no chance for divergence (see Section 3), i.e. divergence is not possible. The author’s conjecture is that at least two fixed points on a vertical line \( p = p^* = \text{const.} \) (with the constant equal to zero included) will qualify for a vertical invariant line. But this subject is irrelevant when pursuing divergence.

It remains to formulate a rule for the expansion of Bairstow’s algorithm to avoid divergence of even degree polynomials as given by Theorem II.

**Rule 2.** In the event where a quadratic factor is calculated for an even polynomial and the counter for the number of allowable iterations is exceeded while being bound to \( p = 0 \) or \( p = c = \text{const.} \new \) initial values \( (p_0 \neq 0 \) or \( c \neq 0 \) respectively, \( q_0 \) arbitrary) have to be selected and the iteration restarted. Since it is thinkable that the new trial has some chance of entering the invariant line once again, a set of initial values with larger distance from the invariance line should be prepared for reruns. If polynomial reduction is applied (see Rule 3, Section 3), only \( p = 0 \) may be the cause of divergence.

Some closing remarks regarding both invariance theorems and their invariant straight lines in the \( p, q - \text{plane} \) under Bairstow’s method should be added. Theorem I includes the \( p - \text{axis} \), i.e. \( q = 0 \), as invariant line for the special case of a real root \( x = k = 0 \). For real roots \( x = k \neq 0 \), thus depending on sign and value of \( k \), the whole \( p, q - \text{plane} \) above the area \( q < -p^2/4 \) is covered by the appropriate invariant lines \( q = -kp + k^2 \). These exclude only the \( q - \text{axis} \), i.e. \( p = 0 \), which theoretically would be approached only by an infinite slope corresponding to an infinite root \( k \). But this case, i.e. the inclusion of the \( q - \text{axis} \), is just covered by Theorem II, for invariant line \( p = 0 \). For the invariant lines \( p = \text{const.} \) refer to the statements further above and Theorem III valid for cubics below (Section 4).

We are going back to (33), (34), we add now the missing formula for odd
non algebraic factors can be determined by observing Pascal’s triangle

\[ A_{(n_o)}(p, q) = \sum_{\mu=0}^{(n_o-1)/2} q^\mu \sum_{i=0}^{n_o-(2\mu+1)} \binom{i+\mu}{i} a_{i+2\mu+2} p^i, \quad (41) \]

\( n_o = 3, 5, 7, \ldots \) (odd), the explicit representation of \( A_{(n_o)}(p, q) \) and the so far omitted induction proof:

Polynomial long division of (1) by (2) using \( a_{n+1} = 1 \) yields:

\[ A_{(3)}(p, q) = p^2 + a_3 p + a_2 + q, \quad (42) \]

\[ A_{(4)}(p, q) = p^3 + a_4 p^2 + a_3 p + a_2 + q(2p) + a_4 q, \quad (43) \]

\[ A_{(5)}(p, q) = p^4 + a_5 p^3 + a_4 p^2 + a_3 p + a_2 + q(3p^2 + 2a_5 p) + q^2 + a_4 q, \quad (44) \]

\[ A_{(6)}(p, q) = p^5 + a_6 p^4 + a_5 p^3 + a_4 p^2 + a_3 p + a_2 + q(4p^3 + 3a_6 p^2 + 2a_5 p) + q^2(3p) + a_6 q^2 + a_4 q, \quad (45) \]

\[ A_{(7)}(p, q) = p^6 + a_7 p^5 + \cdots + a_3 p + a_2 + q(5p^4 + 4a_7 p^3 + 3a_6 p^2 + 2a_5 p) + q^2(6p^2 + 3a_7 p) + q^3 + a_6 q^2 + a_4 q, \quad (46) \]

\[ A_{(8)}(p, q) = p^7 + a_8 p^6 + \cdots + a_3 p + a_2 + q(6p^5 + 5a_8 p^4 + 4a_7 p^3 + 3a_6 p^2 + 2a_5 p) + q^2(10p^3 + 6a_8 p^2 + 3a_7 p) + q^3(4p) + a_8 q^3 + a_6 q^2 + a_4 q, \quad (47) \]

\[ A_{(9)}(p, q) = p^8 + a_9 p^7 + \cdots + a_3 p + a_2 + q(7p^6 + 6a_9 p^5 + \cdots + 3a_6 p^2 + 2a_5 p) + q^2(15p^4 + 10a_9 p^3 + 6a_8 p^2 + 3a_7 p) + q^3(10p^2 + 4a_9 p) + q^4 + a_8 q^3 + a_6 q^2 + a_4 q, \quad (48) \]

\[ A_{(10)}(p, q) = p^9 + a_{10} p^8 + \cdots + a_3 p + a_2 + q(8p^7 + 7a_{10} p^6 + \cdots + 3a_6 p^2 + 2a_5 p) + q^2(21p^5 + 15a_{10} p^4 + 10a_9 p^3 + 6a_8 p^2 + 3a_7 p) + q^3(20p^3 + 10a_{10} p^2 + 4a_9 p) + q^4(5p) + a_{10} q^4 + a_8 q^3 + a_6 q^2 + a_4 q. \quad (49) \]

(42) through (49) is transformed into (33), (34) and (41). The indices of the polynomial coefficients and the powers of \( p \) and \( q \) are easily found, and the non algebraic factors can be determined by observing Pascal’s triangle \( \binom{m}{r} \).
where \( m, r = 0, 1, 2, 3, 4, 5, \ldots \). For the validity of (33) it has to be shown by algebraic identical transformation that the induction holds:

\[
A_{(n+1)}(p, q) = a_2 + pA_{(n)}(p, q) \downarrow a_\mu := a_{\mu+1},
\]

\[
+ qA_{(n-1)}(p, q) \downarrow a_\mu := a_{\mu+2},
\]

which follows directly from the double row Horner scheme as derived in (15), (16). In (50) \( A_{(n)}(p, q) \downarrow a_\mu := a_{\mu+1} \) has the meaning that one must use the formula for \( A_{(n)}(p, q) \) but the index of the polynomial coefficients being raised by one (\( \mathrm{ALGOL} \)-sign for \( \mathrm{assignment} \) which is equivalent to \( \mathrm{ZUSE's} \) “\( \mathrm{Ergibt} \)”-sign \( \Leftarrow \)).

Because the algebraic identical transformation is rather elaborate only the final result

\[
A_{(n+1)}(p, q) = a_2 + \sum_{\nu=0}^{n-1} a_{\nu+3}p^{\nu+1} + \sum_{\nu=1}^{[(n-1)/2]} a_{2\nu+3}q^{\nu}p
\]

\[
+ \sum_{\mu=1}^{[(n-2)/2]} \sum_{i=1}^{n-(2\mu+1)} q^\mu \sum_{i=1}^{n-(2\mu+1)} \left( \begin{array}{c} i + \mu \\ i \end{array} \right) a_{i+2\mu+3}p^{i+1} + \sum_{\nu=0}^{n-2} a_{\nu+2}p^\nu q
\]

\[
+ \sum_{\nu=1}^{[(n-2)/2]} a_{2\nu+4}q^{\nu+1} + \sum_{\mu=1}^{[(n-3)/2]} q^\mu + \sum_{i=1}^{n-(2\mu+2)} \sum_{i=1} \left( \begin{array}{c} i + \mu \\ i \end{array} \right) a_{i+2\mu+4}p^i.
\]

(51)

will be given here, which represents formula (33), if \( n \) is replaced by \( n + 1 \), the formulas (34) and (41) being included in (33). This concludes successfully the induction proof required for Theorem II.

For completeness the corresponding explicit representation of the constant remainder \( B_{(n)}(p, q) \) is made available dependent on the polynomial degree \( n \):
\[ n \geq 3. \text{ From (53) for } n = n_e \text{ (even):} \]
\[
B_{(n_e)}(p, q) = a_1 + \sum_{\mu=1}^{n_e/2} q^\mu \sum_{i=0}^{n_e-2\mu} \left( \frac{i + \mu - 1}{i} \right) a_{i+2\mu+1} p^i, \tag{54}
\]

\[ n_e = 4, 6, 8, \ldots, \text{ and if } n = n_o \text{ (odd):} \]
\[
B_{(n_o)}(p, q) = a_1 + \sum_{\mu=1}^{(n_o-1)/2} q^\mu \sum_{i=0}^{n_o-2\mu} \left( \frac{i + \mu - 1}{i} \right) a_{i+2\mu+2} p^i, \tag{55}
\]

\[ n_o = 3, 5, 7, \ldots \text{ hold. By polynomial long division of (1) and (2) using } a_{n+1} = 1 \]
\[ \text{for } n = 3, 4, \ldots, 10 \text{ the constant remainders will be received:} \]
\[
B_{(3)}(p, q) = a_1 + a_3 q + q(p), \tag{56}
\]
\[
B_{(4)}(p, q) = a_1 + a_3 q + q^2 + q(a_4 p + p^2), \tag{57}
\]
\[
B_{(5)}(p, q) = a_1 + a_3 q + a_5 q^2 + q(a_4 p + a_5 p^2 + p^3) + q^2(2p), \tag{58}
\]
\[
B_{(6)}(p, q) = a_1 + a_3 q + a_5 q^2 + q^3 + q(a_4 p + a_5 p^2 + a_6 p^3 + p^4) + q^2(2a_6 p + 3p^2), \tag{59}
\]
\[
B_{(7)}(p, q) = a_1 + a_3 q + a_5 q^2 + a_7 q^3 + q(a_4 p + a_5 p^2 + a_6 p^3 + a_7 p^4 + p^5) + q^2(2a_6 p + 3a_7 p^2 + p^3) + q^3(3p), \tag{60}
\]
\[
B_{(8)}(p, q) = a_1 + a_3 q + a_5 q^2 + a_7 q^3 + q^4 + q(a_4 p + a_5 p^2 + \ldots + a_8 p^5 + p^6) + q^2(2a_6 p + 3a_7 p^2 + 4a_8 p^3 + 5p^4) + q^3(3a_8 p + 6p^2), \tag{61}
\]
\[
B_{(9)}(p, q) = a_1 + a_3 q + \ldots + a_9 q^3 + a_9 q^4 + q(a_4 p + a_5 p^2 + \ldots + a_9 p^6 + p^7) + q^2(2a_6 p + 3a_7 p^2 + 4a_8 p^3 + 5a_9 p^4 + 6p^5) + q^3(3a_8 p + 6a_9 p^2 + 10p^3) + q^4(4p), \tag{62}
\]
\[
B_{(10)}(p, q) = a_1 + a_3 q + \ldots + a_9 q^4 + q^5 + q(a_4 p + a_5 p^2 + \ldots + a_{10} p^7 + p^8) + q^2(2a_6 p + 3a_7 p^2 + 4a_8 p^3 + 5a_9 p^4 + 6a_{10} p^5 + 7p^6) + q^3(3a_8 p + 6a_{10} p^2 + 10p^3) + q^4(4p), \tag{63}
\]
(53) through (55) have been established from the equalities (56) through (63). Again their validity asks for the induction proof. As before it follows from the double row Horner scheme (nomenclature explained above in (50) and following lines):

$$B_{(n+1)}(p, q) = a_1 + q A_{(n)}(p, q) \downarrow a_\nu := a_{\nu+1}.$$  

Applying (53) to (64) yields

$$B_{(n+1)}(p, q) = a_1 + q \sum_{\nu=1}^{n-1} a_{\nu+3} p^{\nu} + \sum_{\nu=1}^{[(n-1)/2]} a_{2\nu+3} q^{\nu+1}$$

$$+ a_3 q + \sum_{\mu=1}^{[(n-2)/2]} q^{\mu+1} \sum_{i=1}^{n-(2\mu+1)} \left( {i + \mu \atop i} \right) a_{i+2\mu+3} p^i$$

$$= a_1 + \sum_{\nu=1}^{[(n+1)/2]} a_{2\nu+1} q^{\nu} + \sum_{\mu=1}^{[n/2]} q^{\mu} \sum_{i=1}^{n+1-2\mu} \left( {i + \mu - 1 \atop i} \right) a_{i+2\mu+1} p^i. \quad (65)$$

Relatively simple and short algebraic identical transformations of (65) lead to (66):

$$B_{(n+1)}(p, q) = \sum_{\nu=0}^{[(n+1)/2]} a_{2\nu+1} q^{\nu} + \sum_{\mu=1}^{[n/2]} q^{\mu} \sum_{i=1}^{n-2\mu} \left( {i + \mu - 1 \atop i} \right) a_{i+2\mu+1} p^i. \quad (66)$$

This concludes the induction proof.

The next Section is dedicated to the reduction of given polynomials (1), how the polynomial coefficient of $x^{n-1}$, i.e. $a_n$, can be forced to zero by transformation and how an algorithm like Bairstow’s iteration may numerically and algebraically benefit from this transformation and how polynomial conditioning will be influenced.

### 3. Considerations on the Reduction of Polynomials to the Form with Vanishing Root Sum ($b_n = 0$)

A polynomial $f(x)$, refer to (1), with the condition of meeting $a_n \neq 0$, which is n-times applied to polynomial long division by the factor $(x + a_n/(a_{n+1}n))$ will
be reduced into a form \( f(y) \), where the polynomial coefficient of \( y^{n-1} \) vanishes. This is best done using synthetic division in a continued or complete Horner Scheme about the point \( x = -a_n / (a_{n+1}) \). Thus we are shifting the axis of polynomial (1) by the substitution

\[
 x = y - a_n / (a_{n+1}) \quad \text{obtaining the new reduced polynomial} \quad (67)
\]

\[
 f(y) = b_{n+1} y^n + 0 y^{n-1} + b_{n-1} y^{n-2} + \cdots + b_2 x + b_1 ,
\]

where \( b_{n+1} = a_{n+1} \) and \( b_n = 0 \). The appropriate complete Horner Scheme reads if \( a_{n+1} = 1 \):

\[
\begin{array}{c|cccccc}
\text{const} & 1 & a_n & a_{n-1} & \ldots & a_2 & a_1 \\
\hline
\text{Faktor} & \frac{a_{n+1}}{n} & \frac{a_n}{n} & \frac{a_{n-1}}{n} & \ldots & \frac{a_2}{n} & \frac{a_1}{n} \\
\hline
x = -\frac{a_n}{n} & 1 & \frac{a_n}{n} & \frac{(n-1)a_n}{n} & \ldots & \frac{a_2}{n} & \frac{a_1}{n} \\
\hline
x = -\frac{a_n}{n} & 1 & \frac{a_n}{n} & \frac{(n-2)a_n}{n} & \ldots & \frac{a_2}{n} & f'(-\frac{a_n}{n}) \\
\hline
x = -\frac{a_n}{n} & 1 & \frac{a_n}{n} & \frac{(n-1)a_n}{n} & \ldots & \frac{a_2}{n} & b_1 = f(-\frac{a_n}{n}) \\
\hline
x = -\frac{a_n}{n} & 1 & \frac{a_n}{n} & \frac{a_{n-1}}{n} = \frac{f(n-2)(-\frac{a_n}{n})}{(n-2)!} \\
\hline
x = -\frac{a_n}{n} & 1 & \frac{a_n}{n} & \frac{a_{n-1}}{n} = \frac{f(n-1)(-\frac{a_n}{n})}{(n-1)!} \\
\hline
\end{array}
\]

This polynomial reduction applied to (1) will mean for the formulas (42) through (49) and (56) through (63) that the coefficient \( b_i = 0 \) and all other coefficients changed to a new value \( b_i \), where \( i = 1, 2, \ldots, n-1, a_{n+1} = b_{n+1} = 1 \). The values \( b_i \) are in the case of \( n = 3 \).

The transformation \( x = y - a_3 / 3 \) changes \( f(x) = x^3 + a_3 x^2 + a_2 x + a_1 \) into the shifted polynomial

\[
 f(y) = y^3 + 0 y^2 + b_2 x + b_1 ,
\]

where

\[
 b_4 = 1, \quad b_3 = 0, \quad \text{and if} \quad a_4 \neq 1 : \quad b_4 = a_4, \quad b_3 = 0, \\
 b_2 = a_2 - \frac{a_2^2}{3}, \quad b_2 = \frac{3 a_2 a_4 - a_2^2}{3 a_4}, \quad b_2 = \frac{3 a_2 a_4 - a_2^2}{3 a_4}, \\
 b_1 = a_1 - \frac{a_2^2}{3} \left[ a_2 - 2 \left( \frac{a_3}{3} \right)^2 \right], \quad b_1 = \frac{a_1 a_4^2 - a_2 a_4 \left( \frac{a_3}{3} \right) + 2 \left( \frac{a_3}{3} \right)^3}{a_4^3}, \quad (69)
\]
In algebraic investigations it is the most natural thing to start from the reduced form of any polynomial. This way the dependence of one supplementary parameter is avoided. The transformation through appropriate axis shifting is the mean to get rid of at least one polynomial coefficient. The roots of the original polynomial $f(x)$ are determined by the simple operation of constant addition/subtraction according to (67).

What is in algebra an obvious necessity, i.e. if possible reducing the number of parameters, has in numerical calculations on the digital computer associated with finite mantissa length floating point operations a correspondence. The rounding error of multiplication with null and summation/subtraction with null are free of round-off errors. For numerical stability alone it will be highly desired to transform a polynomial from a form with no coefficient being zero into one with at least one or more being zero. If the transformation also has an influence on the size of the coefficients to a more moderate number distribution, numerical stability will absolutely benefit from it.

In order to follow up on what has been said above we will look first at Wilkinson’s famous example [8], [9], referenced as well in Matlab and to be found when entering `help poly`:

$$f(x) = (x + 1)(x + 2)(x + 3) \ldots (x + 20).$$

(70)

$f(x) = x^{20} + 210x^{19} + 20615x^{18} + 1256850x^{17} + 53327946x^{16} + 1672280820x^{15} + 40171771630x^{14} + 11310278995381x^{13} + 3113364316139660x^{12} + 6035851865511450x^{11} + 8037811822645052400x^{10} + 359997517947607000x^9 + 8037811822645052400x^8 + 3113364316139660x^7 + 1206647803780373200x^6 + 359997517947607000x^5 + 8037811822645052400x^4$
\{the coeffic. of (70) calculated by poly(-1:-1:-20)\}.

The transformation of \( f(x) \) to \( f(y) \), where

\[
x = y - \frac{a_n}{n} = y - \frac{n(n + 1)}{2n} = y - \frac{n + 1}{2}, \quad \text{for } n = 20 : \quad x = y - 10.5,
\]

could be accomplished by a self-generated Matlab function with a function name shortened from “polynomial reduction”. In this case here we have followed the path that avoids rounding error considerations of such a Matlab function that has to calculate zero valued coefficients:

\[
n = 2 : \quad f_2(x) = (x + 1)(x + 2) = x^2 + 3x + 2 \quad \leftarrow \text{(means calculated by:) poly(-1:-1:2)},
\]

\[
f_2(y) = (y + 0.5)(y - 0.5) = y^2 + 0y - 0.25 \quad \leftarrow \text{poly(-0.5:1:0.5), where } x = y - 1.5.
\]

Thus for \( n = 2, 4, \ldots \) (even):

\[
f_n(x) = (x + 1)(x + 2) \ldots (x + n) \quad \leftarrow \text{poly(-1:-1:-n)},
\]

\[
f_n(y) = (y + 0.5)(y - 0.5)(y + 1.5)(y - 1.5) \ldots \quad (y + (n/2 - 0.5))(y - (n/2 - 0.5)) =
\]

\[
\leftarrow \text{poly}(-(n/2 - 0.5) : 1 : (n/2 - 0.5)), \quad \text{where } x = y - (n + 1)/2. \quad (71)
\]

Using \( n = 20 \) yields:\n\[
px20 = \text{poly}(-1:-1:-20); \quad py20 = \text{poly}(-9.5:1:9.5); \quad x = y - 10.5, \quad (72)
\]

with the following Matlab results \( py20 = f(y) \) and the determination of the roots of \( px20, py20 \):

\[
f(y) = 1y^{20} + 0y^{19} - 333y^{18} + 0y^{17} + 45361y^{16} + 0y^{15} - 3297224y^{14} + 0y^{13}
\]

\[
+ 138891340y^{12} + 0y^{11} - 3458636995y^{10} + 0y^9 + 49853702915y^8 + 0y^7
\]

\[
- 390413791459y^6 + 0y^5 + 1459784595776y^4 + 0y^3 - 1976557769122y^2 + 0y^1
\]

\[
+ 408811723376, \quad (73)
\]

\[
\text{roots(poly(-1:-1:-20)) \quad roots(py20) \quad roots(py20)-10.5} \quad (74)
\]
To check the calculation of $a_1$ to be 20 factorial in Matlab:

```matlab
a1=1; for i=1:20,a1=a1*i;end,a1
```
yields $a_1 = 20! = 2432902008176640000$

and the root sum according to Gauss

$$a_n = \frac{n(n+1)}{2}, \text{ for } n = 20: \quad a_{20} = 10 \times 21 = 210.$$
\[ f_{(n)}(y) = y(y - 1)(y + 1)(y - 2)(y + 2) \cdots (y - (n - 1)/2)(y + (n - 1)/2) = \]
\[ \leftarrow \text{poly}(-(n - 1)/2 : 1 : (n - 1)/2) , \text{ where } x = y - (n + 1)/2 . \]  

Wilkinson has denoted (70) as “a polynomial of moderate degree with roots which certainly not be regarded pathologically close”, but it is regardless an ill-conditioned polynomial. Just because in (74) column 1 the roots of (70) cannot be determined to an accuracy of sometimes more than 3 out of 16 decimal digits. In other words his observation made is such that although the multiple root case with well known numerical instability effects are not present for this polynomial (70), the accuracy of the roots could only be improved by using a floating point arithmetic of a mantissa length well increased from the 52 bit in effect by IEEE floating point double precision arithmetic on the PC. By comparing (74) column 1 with column 2 and 3, where the roots are determined from a polynomial created by axis shifting or Taylor series expansion about an appropriate point, it is clear that for numerical stability alone the root finder method, no matter which one, should be applied to the reduced form instead of the original polynomial. In the case of equidistant roots, no matter whether the roots are of type integer or real, polynomial reduction as considered here will lead to polynomials with every second polynomial coefficient being zero. This means a lot of rounding error free arithmetic operations when applying the reduced polynomial to Bairstow’s method. Beside of this benefit in regard of rounding errors it has to be pointed out that the reduced polynomial has a much more favourable polynomial coefficient number range. This is being well confirmed by the comparison of the results for the roots as presented by (74). It is concluded that polynomial reduction applied to (70) has transformed an ill-conditioned polynomial into a well-conditioned one.

Here is still another reason for promoting polynomial reduction and not to do without it. We look at the polynomial of degree \( n \) having a \( n \)-fold root, therefore
\[ f(x) = (x + a)^n = 0 . \]  

Applying polynomial reduction, where
\[ x = y - a , \]  

yields
\[ f(y) = y^n + 0y^{n-1} + 0y^{n-2} + \cdots + 0y^2 + 0y^1 + 0 = 0 . \]  

This case of the \( n \)-fold root produces while transforming the original polynomial already \( n \)-times a root \( y = 0 \) and applied to (77) \( n \)-times the root \( x = -a \).
Polynomials such as (76) will not at all be applied to Bairstow’s method, if we keep this course of action and do polynomial reduction first. As far as Bairstow’s iterative process is concerned, there will be only an academic interest in the question of what Bairstow’s algorithm properties would be in comparison to the distinct root case and what convergence rate would be in the \( n \)-fold root case. This will be looked at in the Section 5 on convergence and its rate.

For this Section 3 the following rule for the expansion of Bairstow’s algorithm to improve numerical stability, i.e. minimizing rounding errors, and other reasons is being formulated:

**Rule 3.** It is highly recommended to do polynomial reduction first. This way Bairstow’s method would be applied only to polynomials with at least one polynomial coefficient being zero. Such action assures improved numerical stability. If the polynomial represents the characteristic equation of a dynamic system whose eigenvalues are to be determined, we would in most cases expect all roots to be located in the left half plane of the complex number plane. Especially in these cases will polynomial reduction provide a distribution of roots to the left and right hand of the imaginary axis leading to a more moderate, i.e. smaller number range of the polynomial coefficients. This effect helps numerical stability as well. In regard to Theorem II it has to be stated that only through polynomial reduction it may be discovered that Theorem II’s terms are valid and divergence for invariant line \( p = 0 \) may apply. Refer to the statement on the divergence example of [3] presented in Section 5, right after presenting Theorem Ve.

4. **Bairstow’s Method and Cubics: Invariant Lines Mapping into a Fixed Point in one Step**

The following theorem shows from which points of the \( p,q - plane \) one can succeed to reach a problem solution or fixed point \((p^*,q^*)\) under Bairstow’s method in a single iterative step. Bairstow’s iterative technique has also this property (the closed-form formula or single step theorem).

**Theorem III.** If the polynomial

\[
f(x) = x^3 + a_3x^2 + a_2x + a_1
\]

(79)

has three distinct real roots \( x = k_i \), where \( i = 1, 2, 3 \), and the sum of root pairs are therefore \( p_i^* = k_1 + k_2 \), \( p_2^* = k_1 + k_3 \), \( p_3^* = k_2 + k_3 \) or for the case where \( a_3 = k_1 + k_2 + k_3 = 0 \) the sum of root pairs yield \( p_i^* = -k_i \), \( i = 1, 2, 3 \) it will be
valid for every $p^* = p_i^* = \text{const.} \ (i = 1, 2, 3)$ and any initial value

$$-\infty < q(p^*) < +\infty,$$

that the fixed point $p^*$, $q^*$ is obtained in a single iterative step, i.e. in one step like with a closed-form formula.

There exist only a reduced number of vertical straight lines for all other root distributions $p^* = p_i^* = \text{const.} \ (i = 1 \text{ oder } i = 1, 2)$ for which this property is valid.

For the proof use (42) and (56) and assume the reduced form of the polynomial, thus $a_3 = 0$,

$$A_{(3)}(p, q; a_3 = 0) = p^2 + a_2 + q, \ A_p = 2p, \ A_q = 1, \quad (81)$$

$$B_{(3)}(p, q; a_3 = 0) = a_1 + qp, \ B_p = q, \ B_q = p. \quad (82)$$

Using (12) and (81),(82) reveals

$$J = 2p^2 - q, \ D_1 = -(p^3 + a_2p - a_1), \ D_2 = q^2 - (p^2 - a_2)q - 2a_1p. \quad (83)$$

$D_1$ yields using $p_i^* = -k_i, \ i = 1, 2, 3$,

$$D_1 = -(p^3 + a_2p - a_1) = -(p - p_1^*)(p - p_2^*)(p - p_3^*)$$

$$= -(p + k_1)(p + k_2)(p + k_3), \quad (84)$$

therefore, since $p_i^* = -k_i, \ i = 1, 2, 3$:

$$\Delta p = p_{\nu+1} - p_{\nu} = \frac{D_1}{J} = 0. \quad (85)$$

It remains to prove that for all $-\infty < q(p^*) < +\infty$ the value $q^*$ will be calculated in a single iteration. Assume $\nu = 0, \ p = p^* = p_i, \ i = 1, 2, 3$ und $q = q(p_i)$:

$$\Delta q = q_{\nu+1} - q_{\nu} = \frac{D_2}{J} = \frac{q^2 - (p_i^2 - a_2)q - 2a_1p_i}{q - 2p_i^2} = - \left[ q + (p_i^2 + a_2) \right], \quad (86)$$

because polynomial long division yields:

$$\begin{array}{c|c|c|c}
| \hline
q^2 & -(p_i^2 - a_2)q & -2a_1p_i & (q - 2p_i^2) = q + (p_i^2 + a_2) \\
\hline
0 & (p_i^2 + a_2)q & & \\
- & (p_i^2 + a_2)q & -2p_i(p_i^2 + a_2) & \\
- & 0 & 2p_i(p_i^2 + a_2p_i - a_1) & 0 \\
\hline
\end{array}$$
\[(\Delta q)_0 = q_1 - q_0 = -q_0 - ((p_i^*)^2 + a_2), \quad (87)\]
\[q_1 = -(p_i^*)^2 - a_2. \quad (88)\]

The right hand side of (88) is identical to \(q^* = q^*(p_i^*)\) which is determined from (81) for \(A = 0:\)
\[q_1 = -(p_i^*)^2 - a_2 \equiv q^*(p_i^*). \quad (89)\]

If this has not been observed, the second iteration yields
\[(\Delta q)_1 = q_2 - q_1 = -q_1 - ((p_i^*)^2 + a_2) = 0, \quad (90)\]
thus \(q_2 = q_1 = q^*\) (fixed point!), which concludes the proof.

In calculations with finite mantissa length on the PC it will take occasionally more than one iterative step. This is for example the case, if this mantissa length has not a sufficient number of bits to take all digits required, such as \(q_0 = +1.e + 20\) and \(q^* = -4\). One should have reached in one step \(q^*\) with \(\Delta q = -1.e + 20 - 4\), but the subtrahend 4 is outside the 16 decimal digits that the computer arithmetic can take into account. Consequently the first iteration calculates only \((\Delta q)_0 = q_1 - q_0 = -1.e + 20\) without inclusion of -4 yielding \(q_1 = q_0 + (\Delta q)_0 = 0 \neq q^*\). In the now necessary second iteration step one will start from \(q_1 = 0\) and calculate \((\Delta q)_1 = q_2 - q_1 = -4\) and will naturally find the correct answer \(q_2 = 0 - 4 = -4 = q^*\) at the cost of a supplementary iteration. This is another example where the scientist could be misled by looking at computer results taking them at first glance for real (!) thinking Bairstow’s method needs two iterations but it is the sole influence of computer arithmetic finite mantissa length.

5. Convergence Rate (Quadratic and Linear), Divergence under Bairstow’s Iterative Algorithm

Knowing the general convergence condition for the conventional Newton method for functions of a single variable the comparable convergence conditions for Bairstow’s method (13) are:
\[
\left| \frac{\partial \varphi_1 (p^*, q^*)}{\partial p} \right| < 1, \quad \left| \frac{\partial \varphi_2 (p^*, q^*)}{\partial q} \right| < 1, \quad (91)
\]
where \(\varphi_1 = p + \frac{d_1(p,q)}{J(p,q)}\); \(\varphi_2 = q + \frac{d_2(p,q)}{J(p,q)}\).
Using the notation \( \overline{\varphi} = \varphi_1(p^*, q^*), \ldots \), where \( (p^*, q^*) \) any fixed point under Bairstow’s method, and consequently \( \overline{A} = \overline{B} = \overline{D}_1 = \overline{D}_2 = 0 \), yields:

\[
\begin{align*}
\frac{\partial \overline{\varphi}_1}{\partial p} &= \frac{\overline{B} \overline{A}_{pq} - \overline{A} \overline{B}_{pq}}{\overline{J}} - \frac{\overline{J}_p \overline{D}_1}{\overline{J}^2} = 0, \\
\frac{\partial \overline{\varphi}_2}{\partial q} &= \frac{\overline{A} \overline{B}_{pq} - \overline{B} \overline{A}_{pq}}{\overline{J}} - \frac{\overline{J}_q \overline{D}_2}{\overline{J}^2} = 0,
\end{align*}
\]

if \( \overline{J} \neq 0 \). (92)

Nonsingularity of the Jacobian matrix is valid at each existing fixed point \( (p^*, q^*) \) if and only if all roots are distinct. In this case it is to be learned from (92) that all fixed points are reached in the favourable quadratic convergence rate, because the convergence factor is zero. This is meant in the sense of local convergence.

From (13), (14) and Lagrange’s Mean Value Theorem follows when taking into account the case where the convergence factor becomes zero:

\[
\begin{align*}
\left| \frac{\partial \varphi_1(p^*_i, q^*_i)}{\partial p^r} \right| &= \lim_{\nu \to \infty} \left| \frac{p_{\nu+1} - p^*_i}{(p_{\nu} - p^*_i)^r} \right| \leq c, \quad 0 < c < 1, \\
\left| \frac{\partial \varphi_2(p^*_i, q^*_i)}{\partial q^r} \right| &= \lim_{\nu \to \infty} \left| \frac{q_{\nu+1} - q^*_i}{(q_{\nu} - q^*_i)^r} \right| \leq d, \quad 0 < d < 1,
\end{align*}
\]

(93)

where \( i = i \)-th is fixed point, \( \nu = \nu \)-th iteration, and \( r \) being the convergence velocity or convergence order, which is the velocity with which the members of the series \( p_{\nu} \) and \( q_{\nu} \) are approaching the fixed point or the limit values \( p^* \) and \( q^* \). The power of the \( r \)-th convergence order represents for \( r = 1 \) linear convergence and for \( r = 2 \) quadratic convergence. Quadratic convergence will mean for IEEE double precision floating point arithmetic on the PC that, after reaching the first correct decimal digit of a fixed point, each succeeding iteration is doubling the correct number of digits and thus a total of 5 iterations are needed in order to meet the 16 decimal digits of the computer’s complete mantissa length. Speaking of linear convergence this number will be increased to approximately 48 iterations, which means about 3 iterations only for each of the 16 decimal digits. Although Bairstow’s method is presented in almost any standard textbook on calculus and numerical analysis the question on convergence, its rate and nonconvergence is left open. Same holds true with terms and circumstances of linear convergence rate. In order to shed light on how quadratic convergence may be lost to the unfavourable case of linear convergence or even to divergence from a polynomial with only distinct roots just by changing the root distribution the following investigation is pursued. Starting from the invariant lines of the two Theorem I and Theorem II and the lines
$p^* = \text{const.}$ containing more than 1 fixed point are a mean allowing by skilful
use of the number zero to generate closed-form formulas of Bairstow's method
for all invariant lines. Algebraic closed-form formulas permit at first glance to
judge on convergence rate linear or quadratic and divergence as well. Since, in
its last stage, Bairstow's method has to go in any application through solving
a cubic for odd polynomials, or through a quartic for even polynomial degree,
Theorem IV will handle convergence rate for the cubics and Theorem V for the
quartics in regard to invariant lines. Both theorems will be divided in parts,
each part covering a root distribution of special interest. This action is sort of
a strategic plan of doing algorithmic validation and verification of the method
under investigation.

For cubics exist four different cases that depend on kind of real or complex
root and multiplicity of real roots which require the partition of Theorem IV into
the parts a to d. Convergence order is presented for all existing invariant lines
stemming from Theorem I, Theorem II is not applicable because it is concerned
with even polynomial degree, Theorem III's "one step invariant line(s)" is (are),
of course, valid.

**Theorem IVa.** The cubics $f(x) = x^3 + a_2 x + a_1$ with three distinct real roots
are having three invariant lines under Theorem I, $q = -k_i p + k_i^2$, where $k_i = -p_i^*$
($i = 1, 2, 3$), with two fixed points $[p_i^*, q_i^*]$, $[p_j^*, q_j^*]$ ($i \neq j$ and $i, j = 1, 2, 3$) for
each line on which global convergence is quadratic. The functions $D_1(p, q) = 0$
and $D_2(p, q) = 0$ (12) intersect at the three fixed points $p^*$, $q^*$ orthogonal, $D_1$
with infinite slope, $D_2$ with zero slope. The Jacobian determinant is nonzero
at all three fixed points.

For the proof use (12), (81) to (83) and the polynomial

$$f(x) = (x + a)(x - a)x = x(x^2 - a^2) = x^3 + 0x^2 - a^2 x + 0 = 0,$$

where $a_1 = 0$, $a_2 = -a^2$, $a_3 = 0$, $k_1 = 0 = p_1^*$, $k_2 = +a = -p_2^*$, $k_3 = -a = -p_3^*$,

$$\Delta p = p_{\nu+1} - p_\nu = \frac{D_1(a_1 = 0, a_2 = -a^2)}{J} = \frac{-p(p + a)(p - a)}{2p^2 - q},$$

$$\Delta q = q_{\nu+1} - q_\nu = \frac{D_2(a_1 = 0, a_2 = -a^2)}{J} = \frac{q(q - p^2 - a^2)}{2p^2 - q}.$$  

Entering the three invariant lines one by one into (95), (96) will result in closed-
form formulas for Bairstow's iterative process valid for those straight lines:
1. **Straight Line.** $q(k = 0) = 0$ as entry area into the two fixed points \[ p_{1,2}^* = \pm a, \quad q_{1,2}^* = 0 \]

\[
\Delta p(q = 0) = \frac{-p(p + a)(p - a)}{2p^2} = \frac{-(p + a)(p - a)}{2p} = \frac{-p^2 - a^2}{2p}. \quad (97)
\]

The *quadratic convergence* or convergence velocity $r = 2$ is viewed, if we form the derivative

\[
\frac{d\Delta p(q = 0)}{dp} = -\frac{p^2 + a^2}{2p^2}. \quad (98)
\]

As expected, (96) yields

\[
\Delta q(q = 0) = 0 = 0. \quad (99)
\]

For a trial: $-\frac{\Delta q}{\Delta p} = \frac{0}{\Delta p} = 0 = k$ being the slope of the invariant line under consideration.

2./3. **Straight Line.** $q(k = \mp a) = \pm ap + a^2 = \pm a(p \pm a)$ as entry area into the two fixed points \[ p^* = \mp a, \quad q^* = 0 \]; \[ p^* = 0, \quad q^* = a^2 \], where upper sign for 2. line, lower sign 3. line.

\[
\Delta p(q = \mp ap + a^2) = -\frac{p(p + a)(p - a)}{2(p \mp a)(p \pm a/2)} = \frac{-p(p \pm a)}{2(p \mp a/2)}. \quad (100)
\]

Because the denominator keeps a factor that does not cancel out the *quadratic convergence* is kept. Same for $\Delta q$:

\[
\Delta q(q = \pm ap + a^2) = -\frac{ap(p + a)(p - a)}{2(p \mp a)(p \pm a/2)} = \frac{-ap(p \pm a)}{2(p \mp a/2)} = \pm a\Delta p
\]

\[ (q = \pm ap + a^2). \quad (101) \]

For a trial: $-\frac{\Delta q}{\Delta p} = \mp a\frac{\Delta p}{\Delta p} = \mp a = k$ being the slope of the invariant line under consideration

**Orthogonality proof of** $D_1(p^*, q^*), \ D_2(p^*, q^*)$:

It is already known from (83), (84) that the vertical lines $p^* = p^*_i =$const., $(i = 1, 2, 3)$ coincide with $D_1(p^*, q^*) = 0$. Therefore it is only to show that $D_2(p^*, q^*) = 0$ has an extreme value in each of the three fixed points. Differentiating $D_2 = q^2 - (p^2 - a_2)q - 2a_1p$ yields

\[
2qq' - (p^2 - a_2)q' - 2pq - 2a_1 = 0 \quad \Rightarrow \quad q' = \frac{2pq + 2a_1}{2q - p^2 + a_2}. \quad (102)
\]
\[ q'(p = p^*, q = q^*) = -\frac{a_1}{p^*} = \frac{2p^* q^* + 2a_1}{2q^* - p^* + a_2} = p^* + \frac{p^* 3 - a_2 p^* + 2a_1}{2a_1/p^* - p^* + a_2} = p^* - \frac{p^* 3 - a_2 p^* + 2a_1}{p^* 3 - a_2 p^* + 2a_1} = 0. \]

**Theorem IVb.** The cubics \( f(x) = x^3 + a_2 x + a_1 \) with a double real root \( x_{1,2} = k_d = -\frac{a_1}{2a_2} \) and a supplementary single root \( x_3 = k_s = -2k_d = +\frac{a_1}{a_2} \) are having two invariant lines under Theorem I (103) and (104),

\[ q(k_d) = -k dp + k_d, \text{ with two fixed points} \quad (103) \]

\[ \left[ p_1^* = -3 \frac{a_1}{a_2}; q_1^* = -\frac{9}{4} \left( \frac{a_1}{a_2} \right)^2 \right] \text{ and } \left[ p_2^* = \frac{3}{2} \frac{a_1}{a_2}; q_2^* = \frac{9}{4} \left( \frac{a_1}{a_2} \right)^2 \right], \]

on which global convergence is quadratic. Only the approach on invariant line (103) into the fixed point \( p_2^*, q_2^* \) will terminate the series of \( p_\nu \) and \( q_\nu \), with an undefined numerical result for \( \Delta p \) and \( \Delta q = 0/0 \), because of \( J(p_2^*, q_2^*) = 0 \).

\[ q(k_c) = -k_c p + k_c^2, \text{ with a single fixed point} \quad (104) \]

\[ [p^* = p_2^*; q^* = q_2^*] \text{ on which global convergence is linear. The approach on invariant line (104) into the single fixed point } p_2^*, q_2^* \text{ will terminate the series of } p_\nu \text{ and } q_\nu \text{ with an undefined numerical result for } \Delta p \text{ and } \Delta q = 0/0, \text{ because of } J(p_2^*, q_2^*) = 0. \]

The functions \( D_1(p, q) = 0 \) and \( D_2(p, q) = 0 \) intersect orthogonal only in the fixed point \( p_1^*, q_1^*, D_1 = 0 \) with infinite slope and \( D_2 = 0 \) with zero slope.

For the proof of the convergence order of invariant line (103), replace \( q \) in (83) and (85) respectively as given by (103):

\[ \Delta p(q(k_d)) = -\frac{(p - \frac{3 a_1}{2 a_2})^2 (p + \frac{3 a_1}{a_2})}{2 \left[ p^2 - \frac{3 a_1}{4 a_2} p - \frac{9}{8} \left( \frac{a_1}{a_2} \right)^2 \right]} = -\frac{(p - \frac{3 a_1}{2 a_2}) (p - \frac{3 a_1}{2 a_2}) (p + \frac{3 a_1}{a_2})}{2 (p - \frac{3 a_1}{2 a_2}) (p + \frac{3 a_1}{a_2})}, \]

\[ \Delta p(q(k_d)) = \frac{-(p - \frac{3 a_1}{2 a_2}) (p + \frac{3 a_1}{a_2})}{2 (p + \frac{3 a_1}{a_2})}, \quad (105) \]

\[ \Delta q(q(k_d)) = \frac{3 a_1}{2 a_2} \Delta p(q(k_d)), \quad (106) \]

i.e. the denominator of (105) has a factor that cannot be cancelled out, therefore global convergence on entry area (103) into two fixed points (the zeros of (105)!) is **quadratic.**
For the proof of the convergence order of invariant line (104), replace \( q \) in (83) and (85) respectively as given by (104):

\[
\Delta p(q(k_e)) = \frac{-(p - \frac{3a_1}{2a_2})^2 (p + 3\frac{a_1}{a_2})}{2 \left[ p^2 + 3\frac{a_1}{2a_2} p - \frac{9}{4} \left( \frac{a_1}{a_2} \right)^2 \right]} = \frac{-(p - \frac{3a_1}{2a_2})(p - \frac{3a_1}{2a_2})(p + 3\frac{a_1}{a_2})}{2(p - \frac{3a_1}{2a_2})(p + 3\frac{a_1}{a_2})},
\]

\[
\Delta p(q(k_e)) = -\frac{1}{2} (p - \frac{3a_1}{2a_2}), \quad (107)
\]

\[
\Delta q(q(k_e)) = -3\frac{a_1}{a_2} \Delta p(q(k_e)). \quad (108)
\]

Since the denominator, the Jacobian determinant, cancels completely out against the numerator of (107), thus the global convergence on entry area (104) into the single fixed point (the zero of (107) \( p^*_2 \)) is linear.

The proof of \( J(p^*_2, q^*_2) = 0 \) is immediately achieved from (83), omitted for simplicity reasons.

For the proof of the orthogonality of \( D_1(p^*_1, q^*_1) \) and \( D_2(p^*_1, q^*_1) \) we commence from (102) and have to prove only that the derivative of \( q \) will vanish:

\[
q'(p^*_1) = p^*; q^*_1 = -p^*/4 = \frac{2p^*(-p^*/4) + 2a_1}{2(-p^*/4) - p^2 + a_2} = -\frac{p^*3 + 4a_1}{p^*2 - 2a_2}.
\]

Replacing \( p^* = p^*_1 \) by \(-3a_1/a_2 \) and from elementary calculations is obtained:

\[
q' = \frac{a_1}{a_2} \frac{27a_1^2 + 4a_3^2}{9a_1^2 - 2a_2} = 0,
\]

because the discriminant for the double root of the cubic with \( a_3 = 0 \) reads:

\[
d = 27a_1^2 + 4a_3^2 = 0
\]

which concludes the proof.

Numerically speaking, the fixed point \( p^*_2, q^*_2 \) of Theorem IVb is so far the most interesting one. The fixed point \( p^*_2, q^*_2 \) is limit of the series \( p_\nu \) and \( q_\nu \) for a total of three different invariant lines, each direction having a different convergence rate from single step, quadratic to linear, which is sort of a threefold directionally dependent convergence order:

1. invariant line \( q(k_d) = -k_dp + k_d^2 \) that approaches \( p^*_2, q^*_2 \) with quadratic convergence order
2. invariant line \( q(k_e) = -k_ep + k_e^2 \) that approaches \( p^*_2, q^*_2 \) with linear convergence order
3. from the vertical line \( p^*_2 \) that approaches \( p^*_2, q^*_2 \) in a single step (Theorem III, Section 4).
The vanishing Jacobian determinant in the considered fixed point appears not to be a problem, but loosing quadratic convergence to the unfavourable linear convergence in the last stage of the iterative process by switching from one invariant line through rounding errors to another one with negative effects on a finally achieved accuracy and largely increased count on used iterations. This is visualized in Figure 2 using the polynomial \( f(x) = (x + 1)^2(x - 2) \) and initial values \( p_0 = 4 \) and \( q_0 = 5 \) on invariant line \( q(k = -1) = p + 1 \). Six iterations take place with quadratic convergence on this invariant line. From the seventh iteration on Bairstow’s method operates with linear convergence on invariant line \( q(k = +2) = -2p + 4 \). The switch is only explainable through rounding errors when coming close to \( J = 0 \). On the 25-th iteration accuracy is less than the accuracy acquired with the first 6 iterations with quadratic convergence (!).

Boyd’s paper [2] states in his introduction about Bairstow’s method “...because of quadratic convergence in case of simple roots or real roots of multiplicity two,...” which obviously has overlooked the existence of the invariant line through one of the two existing fixed points of cubics where convergence is
linear!

Conclusion: Cubics with roots of multiplicity two yield a partial loss of quadratic convergence to the unfavourable linear convergence and a vanishing Jacobian determinant in one out of two existing fixed points are worsening numerically proper termination of the iterative process. This could be further investigated.

**Theorem IVc.** The cubics \( f(x) = x^3 + a_2 x + a_1 \) with a complex conjugate root and one real root \( x = k \) are having just one invariant line under Theorem I, \( q = -kp + k^2 \), (109) but this invariant line is lacking any fixed point \([p^*; q^*]\). The closed-form formula for \( \Delta p = p_{n+1} - p_n \) and \( \Delta q = q_{n+1} - q_n \) on the invariant line (109) have no real zeroes and therefore the iterative process is searching for a not existing limit \([p^*; q^*]\) and this with formal “quadratic convergence order” and can truly only diverge.

For the proof consider the polynomial with the roots: \( x_1 = -\frac{k}{2} + ai \), \( x_2 = -\frac{k}{2} - ai \), \( x_3 = k \ (i = \sqrt{-1}) \) and using (109) and (83):

\[
\Delta p = D_1 \frac{J}{J} = \frac{(p - \frac{k}{2} + ai)(p - \frac{k}{2} - ai)(p + k)}{2 \left(p^2 + \frac{k^2}{4}p - \frac{k^2}{2}\right)} = -\frac{(p^2 - kp + \frac{k^2}{4} + a^2) [p + k]}{2 \left[p + k\right] (p - \frac{k}{2})},
\]

\[
\Delta p(q = -kp + k^2) = -\left(\frac{p^2 - kp + \frac{k^2}{4} + a^2}{2 \left(p - \frac{k}{2}\right)}\right),
\]

\[
\Delta q(q = -kp + k^2) = -k\Delta p(q = -kp + k^2).
\]

Choosing \( k = 0 \) generates the most simple closed-form formula for divergence of Bairstow’s method on invariant line (109) \( q = 0 \):

\[
\Delta p(q = -kp + k^2 = 0) = -\left(\frac{p^2 + a^2}{2p}\right),
\]

\[
\Delta q(q = -kp + k^2 = 0) = -k\Delta p(q = -kp + k^2) = 0.
\]

The “convergence” order or velocity remains formally at \( r=2 \) as shown by the derivative

\[
\frac{d\Delta p(q = 0)}{dp} = -\frac{p^2 - a^2}{2p^2} = -\frac{(p + a)(p - a)}{2p^2},
\]

(112)
providing two extreme values, a minimum for positive \( \Delta p \) and a maximum for negative \( \Delta p \), i.e. the slope being zero at \( p = \pm a \) and \( \Delta p(q = 0; p = \pm a) = \mp a \). No absolute smaller value than \( a, a > 0 \), can be obtained under Bairstow’s method. The reason being that (111) has in the numerator only complex conjugate roots and no real zero.

As indicated in Section 3 there remains an academic interest to investigate the case of real roots of multiplicity three.

**Theorem IVd.** The cubics \( f(x) = x^3 + 0x^2 + 0x + 0 \) with a triple real root \( x_1 = x_2 = x_3 = k = 0 \) are having a single entry area or invariant line under Theorem I \( q = 0 \) with a single fixed point \( [p^* = 0; q^* = 0] \) on which global convergence is linear. When entering the only fixed point, iterations will be terminated with an undefined expression \( 0/0 \) for \( \Delta p \) and \( \Delta q \) because of a vanishing Jacobian determinant \( J(p^*, q^*) = 0 \).

For the proof calculate from (83):

\[
\Delta p(q = 0, a_1 = a_2 = a_3 = 0) = \frac{D_1}{J} = \frac{-p^3}{2p^2} = -\frac{1}{2}p, \tag{113}
\]

\[
\Delta q(q = 0, a_1 = a_2 = a_3 = 0) = \frac{D_2}{J} = \frac{0}{2p^2} = 0,
\]

yielding linear convergence on the invariant line and \( J(p^* = 0, q^* = 0) = 2p^2 - q = 0 \).

From here on it remains to look at convergence velocity on entry areas for even polynomials of degree \( n = 4 \). This will mean that the invariant lines stemming from Theorem I have to be extended by those valid under Theorem II. Since we do not expect anything different than quadratic convergence for the distinct root cases, the multiple root distributions and divergence will only be evaluated. This takes a division of Theorem V into a total of five parts a through e.

**Theorem Va.** Given any quartic \( f(x) = x^4 + a_4x^3 + a_3x^2 + a_2x + a_1 \) with a double real root and two distinct real roots is having three entry areas under Theorem I,

\[
q = -k_ip + k_i^2 \quad (i = 1, 2, 3),
\]

two fixed points \( [p_i^*, q_i^*], [p_j^*, q_j^*] \) \((i \neq j \text{ and } i, j = 1, 2, 3)\), where the convergence is quadratic. If \( a_4 = a_2 = 0 \) a forth entry area, \( p = 0 \), is given under Theorem II which contains two fixed points and having quadratic convergence as well. Numerically speaking, the double root is only worsening the situation that
there are fixed points with vanishing Jacobian determinant, no degradation to linear convergence rate takes place.

For the proof consider the quartic \( f(x) = x^2(x^2+a_3) = x^4+a_3x^2+0x+0 = 0 \) with the roots \( x_1 = x_2 = 0 \) and \( x_{3,4} = \pm \sqrt{-a_3} \), where \( a_3 < 0 \) for \( x_{3,4} \) to be real. Again, the skilful use of the number zero shall help to generate a closed-form formula of Bairstow’s method. Using (43) and (57) with \( a_4 = 0 \) into (12) yields:

\[
J(p, q; a_4 = 0) = 4 \left( q^2 + (a_3 + p^2)q + \frac{3}{4}p^4 + a_3p^2 + \left( \frac{a_3}{2} \right)^2 \right), \tag{114}
\]

\[
D_1(p, q; a_4 = 0) = -2p q^2 - 2 \left( p^3 + a_3p + a_2 \right) q - p \left( p^4 + 2a_3p^2 + a_2p + a_2^2 - 2a_1 \right) - a_2a_3. \tag{115}
\]

The entry area \( q(k = 0) = 0 \) and \( a_1 = a_2 = a_4 = 0 \) put in (114) and (115) gives:

\[
J(p; q = 0, a_1 = a_2 = a_4 = 0) = 4 \left( \frac{3}{4}p^4 + a_3p^2 + \left( \frac{a_3}{2} \right)^2 \right) = 3 \left( p + \sqrt{-\frac{a_3}{3}} \right) \left( p - \sqrt{-\frac{a_3}{3}} \right) \left( p + \sqrt{-a_3} \right) \left( p - \sqrt{-a_3} \right), \tag{116}
\]

\[
D_1(p; q = 0, a_1 = a_2 = a_4 = 0) = -p \left( p^4 + 2a_3p^2 + a_3^2 \right) = -p \left( \left( p + \sqrt{-a_3} \right)^2 \left( p - \sqrt{-a_3} \right)^2 \right), \tag{117}
\]

yielding the closed-form formulas of the iterative process:

\[
\Delta p(p; q(k = 0) = 0) = \frac{D_1}{J} = \frac{p \left( p + \sqrt{-a_3} \right) \left( p - \sqrt{-a_3} \right)}{3 \left( p + \sqrt{-\frac{a_3}{3}} \right) \left( p - \sqrt{-\frac{a_3}{3}} \right)}, \tag{118}
\]

\[
\Delta q(p; q(k = 0) = 0) = -k \Delta p(p; q(k = 0) = 0) = 0, \tag{119}
\]

i.e. quadratic convergence on invariant line \( q = 0 \), no cancellation of the denominator’s factors is taking place to reduce the convergence order. The Jacobian determinant (116) vanishes at the two fixed points \( (p^* = \pm \sqrt{-a_3}, q^* = 0) \) without much numerical impact as can be seen from (118) and the cancellation. Boyd’s statement \([2]\) on double root influence and quadratic convergence holds true in this case for quartics, but not cubics, as already commented on.

The convergence velocity for entry area \( q(k = \pm \sqrt{-a_3}) = \mp \sqrt{-a_3}p - a_3 \) is determined using (114), (115) where only the upper sign is carried through:
\[ J(p; q = -\sqrt{-a_3}p - a_3, a_1 = a_2 = a_4 = 0) \]
\[ = 3 \left( p^4 - \frac{4}{3} \sqrt{-a_3}p^3 - \frac{4}{3}a_3p^2 + \frac{4}{3}a_3\sqrt{-a_3}p + \frac{1}{3}a_3^2 \right) \]
\[ = 3 \left( p^2 - a_3 \right) \left( p - \sqrt{-a_3} \right) \left( p - \frac{1}{3}\sqrt{-a_3} \right), \quad (120) \]

\[ D_1(p; q = -\sqrt{-a_3}p - a_3, a_1 = a_2 = a_4 = 0) \]
\[ = -p \left[ p^4 - 2\sqrt{-a_3}p^3 - 2a_3p^2 + 2a_3\sqrt{-a_3}p + a_3^2 \right] \]
\[ = -p \left( p^2 - a_3 \right) \left( p - \sqrt{-a_3} \right) \left( p - \sqrt{-a_3} \right). \quad (121) \]

The closed-form formula of the iterative process being:

\[ \Delta p(p; q = -\sqrt{-a_3}p - a_3, a_1 = a_2 = a_4 = 0) = \frac{D_1}{J} \]
\[ = -\frac{p}{3} \left( p - \frac{1}{3}\sqrt{-a_3} \right), \quad (122) \]

\[ \Delta q(p; q = -\sqrt{-a_3}p - a_3, a_1 = a_2 = a_4 = 0) \]
\[ = -\sqrt{-a_3} \Delta p(p; q = -\sqrt{-a_3}p - a_3, a_1 = a_2 = a_4 = 0), \quad (123) \]
i.e. quadratic convergence order into two fixed points. These facts do not change using the corresponding other sign of the invariant line.

It remains to prove quadratic convergence for entry area \( p = 0 \) under Theorem II. Inserting \( p = 0 \) into (114), (115):

\[ J(q; p = 0, a_1 = a_2 = a_4 = 0) = 4 \left( q^2 + a_3q + \left( \frac{a_3}{2} \right)^2 \right) \]
\[ = 4 \left( q + \frac{a_3}{2} \right)^2, \quad (124) \]

\[ D_1(q; p = 0, a_1 = a_2 = a_4 = 0) = 0. \quad (125) \]

From (43), (57) and (12)

\[ D_2(q; p = 0, a_1 = a_2 = a_4 = 0) = -2 \left[ q^3 + \frac{3}{2}a_3q^2 + \frac{1}{2}a_3^2q \right] \]
\[ = -2q \left( q + \frac{a_3}{2} \right) \left( q + a_3 \right). \quad (126) \]
Using (124), (126):

\[ \Delta p(q; p = 0, a_1 = a_2 = a_4 = 0) = \frac{D_1}{J} = 0, \tag{127} \]

\[ \Delta q(q; p = 0, a_1 = a_2 = a_4 = 0) = \frac{D_2}{J} = -\frac{1}{2} \frac{q (q + a_3)}{q + a_1} \tag{128} \]

The entry area \( p = 0 = p^* \) contains two fixed points \( q^* = 0 \) and \( q^* = -a_3 \), the convergence rate is quadratic which concludes the proof.

It is to be learned that an expected lowering of quadratic to linear convergence of quartics with double real root did not take place for any of the invariant lines. Therefore we try next the two fold double real root.

**Theorem Vb.** Given any quartic \( f(x) = x^4 + a_4 x^3 + a_3 x^2 + a_2 x + a_1 \) with a two fold double real root is having two entry areas under Theorem I, \( q = -k_i p + k_i^2 (i = 1, 2) \), each with two fixed points \((p_i^*, q_i^*)\) on which convergence is quadratic if and only if the fixed point to be approached by the iterative process has a nonzero Jacobian determinant. If \( a_4 = a_2 = 0 \) a third entry area, \( p = 0 \), is given by Theorem II which contains a single fixed point \((p_3^*, q_3^*)\) on which convergence is linear. The Jacobian determinant of \((p_3^*, q_3^*)\) is zero and approaches of this fixed point from the two invariant lines under Theorem I are of linear convergence (i.e. convergence rate is split between linear and quadratic depending on which of the two possible fixed points are determined).

For the proof we consider the polynomial \( f(x) = x^2(x + a)^2 = x^4 + 2ax^3 + a^2 x^2 + 2x + 0 \), where the transformation \( x = y - a/2 \) yields \( f(y) = y^4 + 0y^3 - \frac{1}{2}a^2 y^2 - 0y + (\frac{a}{2})^4 = 0 \) with the solutions:

\[ y_1 = y_2 = a/2, \ y_3 = y_4 = -a/2; \ p_1^* = y_1 + y_2 = a, \ q_1^* = -(a/2)^2, \]

\[ p_2^* = y_3 + y_4 = -a, \ q_2^* = -(a/2)^2; \ p_3^* = y_1 + y_3 = 0, \ q_3^* = +(a/2)^2. \]

Using (114):

\[ J(p, q; a_1 = (a/2)^4, a_3 = -2(a/2)^2, a_2 = a_4 = 0) \]

\[ 4 \left( q^2 + (-2(a/2)^2 + p^2)q + \frac{3}{4}p^4 - 2(a/2)^2p^2 + \left(\frac{a}{2}\right)^4 \right) \tag{129} \]

and (43), (57), (12):

\[ D_2(p, q; a_1 = (a/2)^4, a_3 = -2(a/2)^2, a_2 = a_4 = 0) \]
\begin{align*}
&= -2 \left[ q^3 + \frac{1}{2} [p^2 - 6(a/2)^2] \right] q^2 + \frac{1}{2} \left[ p^4 - 4(a/2)^2 p^2 + 6(a/2)^4 \right] q \\
&\quad + \frac{3}{2} (a/2)^4 \left[ p^2 - \frac{2}{3} (a/2)^2 \right]. \quad (130)
\end{align*}

Entering the entry area under Theorem II \( p = 0 \) into (129), (130):

\[ J(p = 0; q; a_1 = \frac{a}{2}^4, a_3 = -2(a/2)^2, a_2 = a_4 = 0) \]
\[ = 4 \left( q^2 - 2(a/2)^2 q + \left( \frac{a}{2} \right)^4 \right) = 4 \left( q - (a/2)^2 \right)^2, \quad (131) \]

\[ D_2(p = 0; q; a_1 = \frac{a}{2}^4, a_3 = -2(a/2)^2, a_2 = a_4 = 0) \]
\[ = -2 \left[ q^3 - 3(a/2)^2 q^2 + 3(a/2)^4 q + (a/2)^6 \right] = -2 \left( q - (a/2)^2 \right)^3, \quad (132) \]

\[ \Delta p(p = 0; q; a_1 = \frac{a}{2}^4, a_3 = -2(a/2)^2, a_2 = a_4 = 0) = 0 \quad (\text{Theorem II}) \]

\[ \Delta q(p = 0; q; a_1 = \ldots) = \frac{D_2}{J} = -\frac{2}{4} \left( q - (a/2)^2 \right)^3 = -\frac{1}{2} \left( q - (a/2)^2 \right). \quad (133) \]

The entry area \( p=0 \) yields according to (133) \textit{linear convergence} into the sole fixed point \( p^* = 0; q^* = +(a/2)^2 \) on this invariant line.

Now we are investigating Theorem I’s entry area \( q(k = -a/2) = \frac{a}{2} p + (a/2)^2 \) and its convergence rate using (115):

\[ D_1(p, q; a_1 = \frac{a}{2}^4, a_3 = -2(a/2)^2, a_2 = a_4 = 0) \]
\[ = -p \left[ 2q^2 + 2 \left( p^2 - 2(a/2)^2 \right) q + p^4 - 4(a/2)^2 p^2 + 2(a/2)^4 \right], \quad (134) \]

\[ D_1(p, q(k = -a/2) = \frac{a}{2} p + (a/2)^2; a_1 = \frac{a}{2}^4, a_3 = -2(a/2)^2, a_2 = a_4 = 0) \]
\[ = -p \left[ 2 \left( \frac{a}{2} p + (a/2)^2 \right)^2 + 2 \left( p^2 - 2(a/2)^2 \right) \left( \frac{a}{2} p + (a/2)^2 \right) + p^4 - 4(a/2)^2 p^2 + 2(a/2)^4 \right] = -p^4 (p + a). \quad (135) \]

We find for the Jacobian determinant on the same entry area using (129):

\[ J(p, q = \frac{a}{2} p + (a/2)^2; a_1 = \frac{a}{2}^4, a_3 = -2(a/2)^2, a_2 = a_4 = 0) \]
and therefore

\[
\Delta p(p, q) = \frac{a}{2} p + (a/2)^2; a_1 = (a/2)^4, a_3 = -2(a/2)^2, a_2 = a_4 = 0
\]

\[
\Delta q(p, q(k = -a/2); a_1 = \ldots) = \frac{a}{2} \Delta p(p, q(k = -a/2); a_1 = \ldots).
\]  (137)

From (137) is at first glance the conclusion made that convergence is quadratic into the two limits of the iterative process (the zeros of (137)) \( p^* = 0 \) and \( p^* = -a \), since there remains a linear factor in the denominator of (137) that cannot be cancelled. In practical applications, the software validation and verification efforts, it has come to light that the invariant line \( q(k = -a/2) = (a/2) p + (a/2)^2 \) will produce only linear convergence into \( (p_3^* = 0, q_3^* = +(a/2)^2) \) and, confirming the results of the proof, quadratic convergence into the other fixed point \( (p_2^* = -a, q_2^* = -(a/2)^2) \) located on the invariant line. The difference between the two fixed points is that the first has a zero Jacobian determinant, the second is nonzero. For the first time we have encountered an invariant line that has sections of linear convergence and another section of quadratic convergence.

In order to obtain the results of the third entry area \( q(k = +a/2) = -(a/2) p + (a/2)^2 \) replace in (137) “a” by “-a”. That concludes the proof. Figure 3 is a visualization of the situation for invariant lines having linear and quadratic convergence.

We continue with the convergence properties of entry areas for the threefold root:

**Theorem Vc.** Given any quartic \( f(x) = x^4 + a_4 x^3 + a_3 x^2 + a_2 x + a_1 \) with a threefold real root \( k = x_1 = x_2 = x_3 \neq x_4 \) is having two entry areas under Theorem I, that is invariant line \( q = -kp + k^2 \) with two fixed points \( (p_1^* = 2k, q_1^* = -k^2), (p_2^* = k + x_4, q_2^* = -kx_4) \) and quadratic convergence, that is also invariant line \( q = -x_4 p + x_4^2 \) with one fixed point \( (p_2^* = k + x_4, q_2^* = -kx_4) \), where the convergence is linear. Both fixed points are having a vanishing Jacobian determinant.

For the proof consider the polynomial \( f(x) = x^3(x + a_4) = x^4 + a_4 x^3 + 0x^2 + 0x + 0 \), where the solutions are \( k = x_1 = x_2 = x_3 = 0, x_4 = -a_4; p_1^* = 2k = 0, q_1^* = 0; p_2^* = x_4 = -a_4, q_2^* = 0. \) Because \( a_4 \neq 0 \) we have to go back
to (43), (57) and (12) and calculate for the invariant line, \( q = 0 \), characterised by the threefold root

\[
J (p, q = 0 ; a_1 = a_2 = a_3 = 0) = 3p^2 \left( p + \frac{2}{3}a_4 \right), \tag{138}
\]

\[
D_1 (p, q = 0 ; a_1 = a_2 = a_3 = 0) = -p^3 \left( p + a_4 \right) \left( p + a_4 \right), \tag{139}
\]

\[
\Delta p (p, q = 0 ; a_1 = a_2 = a_3 = 0) = -\frac{p^3 \left( p + a_4 \right) \left( p + a_4 \right)}{3p^2 \left( p + a_4 \right) \left( p + \frac{3}{2}a_4 \right)} = -\frac{p \left( p + a_4 \right)}{3 \left( p + \frac{3}{2}a_4 \right)}, \tag{140}
\]
\[ \Delta q(p, q(k = 0) = 0; a_1 = a_2 = a_3 = 0) = -k \Delta p(p, q = 0; a_1 = a_2 = a_3 = 0) = 0. \]  

As shown by (140) the entry area \( q(x_1 = x_2 = x_3 = 0) = 0 \) has global quadratic convergence into the two fixed points or zeros of (140) \( p = p_1^* = 0 \) and \( p = p_2^* = -a_4 \).

This leads us to the proof of linear convergence on entry area \( q(k = -a_4) = +a_4p + a_4^2 \).

Using (43), (57) and (12) and elementary calculations provide

\[ J(p, q = a_4p + a_4^2; a_1 = a_2 = a_3 = 0) = 3(p^4 + 3a_4p^3 + 4a_4^2p^2 + 3a_4^3p + a_4^4) = 3(p + a_4)(p + a_4)(p^2 + a_4p + a_4^2), \] (142)

\[ D_1(p, q = a_4p + a_4^2; a_1 = a_2 = a_3 = 0) = -(p^5 + 4a_4p^4 + 7a_4^2p^3 + 7a_4^3p^2 + 4a_4^4p + a_4^5) = -(p + a_4)(p + a_4)(p + a_4)(p^3 + a_4p + a_4^2), \] (143)

\[ \Delta p(p, q = a_4p + a_4^2; a_1 = a_2 = a_3 = 0) = \frac{(p + a_4)(p + a_4)(p + a_4)(p^2 + a_4p + a_4^2)}{3(p + a_4)(p + a_4)(p^2 + a_4p + a_4^2)} = \frac{(p + a_4)}{3}, \] (144)

\[ \Delta q(p, q(k = -a_4) = a_4p + a_4^2; a_1 = a_2 = a_3 = 0) = -k \Delta p(p, q = a_4p + a_4^2; a_1 = a_2 = a_3 = 0) = a_4 \Delta p(p, q = a_4p + a_4^2; a_1 = a_2 = a_3 = 0). \] (145)

It is learned from (144) that on the entry area \( q(k = -a_4) = +a_4p + a_4^2 \) through linear factor cancellation the convergence order has worsened to linear and the zero \( p = p_1^* = -a_4 \) is the sole limit of the series of iterations for this invariant line. It is simple to prove that \( J(p_1^* = 0, q_1^* = 0) = J(p_2^* = -a_4, q_2^* = 0) = 0 \).

It is again academic interest that leads to next part of this theorem looking at the real roots of multiplicity four.

**Theorem Vd.** Given the quartic \( f(x) = x^4 + 0x^3 + 0x^2 + 0x^1 + 0 \) with a fourfoldreal root \( k = x_1 = x_2 = x_3 = x_4 = 0 \) is having two entry areas, one under Theorem I, that is invariant line \( q(k = 0) = -kp + k^2 = 0 \), another under Theorem II, invariant line \( p = 0 \), where both invariant lines have the
same fixed point \((p^* = 0, q^* = 0)\) and convergence being linear. The Jacobian determinant is zero at the fixed point.

For the proof use (138) setting \(q = 0\) (invariant line) and \(a_4 = 0\):

\[ J(p, q = 0; a_1 = a_2 = a_3 = a_4 = 0) = 3p^4. \tag{146} \]

Same settings in (139):

\[ D_1(p, q = 0; a_1 = a_2 = a_3 = a_4 = 0) = -p^5. \tag{147} \]

Using (146), (147):

\[ \Delta p(p, q = 0; a_1 = a_2 = a_3 = a_4 = 0) = -\frac{p^5}{3p^4} = -\frac{1}{3}p, \tag{148} \]
\[ \Delta q(p, q = 0; a_1 = a_2 = a_3 = a_4 = 0) = 0. \tag{149} \]

The linear convergence on invariant line \(q = 0\) into fixed point \((p^* = 0, q^* = 0)\) is proven by (148), (149).

For the proof of convergence on invariant line \(p = 0\) enter \(p = a_1 = a_2 = a_3 = a_4 = 0\) in (43), (57), their partial derivatives and calculate according to (12):

\[ J(p = 0, q; a_1 = a_2 = a_3 = a_4 = 0) = 4q^2, \tag{150} \]
\[ D_1(p = 0, q; a_1 = a_2 = a_3 = a_4 = 0) = 0, \tag{151} \]
\[ D_2(p = 0, q; a_1 = a_2 = a_3 = a_4 = 0) = -2q^3, \tag{151} \]
\[ \Delta p(p = 0, q; a_1 = a_2 = a_3 = a_4 = 0) = \frac{D_1}{J} = 0, \tag{152} \]
\[ \Delta q(p = 0, q; a_1 = a_2 = a_3 = a_4 = 0) = \frac{D_2}{J} = -\frac{2q^3}{4q^2} = -\frac{1}{2}q. \tag{152} \]

The linear convergence on invariant line \(p = 0\) into fixed point \((p^* = 0, q^* = 0)\) is proven by (152).

**Theorem Ve.** Given any quartic \(f(x) = x^4 + 0x^3 + a_3x^2 + 0x + a_1\) with sole complex conjugate roots is having just one invariant line or “entry area” under Theorem II, \(p = 0\). This invariant line is not having any fixed point \((p^* = 0, q^*)\), if and only if \(a_3^2 < 4a_1\), therefore also \(a_1 > 0\). For the case where \(a_3 = 0\): The only condition that remains for divergence on \(p = 0\) is \(a_1 > 0\). In such a case the formula \(\Delta q(p = 0, q) = q_{n+1} - q_n\) \((\Delta p = 0)\) of the iterative process is lacking any real zero, the series of iterations is aimed at a non-existing limit or
fixed point and this with formal quadratic “convergence” order and cannot do anything else but diverge.

For the proof use (43), (57) and its partial derivatives and set \( p = a_2 = a_4 = 0 \) in (12):

\[
J(p = 0, q; a_2 = a_4 = 0) = 4 \left( q + \frac{a_3}{2} \right) \left( q + \frac{a_3}{2} \right),
\]

(153)

\[
D_1(p = 0, q; a_2 = a_4 = 0) = 0,
\]

(154)

\[
D_2(p = 0, q; a_2 = a_4 = 0) = -2 \left( q^3 + \frac{3}{2}a_3q^2 + \frac{1}{2} \left( a_3^2 + 2a_1 \right)q + \frac{1}{2}a_1a_3 \right)
\]

\[
= -2 \left( q + \frac{1}{2}a_3 \right) \left( q^2 + a_3q + a_1 \right).
\]

(155)

The closed-form formulas of Bairstow’s iterative technique on invariant line \( p = 0 \):

\[
\Delta p(p = 0, q; a_2 = a_4 = 0) = \frac{D_1}{J} = 0,
\]

(156)

\[
\Delta q(p = 0, q; a_2 = a_4 = 0) = \frac{D_2}{J} = -\frac{2 \left( q + \frac{1}{2}a_3 \right) \left( q^2 + a_3q + a_1 \right)}{4 \left( q + \frac{1}{2}a_3 \right) \left( q + \frac{1}{2}a_3 \right)}
\]

\[
= -\frac{q^2 + a_3q + a_1}{2 \left( q + \frac{1}{2}a_3 \right)}.
\]

(157)

Formula (157) confirms the statements made in Theorem V e on divergence.

The most simple case for divergence of quartics is obtained by demanding in (157) \( a_3 = 0 \):

\[
\Delta p(p = 0, q; a_2 = a_3 = a_4 = 0) = D_1/J = 0,
\]

(158)

\[
\Delta q(p = 0, q; a_2 = a_3 = a_4 = 0)
\]

\[
= \frac{D_2}{J} = -\frac{2q \left( q^2 + a_1 \right)}{4q^2} = -\frac{q^2 + a_1}{2q},
\]

(159)

where the zeros for \( \Delta q \) are complex conjugates, if and only if \( a_1 > 0 \). This concludes the proof.

The other root combinations with only one invariant line under Theorem II and convergence that we did not look at is the case of double complex conjugate
root, i.e. quartics that meet the supplementary coefficient condition $a_3^2 = 4a_1$. In this case we are finding on invariant line $p=0$ that convergence is linear, and that has been expected. Although we will remember the case of a sole double real root did not generate invariant lines with linear but only quadratic convergence!

Further more the case with two distinct complex conjugate roots, i.e. $a_3^2 > 4a_1$, shows well within expectations on invariant line $p=0$ quadratic convergence.

In the context of Theorem II and Theorem Ve, a statement is to be made on the divergence example given by Fiala/Krebsz [3], p. 480,

$$f(x) = x^4 + 6x^3 + 12x^2 + 9x + 3.$$  \hspace{1cm} (160)

The quartic (160) shall be examined for divergence under Theorem II. Naturally this cannot be decided from the polynomial we have to hand, but this can be changed by the transformation $x = y - 6/4 = y - 3/2$, see Section 3, (67). This is yielding

<table>
<thead>
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<th>const</th>
<th>-3/2</th>
<th>+9/2</th>
<th>-3/2</th>
<th>+6/2</th>
<th>+3/2</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Faktor</td>
<td>12/4</td>
<td>12</td>
<td>9</td>
<td>9</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>-3/2</td>
<td>+21/4</td>
<td>-18/4</td>
<td>-9/4</td>
<td>-9/4</td>
<td>1</td>
</tr>
<tr>
<td>$x = -3/2$</td>
<td>-3/2</td>
<td>+9/8</td>
<td>+9/8</td>
<td>+21/16</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$x = -3/2$</td>
<td>-3/2</td>
<td>+3/4</td>
<td>-9/4</td>
<td>-3/2</td>
<td>-3/2</td>
<td>1</td>
</tr>
<tr>
<td>$x = -3/2$</td>
<td>0</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

the following result

$$f(y) = y^4 + 0y^3 - \frac{3}{2}y^2 + 0y + \frac{21}{16}.$$  \hspace{1cm} (161)

Since the coefficients of polynomial $f(y)$, (161), $b_2 = b_4 = 0$, we can examine (161) for divergence under Theorem Ve, i.e. whether $b_3^2 < 4b_1$ is valid:

$$b_3^2 = 9/4 < 4b_1 = 4 \frac{21}{16} = 21/4.$$  \hspace{1cm} 

Which clarifies that $f(y)$ will diverge on $p = 0$! Now we ask, where the invariant line $p=0$ is located for the original quartic $f(x)$, (160), i.e. before shifting the axis by $x = y - 3/2$. If $p_1^*$ und $p_2^*$ are fixed points of (160) the invariant line for divergence is $p = (p_1^* + p_2^*)/2 = -3$, where $p_1^* = -1.052877033292987e+000,$
\[ p_2^* = -4.947122966707013e+000. \] More simple could be calculated \( p = -a_4/2 = -3 \), where from (160) \( a_4 = 6 \). In [3] this result corresponds to \( u = +3 \), the difference in the sign originates from the fact that the quadratic factor is not defined by \( h(x) = x^2 - px - q \) but by \( h(x) = x^2 + ux + w \).

For this Section 5 the following rule for the expansion of Bairstow’s algorithm to handle matters of convergence, unfavourable linear convergence velocity, zero Jacobian determinant at fixed points, and other reasons is being formulated:

**Rule 4.** Multiple root cases worsen the numerical conditions under which Bairstow’s method has to operate. It starts with the existence of the Jacobian determinant being zero at a fixed point. Next step of worsening is loss of quadratic convergence for certain entry areas and possibly an increasing number of fixed points with vanishing Jacobian determinant. This effect will intensify to a total loss of quadratic convergence order and the Jacobian determinant being zero at the fixed point(s). The negative effects of multiple roots could be eliminated by separation of the greatest common divisor of \( f(x) \) and its derivative \( f'(x) \). This would require software calculating the chain of Sturm’s functions with special care to round-off errors. Bairstow’s method would then only be applied to polynomials with distinct roots. If this route is not wanted, the question on how to terminate iterations under Bairstow’s method needs further investigations, where the fixed point being approached has a vanishing Jacobian determinant and a directionally dependent convergence rate (see Theorem IVb and its proof). The number of the maximal allowable iterations would have also to be raised considerably.

This rule will have some strong influence on the expansion of the Bairstow algorithm.

6. Two More Rules Concerning the Expansion of Bairstow’s Iterative Technique

**Rule 5.** Initial values \((p_0, q_0)\) or calculated improved values \((p_\nu, q_\nu)\) may lead to

\[
\Delta p(p_\nu, q_\nu) = 0/0 \quad \text{or} \quad = -\infty \quad \text{or} \quad = +\infty ,
\]

\[
\Delta q(p_\nu, q_\nu) = 0/0 \quad \text{or} \quad = -\infty \quad \text{or} \quad = +\infty .
\]

This is the case, if \( J(p, q) = 0 \) intersects with \( D_1(p, q) = 0 \) and \( D_2(p, q) = 0 \) respectively or \( J(p, q) = 0 \) and \( D_1(p, q) \neq 0 \) and \( D_2(p, q) \neq 0 \) respectively. Whenever this is encountered at \((p_0, q_0)\) or \((p_\nu, q_\nu)\) the value pair has to be changed in their neighbourhood to an appropriate new pair. This holds true,
if and only if $A(p_\nu, q_\nu)$ and $B(p_\nu, q_\nu)$ are not simultaneously approaching zero. In the case where shifting of the pair $(p_\nu, q_\nu)$ is absolutely required, it will be sufficient to shift only one of the two, $p$ or $q$. Knowing Theorem III the author has chosen to shift the value of $q$ only. Thus $\Delta p = 0$ has been chosen and $\Delta q$ is calculated depending on $q_\nu = 0$ yielding $\Delta q = -2^{-3} = -1/8 = -0.125$ or $q_\nu \neq 0$ yielding $\Delta q = -|q| 2^{-3}$:

$$(p_\nu + \Delta p, q_\nu + \Delta q) = (p_\nu, 0.875q_\nu) \text{ if } q_\nu \neq 0,$$

$$(p_\nu + \Delta p, q_\nu + \Delta q) = (p_\nu, -0.125) \text{ if } q_\nu = 0.$$

Because of the arbitrariness of the value two in the power of minus three the programmer may try his own preferred value.

Rule 6. In practical applications of Bairstow’ method it has been occasionally observed that on the last iterations into the fixed point an infinite loop between two value pairs is locked in, that looks like the “cage-effect” or “pendulum-effect” of the introduction equation (19). The difference to (19) being that the correction $\Delta p$ or $\Delta q$ is such that the corrections are a little bit on top of the value for the fixed point. The next iteration reverses the sign for the corrections and ends up with a small underflow of the fixed point and that keeps on going like this indefinitely if not stopped by the maximal value of the iterations count. Therefore precautions have to be made. When detected the final corrections to be used are: $p = p + \Delta p/2$, and $q = q + \Delta q/2$.

7. Algebraic Solution Attempts $A(p^*, q^*) = B(p^*, q^*) = 0$ for Polynomial Degree $n = 3$ and $4$

Bairstow’s method may well be used to find algebraic solutions for cubics $f(x) = x^3 + 0x^2 + a_2x + a_1$ from

$$A_{(3)}(a_3 = 0; p, q) = B_{(3)}(a_3 = 0; p, q) = 0,$$

i.e. closed-form solutions according to (5) for $p^*, q^*$. Using (42) and (56) and elementary algebraic calculations by elimination of $q$ yields

$$f(p) = p^3 + a_2p - a_1 = 0.$$  

By comparison with the polynomial of the starting point $f(x) = x^3 + a_2x + a_1 = 0$, the following is concluded

$$p^*_i = -k_i, \quad i = 1, 2, 3 \text{ for the case of three real roots } x = k,$$
and the equivalent for the other possible root combinations. For cubics with \( a_3 = 0 \) are the fixed points \( p_i^* \) equal to the negative value of the real roots, the evaluation of the real roots does not demand solving the quadratic factor \( x^2 - p^*x - q^* = 0 \). Only exception is the complex conjugate case. Unfortunately there is no more to gain for cubics.

The equivalent evaluation for quartics \( f(x) = x^4 + 0x^3 + a_3x^2 + a_2x + a_1 \),
\[
A_4(a_4 = 0; p, q) = B_4(a_4 = 0; p, q) = 0 \tag{165}
\]
is much more rewarding. We are finding the resolvent cubic, the closed-form formula of the quartics where quartics are reduced to the solution of cubics, a derivation simple, clear and very short. Using (43), (57), (165) and solving (43) for \( q \)
\[
q = -\left[ \frac{1}{2}p^2 + \frac{a_3}{2} + \frac{a_2}{2p} \right], \tag{166}
\]
and replacing in (57) \( q \) by means of (166). Simple elementary algebraic calculations yield the polynomial of degree six for the fixed points:
\[
p^6 + 2a_3p^4 + (a_2^2 - 4a_1)p^2 - a_2^2 = 0. \tag{167}
\]
The special coefficient properties of polynomial (167) allow via the substitution
\[
p^2 = z, \quad p = \pm \sqrt{z} \tag{168}
\]
the closed-form solution for root pairs of the quartics to trace back to the cubics
\[
z^3 + 2a_3z^2 + (a_2^2 - 4a_1)z^1 - a_2^2 = 0. \tag{169}
\]
This is our auxiliary equation or resolvent cubic, which may be solved by the Cardanic formula. In this case polynomial reduction has to be done:
\[
z = y - \frac{2}{3}a_3, \tag{170}
\]
yielding
\[
y^3 + 0y^2 + 3\frac{b_2}{3}y + 2\frac{b_1}{2} = 0, \tag{171}
\]
where
\[
\frac{b_1}{2} = 12\left( \frac{a_1}{3} \right) \left( \frac{a_3}{3} \right) - \left( \frac{a_3}{3} \right)^3 - 9 \left( \frac{a_2}{3} \right)^2, \quad \frac{b_2}{3} = -\left( \frac{a_3}{3} \right)^2 - 4 \left( \frac{a_1}{3} \right),
\]
obtained from the continued Horner scheme:

\[
\begin{array}{c|cccc}
\text{const.} & 1 & 2a_3 & a_3^2 - 4a_1 & -a_2^2 \\
\text{Fakt.} & - & - \frac{2}{3}a_3 & - \frac{8}{9}a_3^2 & -\frac{2}{9}a_3^3 + \frac{8}{9}a_1a_3 \\
\hline
z = -\frac{2}{3}a_3 & 1 & \frac{4}{3}a_3 & \frac{1}{9}a_3^2 - 4a_1 & -\frac{2}{9}a_3^3 - a_2^2 + \frac{8}{9}a_1a_3 = b_1 \\
\hline
z = -\frac{2}{3}a_3 & 1 & + \frac{2}{3}a_3 & -\frac{4}{3}a_3^2 - 4a_1 = b_2 \\
\hline
z = -\frac{2}{3}a_3 & 1 & = b_4 & 0 = b_3
\end{array}
\]

\[
y^3 + 0y^2 + 3\frac{b_2}{3}y + 2\frac{b_1}{2} = 0 ,
\]

(172)

where

\[
\frac{b_1}{2} = 12\left(\frac{a_1}{3}\right)\left(\frac{a_3}{3}\right) - \left(\frac{a_3}{3}\right)^3 - \frac{9}{2}\left(\frac{a_2}{3}\right)^2 , \quad \frac{b_2}{3} = -\left(\frac{a_3}{3}\right)^2 - 4\left(\frac{a_1}{3}\right) .
\]

(173)

Only real numbers for \(p\) are solutions of (168), the appropriate \(q\)-values are for \(p \neq 0\) to be determined by (166). If \(p = 0\) (166) cannot be used because of its singularity and we must turn to (57) \(B_{(4)}(a_4 = 0; p = 0, q) = 0:\)

\[
q^2 + a_3q + a_1 = 0 ,
\]

\[
q_{1,2} = -\frac{a_3 \pm a}{2} , \text{ where } a = \sqrt{a_3^2 - 4a_1} .
\]

(174)

8. Remarks from a Control System Theory Point of View

Control system theory is mathematics, but also the trial to avoid certain problems in mathematics. It is here specifically meant the polynomial factorization.
Any stability analysis of any linear control system could be solved using a polynomial root finder, but just this is tried to be avoided as best seen from the many methods that are aimed “to work around it”: Routh criterion, Hurwitz criterion, root locus by Evans, Bode diagrams or logarithmic frequency amplitude and phase shift response, Nyquist criterion, etc. Apparently, already in the early days, scientists working in the field of feedback control must have been aware of the difficulties of polynomial factorization and therefore have looked for alternatives and this with high inventive spirits. Bairstow appears to be an exception. He worked with colleagues on a stability problem of an aeroplane when in circling flight giving the scientists polynomials of eighth degree with 2 real and three pairs of complex conjugate roots. That caused the idea of factorizing a polynomial into quadratic factors avoiding calculations with complex numbers. He could not be expected to know at the time of the method’s invention that complex conjugate root pairs would also be the reason for divergence of his method which follows from the two invariance Theorem I and Theorem II! Furthermore the polynomial reduction with its positive numeric stability effect is to the author’s knowledge a mathematical law on polynomials completely left out in control system theory. Perhaps, the reason is because it shifts the roots of a given polynomial into the instability range which seems to be, and at first glance, a not useful operation when investigations are made on stability of feedback control systems. Apparently the intermediate use of such a law for numerical stability test by changing an ill-conditioned to a well-conditioned polynomial must have been overseen. The subtraction of a constant to return from the intermediate root distribution to the final one should not prevent using the mean of polynomial reduction in control system theory. It is the author’s intent to have at least one polynomial root finder method understood mathematically so well that it could be an acceptable tool for stability analysis of control systems. Bairstow’s iterative technique being a method of Newton type seems to serve this purpose if studied in sufficient depth.

Acknowledgments

The idea to perform this research into Bairstow’s method goes back to my long time ago diploma thesis “Calculating the Root Locus on the Digital Computer” at the Technical University of Berlin, Professor Dr. Phil. Nat. Gerd Schneider, “Lehrstuhl für elektrische Steuerung und Regelungstechnik”, and his ZUSE Z25 of the “Deutsche Forschungsgemeinschaft”. That have been the days where computer time was a precious thing and computer software were
almost not available yet, what is nowadays hardly imaginable. Throughout all
of this I think with warm feelings of my long-term fellow students at the TU-
Berlin and lifelong friends Dipl.-Ing. Jörg Seguin and Professor Dr.-Ing. Georg
Thiele. They have supported me and know of the difficulties to do scientific
research successfully. The word of our that time mathematics teacher at the
TU-Berlin, Professor Dr. Wolfgang Haack, “Mathematics commences with zero
and infinity” has had noticeable influence on this research or not?

References

[1] L. Bairstow, Reports and memoranda Nr.154 of Advisory committee for
Aeronautics (October, 1914), 239-252 (Investigation of the stability of an
aeroplane when in circling flight, Appendix: The solution of algebraic equa-
tions with numerical coefficients in the case where several pairs of complex
roots exist).

Anal., 14 (1977), 571-574.


[4] W. Gabler, Entry areas for pairs of roots using Bairstow’s method, ZAMM,

Phys., 17 (1938), 55-58.

Wiley and Sons, Inc. (1967),

(1961), 120-124.


