ON COPRIMELY PACKED MULTIPLICATION MODULES

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Abstract: Let $M$ be a non-zero multiplication $R$-module. Then, $M$ is called coprimely packed if every submodule of $M$ is coprimely packed. We generalize the notion of a coprimely packed ring to coprimely packed multiplication modules.

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1. Introduction

Throughout this paper all rings will be commutative with identity and all modules will be non-zero and unitary. A ring $R$ is defined to be compactly packed by prime ideals if whenever an ideal $I$ of $R$ is contained in the union of a family of prime ideals of $R$, $I$ is actually contained in one of the prime ideals of the family. An ideal $I$ of a ring is said to be coprimely packed $I + P_s = R$, where $P_s (s \in S)$ are prime ideals of $R$; then $I \not\subseteq \bigcup_{s \in S} P_s$. A non-empty subset $X$ of the set of prime ideals of $R$ is said to be coprimely packed if whenever an element $P$ of $X$ is coprime to each element of a subset $Y$ of $X$, then $P \not\subseteq \bigcup_{Q \in Y} Q$. If every ideal of $R$ is coprimely packed, then $R$ is a coprimely packed ring, see [3], [5], [4].

Let $M$ be an $R$-module. A proper submodule $K$ of $M$ is called prime if $rm \in K$, for $r \in R$ and $m \in M$, then $m \in K$ or $r \in (K : M)$, where $(K : M) = \{ r \in R : rM \subseteq K \}$. An $R$-module $M$ is called a multiplication module if for each submodule $N$ of $M$, $N = IM$ for some ideal $I$ of $R$. A multiplication module $M$ is defined to be compactly packed by prime submodules if whenever
a submodule $N$ of $M$ is contained in the union of a family of prime submodules of $M$, $N$ is actually contained in one of the prime submodules of the family. A submodule $N$ of $M$ is said to be coprimely packed by prime submodules of $M$ if whenever $N$ is coprime to each element of a family of prime submodules of $M$, $N$ is not contained in the union of prime submodules of the family. We say that $M$ is coprimely packed if every submodule of $M$ is coprimely packed. Here we generalize the notion of a coprimely packed ring to coprimely packed multiplication modules.

We recall from [7], [1] the following facts.

**Theorem 1.** (The Prime Avoidance Theorem) Let $M$ be an $R$-module, $L_1, L_2, ..., L_n$ a finite number of submodules of $M$, and $L$ a submodule of $M$ such that $L \subseteq L_1 \cup L_2 \cup ... \cup L_n$. Assume that at most two of the $L_i$’s are not prime, and that $(L_j : M) \nsubseteq (L_k : M)$ whenever $j \neq k$. Then $L \subseteq L_k$ for some $k$.

Let $L_1, L_2, ..., L_n$ be submodules of an $R$-module $M$. We call a covering $L \subseteq L_1 \cup L_2 \cup ... \cup L_n$ efficient if no $L_k$ is superfluous.

**Theorem 2.** (The Prime Avoidance Theorem For Multiplication Modules) Let $M$ be a multiplication $R$-module, $L_1, L_2, ..., L_n$ a finite number of submodules of $M$, and $L$ a submodule of $M$ such that $L \subseteq L_1 \cup L_2 \cup ... \cup L_n$. Assume that at most two of the $L_i$’s are not prime. Then $L \subseteq L_k$ for some $k$.

**Proof.** We may assume that the covering is efficient. Then $(L_j : M) \nsubseteq (L_k : M)$ whenever $j \neq k$. Otherwise, if $(L_j : M) \subseteq (L_k : M)$, then $L_j = (L_j : M) M \subseteq (L_k : M) M = L_k$. This is a contradiction. Consequently, $L \subseteq L_k$ for some $k$ by Theorem 1. \qed

For any $R$-module $M$, let $\text{Spec}(M)$ denote the collection of all prime submodules of $M$. Note that some modules $M$ have no prime submodules. (i.e., $\text{Spec}(M) = \emptyset$), see [6]. Let $M$ be a multiplication $R$-module. Since $\text{Ann}(M) \neq R$, it follows that there exists a maximal ideal $P$ of $R$ such that $\text{Ann}(M) \subseteq P$ and $M \neq PM$ by [2], Theorem 2.5. $PM$ is a prime submodule of $M$ by [2], Corollary 2.11. Consequently, if $M$ is a multiplication $R$-module, then $\text{Spec}(M) \neq \emptyset$.

**Definition 1.** A multiplication $R$-module $M$ is defined to be compactly packed by prime submodules if whenever a submodule $N$ of $M$ is contained in the union of a family of prime submodules of $M$, $N$ is actually contained in one of the prime submodules of the family.

**Definition 2.** Let $M$ be a multiplication $R$-module. A submodule $N$ of $M$ is said to be coprimely packed by prime submodules of $M$ if whenever $N$ is coprime to each element of a family of prime submodules of $M$, $N$ is not
contained in the union of prime submodules of the family. We say that $M$ is coprimely packed if every submodule of $M$ is coprimely packed.

**Proposition 1.** Let $M$ be a multiplication $R$-module. If $M$ is a compactly packed module, then $M$ is coprimely packed module.

**Proof.** Let $M$ be a compactly packed module and suppose that $M$ is not coprimely packed. Then there is a non-zero submodule $N$ of $M$ and a non-empty subset $X$ of Spec($M$) such that $N + P = M$ for all $P \in X$ and $N \subseteq \bigcup_{P \in X} P$.

Since $M$ is compactly packed, $N \subseteq \bigcup_{P \in X} P$ implies that $N \subseteq P$ for some $P \in X$. This is a contradiction. Therefore, $M$ is coprimely packed.

By a chain of prime submodules of an $R$-module $M$ we mean a finite strictly increasing sequence $N_0 \subset N_1 \subset \cdots \subset N_n$; the dimension of this chain is $n$. We define the dimension of $M$ to be the supremum of the lengths of all chains of prime submodules in $M$. Recall that $M$ is a torsion-free module; if for any $r \in R$, and $m \in M$, $rm = 0$, implies $r = 0$ or $m = 0$.

**Proposition 2.** Let $M$ be a multiplication $R$-module. Let $M$ be a torsion-free module of dimension 1. Then $M$ is compactly packed if and only if it is coprimely packed.

**Proof.** It remains to show that if $M$ is coprimely packed then it is compactly packed. Let $N$ be a non-zero submodule of $M$ and $X$ a non-empty subset of Spec($M$). Suppose that $N \subseteq \bigcup_{P \in X} P$. Since $M$ is coprimely packed, $N + P \neq M$, for some $P \in X$. Hence there is a maximal submodule $T$ of $M$ such that $N + P \subseteq T$ (now in the subset $X$ of Spec($M$), we may as well assume that $0 \notin X$, which does not affect the assumption that $N \subseteq \bigcup_{P \in X} P$). But $M$ is a torsion-free module (and so, $0_M$ is a prime submodule of $M$) of dimension 1. Therefore it follows that $P = T$ and $N \subseteq P$. Hence $M$ is compactly packed module.

**Theorem 3.** Let $M$ be a multiplication $R$-module such that contains only finitely many maximal submodules. Then $M$ is coprimely packed.

**Proof.** Let $N$ be any non-zero submodule of $M$. Suppose that $N + P = M$ for all $P \in X$, where $X$ is any non-empty subset of Spec($M$). We want to show that $N \nsubseteq \bigcup_{P \in X} P$. We know that $M$ contains only finitely many maximal submodules. We may pick a subset $\{M_1, M_2, \ldots, M_n\}$ of MaxSpec($M$) in such a way that for each $M_i$ in $\{M_1, M_2, \ldots, M_n\}$ there exists an element $P$ in $X$ such that $P \subseteq M_i$ and for each $P \in X$ there exists an element $M_i$ in $\{M_1, M_2, \ldots, M_n\}$ such that $P \subseteq M_i$. Since $N + P = M$ for all $P \in X$, it follows that $N + M_i = M$. 

for all $i = 1, 2, \ldots, n$. Hence $N \not\subseteq \bigcup_{i=1}^{n} M_i$ by Theorem 2. But $\bigcup_{P \in X} P \subseteq \bigcup_{i=1}^{n} M_i$. Therefore $N \not\subseteq \bigcup_{P \in X} P$. Therefore, it follows that $M$ is coprimely packed.

**Theorem 4.** Let $M$ be a multiplication $R$-module. Then the following are equivalent:

(i) $M$ is coprimely packed.

(ii) Let $N$ be any submodule in $M$, and let $S$ be any set of maximal submodules in $M$ with $N \subseteq \bigcup \{T \in S\}$. Then $N \subseteq T$ for some $T \in S$.

**Proof.** (i)$\Rightarrow$(ii). If $S$ is a set of maximal submodules and if $N$ is a submodule with $N \subseteq \bigcup \{T \in S\}$, then by (i), for some $T \in S$ we have $N + T \neq M$. Therefore $N \subseteq T$.

(ii)$\Rightarrow$(i). Let $S$ be a set of prime submodules, and let $N$ be a submodule with $N \subseteq \bigcup \{Q \in S\}$. For each $Q \in S$, let $M_Q$ be a maximal submodule containing $Q$. Surely $N \subseteq \bigcup \{M_Q : Q \in S\}$ and so by (ii), $N \subseteq M_Q$ for some $Q \in S$. As also $Q \subseteq M_Q$, we have $N + Q \subseteq M_Q \neq R$. Thus (i) holds.

**References**


