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ON A TRIANGULAR INVERSION FORMULA, ITS APPLICATION AND ORDERED BELL NUMBERS

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Abstract: The paper is devoted to the deriving a general triangular inverse formula between any sequences $\{x_n\}$ and $\{y_n\}$ and an explicit formula for triangular sequence $\{x_n\}$. Further it is shown a way of using this closed form for finding relation for any sequence which is connected with the ordered Bell numbers. For this purpose we derive some new identities for the ordered Bell numbers, too.

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1. Introduction

1.1. The Inversion Formulas

The dual relationship between sequences $\{x_n\}$ and $\{y_n\}$ of the type

$$x_n = \sum_k a_{n,k}\, y_k \quad \text{iff} \quad y_n = \sum_k b_{n,k}\, x_n \tag{1}$$

is called an *inversion formula*. Such pairs occur frequently in many fields of mathematics but especially in combinatorial analysis. In [8] Riordan realized

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that each inverse relation is associated with an orthogonal identity

$$\sum_m a_{n,k}\, b_{k,m} = \delta_{n,m} \ ,$$

where $\delta_{n,m}$ is the Kronecker delta, and further that the coefficients in (1) create infinite lower–triangular matrices $A = (a_{n,k})_{n,k\in\mathbb{N}_0}$ and $B = (b_{n,k})_{n,k\in\mathbb{N}_0}$, i. e. $a_{n,k} = 0$ and $b_{n,k} = 0$ unless $n \ge k$. Thus, the reciprocal formulas are equivalent to the matrix relation $A\,B = I$, where I is the infinite identity matrix. The fact that the linear transformation $f_k = \sum_{i=0}^{k} a_{k,i}\, g_i$, where $k = 0, 1, \ldots, n$, is nonsingular provided that $a_{k,k} \neq 0$ for $k = 0, 1, \ldots, n$, implies that if $a_{n,n} \neq 0$ for all nonnegative integers n it suffices to prove only one of the implications in (1).

Throughout the paper we adopt the conventions that the product over an empty set is 1, that $\lfloor x \rfloor$ represents the greatest integer less than or equal to x and Iverson's notation that

$$[P(k)] = \begin{cases} 1, & \text{if statement } P(k) \text{ is true}, \\ 0, & \text{if statement } P(k) \text{ is false}. \end{cases}$$

For example the following formulas pertain among the well–known inversion formulas (see [1], [3], [5] and [9]):

(a) *Stirling inversion formula.* Let n, k be any integers, $S(n,k)$ be Stirling numbers of second kind and $s(n,k)$ be Stirling numbers of first kind (see e. g. [1] and [2]), then the following holds for arbitrary sequences $\{f_n\}$ and $\{g_n\}$:

$$g_n = \sum_k (-1)^k S(n,k)\, f_k \quad \text{iff} \quad f_n = \sum_k (-1)^k s(n,k)\, g_k \ .$$

(b) *Binomial inversion formulas.* As is known, books [3] and [9] displayed a variety of special inversion formulas, which involve some binomial coefficients. Undoubtedly, the simplest and well–known binomial inversion is the formula

$$g_n = \sum_k \binom{n}{k} (-1)^k f_k \quad \text{iff} \quad f_n = \sum_k \binom{n}{k} (-1)^k g_k \ , \tag{2}$$

for any integers k, n and arbitrary sequences $\{f_n\}$ and $\{g_n\}$. In [4] Gould and Hsu found the general binomial–type inversion and further generalization was done by Krattenhaler in [7], where the following matrix inverse is derived:

$$f_{n,k} = \frac{\prod_{j=k}^{n-1}(a_j + c_k\, b_j)}{\prod_{j=k+1}^{n}(c_j - c_k)} \iff f_{k,l}^{-1} = \frac{a_l + c_l\, b_l}{a_k + c_k\, b_k} \frac{\prod_{j=l+1}^{k}(a_j + c_k\, b_j)}{\prod_{j=l}^{k-1}(c_j - c_k)} \ ,$$

where $\{a_i\}_{i\in\mathbb{Z}}$, $\{b_i\}_{i\in\mathbb{Z}}$, $\{c_i\}_{i\in\mathbb{Z}}$ are arbitrary sequences such that $c_i \neq c_j$ if $i \neq j$.

1.2. Bell Numbers and Ordered Bell Numbers

(a) *The Bell number* b_n is the number of ways to partition a set of n things into subsets. Therefore the Bell numbers are closely connected to the Stirling numbers of second kind $S(n,k)$ by

$$b_n = \sum_{k=0}^{n} S(n,k) \ .$$

(b) *The ordered Bell number* B_n represents the number of preferential arrangements of n labeled elements (see [6]). These numbers appear in some various situations, for example as the number of different trees of a special type, as the number of possible switching of the order of summation in multiple sums and as the number of possible outcomes when n numbers $\{x_1, x_2, \dots, x_n\}$ are compared with each other. But unfortunately notation of these numbers is ambiguous, for example they are called Fubini numbers in [1]. The sequence of these numbers is cited in [10] as the sequence with ID Number A000670.

The numbers B_n are again closely related to the Stirling numbers of second kind by

$$B_n = \sum_{i=0}^{n} i!\, S(n,i)$$

and $B_E(x) = \frac{1}{2-e^x}$ is their exponential generating function (see e.g. [2]). They satisfy for $n > 0$ the recurrence relation

$$B_n = \sum_{i=0}^{n-1} \binom{n}{i} B_i \ , \quad \text{with} \quad B_0 = 1 \ . \tag{3}$$

2. The Main Results

The triangular inversion formula between sequences $\{x_n\}$ and $\{y_n\}$ is present in the next theorem.

Theorem 1. *Let k, n be any nonnegative integers and $\{a_{n,k}\}$ be any multiple sequence of real numbers, with $a_{i,i} \neq 0$ for every nonnegative integer i. Then*

$$y_n = \sum_{i=0}^{n} a_{n,i}\, x_i \tag{4}$$

iff

$$x_n = \frac{1}{a_{n,n}} \times \sum_{i=0}^{n} \left(\frac{a_{n,i}}{a_{i,i}} + \sum_{l=1}^{n-i-1} (-1)^l \sum_{\substack{k_1,k_2,\dots,k_l \\ i<k_l<k_{l-1}<\dots<k_1<n}} \frac{a_{n,k_1} a_{k_l,i}}{a_{k_1,k_1} a_{i,i}} \prod_{m=2}^{l} \frac{a_{k_{m-1},k_m}}{a_{k_m,k_m}} \right) y_i \ . \qquad (5)$$

Theorem 2. *Let i, n be any nonnegative integers and $\{a_{n,i}\}$ be any multiple sequence of real numbers, with $a_{0,0} = 1$. Let $\{x_n\}$ be any sequence of real numbers defined for $n \geq 1$ by the recurrence*

$$x_n = \sum_{i=0}^{n-1} a_{n,i}\, x_i \ ,$$

with $x_0 = \varepsilon$, where ε is any real number. Then the sequence $\{x_n\}$ has the closed form

$$x_n = \varepsilon \left(a_{n,0} + \sum_{l=1}^{n-1} \sum_{\substack{k_1,k_2,\dots,k_l \\ 0<k_l<k_{l-1}<\dots<k_1<n}} a_{n,k_1} a_{k_l,0} \prod_{i=1}^{l-1} a_{k_i,k_{i+1}} \right) . \qquad (6)$$

The inversion formulas between sequences $\{f_n\}$ and $\{g_n\}$, which includes the ordered Bell numbers B_n, are present in the next results.

Theorem 3. *Let s be any positive integer, n be any nonnegative integer and B_n be n–th ordered Bell number. The following inversion formulas for arbitrary sequences $\{f_n\}$ and $\{g_n\}$ hold:*

$$(i) \qquad g_n = f_n - \sum_{k=0}^{n-1} \binom{n}{k} f_k \quad \text{iff} \quad f_n = \sum_{k=0}^{n} \binom{n}{k} B_{n-k}\, g_k$$

and

$$(ii) \quad g_n = f_n + 2 \sum_{j=0}^{n-1} \binom{s-j}{n-j} f_j \ \text{ iff } \ f_n = g_n + 2 \sum_{j=0}^{n-1} (-1)^{n-j} \binom{s-j}{n-j} B_{n-j}\, g_j .$$

Theorem 4. *Let k_0 be any positive integers and B_n be the n–th ordered Bell number. Then*

$$(i) \qquad \sum_{l=1}^{k_0-1} \sum_{\substack{k_1,k_2,\dots,k_l \\ 0<k_l<k_{l-1}<\dots<k_1<k_0}} \prod_{i=0}^{l-1} \binom{k_i}{k_{i+1}} = B_{k_0} - 1 \ ,$$

$$(ii) \quad \sum_{l=1}^{k_0-1} \sum_{\substack{k_1,k_2,\ldots,k_l \\ 0<k_l<k_{l-1}<\cdots<k_1<k_0}} (-2)^l \prod_{i=1}^{l} \binom{n-k_i}{k_{i-1}-k_i} \binom{n}{k_l} = \binom{n}{k_0} \left((-1)^{k_0+1} B_{k_0} - 1\right) .$$

3. The Preliminary Results

In this section we first derive an identity for the ordered Bell numbers. Then we use it for finding a closed form for a special triangular sequence given by a recurrence relation, which includes the binomial coefficients. Further we determine some easy relations for multiple sums, which we will need in the proofs of Theorem 1 and Theorem 2.

Lemma 5. *Let k be any positive integer. For the ordered Bell numbers B_k the following holds*

$$\sum_{i=1}^{k} \binom{k}{i} (-1)^i B_i = \frac{1}{2} \left((-1)^k B_k - 1\right) .$$

Proof. Using (3) and well–known identity $\sum_{i=0}^{k} \binom{k}{i} (-1)^i = 0$, for any integer $k \neq 0$, we get step by step

$$\begin{aligned}
B_k &= \sum_{i=0}^{k-1} \binom{k}{i} B_i - \sum_{i=0}^{k} \binom{k}{i} (-1)^i , \\
B_k + \binom{k}{k} B_k &= \sum_{i=1}^{k-1} \binom{k}{i} B_i - \sum_{i=1}^{k} \binom{k}{i} (-1)^i + \binom{k}{k} B_k , \\
B_k &= \sum_{i=1}^{k} \binom{k}{i} \frac{1}{2} \left(B_i - (-1)^i\right) , \\
B_k &= \sum_{i=1}^{k} \binom{k}{i} (-1)^i \frac{1}{2} \left((-1)^i B_i - 1\right) .
\end{aligned}$$

If we denote $c_n = \frac{1}{2}\left((-1)^n B_n - 1\right)$, for any positive integer n, then the previous relation can be written as $B_k = \sum_{i=1}^{k} \binom{k}{i} (-1)^i c_i$, which can be reverted by using (2) and this completes the proof. □

Lemma 6. *Let k be any nonnegative integer and n be any positive integer. The sequence $\{h_k(n)\}_{k=0}^{\infty}$, with $h_0(n) = -1$, defined for $k > 0$ by the recurrence* *relation*

$$h_k(n) = \binom{n}{k} - 2\sum_{i=1}^{k-1}\binom{n-i}{n-k}h_i(n)\ ,$$

has the closed form $h_k(n) = (-1)^{k+1}\binom{n}{k}B_k$ *for* $k \geq 0$*, where* B_k *is the* k*–th ordered Bell number.*

Proof. The assertion follows from Lemma 5 and the well–known binomial identity $\binom{n}{k}\binom{k}{i} = \binom{n-i}{n-k}\binom{n}{i}$, where i, k, n are any integers, after simplifications. □

Lemma 7. *Let* k, n *be any positive integers and* $\{h_k(n)\}$ *be the sequence from Lemma 6. Then*

$$h_k(n) = \binom{n}{k} + \sum_{l=1}^{k-1}(-2)^l \sum_{\substack{k_1,k_2,\dots,k_l\\ 0<k_l<k_{l-1}<\dots<k_1<k}} \binom{n-k_1}{n-k}\binom{n}{k_l}\prod_{i=1}^{l-1}\binom{n-k_{i+1}}{n-k_i}\ .$$

Proof. Solving the recurrence relation in Lemma 6 using (6) the formula follows. □

Lemma 8. *Let* l, n, k_0, k_1, $\dots$, k_{l+1} *be any positive integers.*
(i) *If* $l > 1$*, then*

$$\sum_{\substack{k_1,k_2,\dots,k_l\\ k_{l+1}<k_l<k_{l-1}<\dots<k_1<k_0}} a_{k_0,k_1}a_{k_1,k_2}\cdots a_{k_l,k_{l+1}} = \sum_{k_1=k_{l+1}+1}^{k_0-1} a_{k_0,k_1} \sum_{\substack{k_2,\dots,k_l\\ k_{l+1}<k_l<k_{l-1}<\dots<k_2<k_1}} a_{k_1,k_2}\cdots a_{k_l,k_{l+1}}$$

and

$$\sum_{\substack{k_1,k_2,\dots,k_l\\ k_{l+1}<k_l<k_{l-1}<\dots<k_1<k_0}} a_{k_0,k_1}a_{k_1,k_2}\cdots a_{k_l,k_{l+1}} = \sum_{k_l=k_{l+1}+1}^{k_0-1} a_{k_l,k_{l+1}} \sum_{\substack{k_1,\dots,k_{l-1}\\ k_l<k_{l-1}<\dots<k_2<k_1<k_0}} a_{k_0,k_1}\cdots a_{k_{l-1},k_l}\ .$$

(ii) For $k_0 \le l + k_{l+1}$ the following holds

$$\sum_{\substack{k_1,k_2,\dots,k_l \\ k_{l+1}<k_l<k_{l-1}<\cdots<k_1<k_0}} a_{k_0,k_1} a_{k_1,k_2} \cdots a_{k_l,k_{l+1}} = 0 \ .$$

Proof. Case (i) directly follows from the notation and case (ii) is a clear consequence of the fact that

$$1 \le k_{l+1} < k_l < k_{l-1} < \cdots < k_1 < k_0 \ . \qquad \square$$

4. The Proofs of the Main Theorems

Proof of Theorem 1. With respect to Lemma 8 it is suffice to consider $\sum_{l=1}^{n-1}$ in place of $\sum_{l=1}^{n-i-1}$ in (5). Replacing $-\frac{a_{i,j}}{a_{i,i}}$ by $a_{i,j}$ and $-\frac{y_i}{a_{ii}}$ by y_i it is possible to rewrite inversion identity (4) $\Leftrightarrow$ (5) to the form

$$y_n = -x_n + \sum_{i=0}^{n-1} a_{n,i}\, x_i \tag{7}$$

iff

$$x_n = -y_n - \sum_{i=0}^{n-1} \left(a_{n,i} + \sum_{l=1}^{n-1} \sum_{\substack{k_1,k_2,\dots,k_l \\ i<k_l<k_{l-1}<\cdots<k_1<n}} a_{n,k_1} \left(\prod_{m=2}^{l} a_{k_{m-1},k_m} \right) a_{k_l,i} \right) y_i \ . \tag{8}$$

Thus, it is suffice to derive the previous inversion identity instead of formula (4). Let us show that identity (8) implies identity (7):

$$- y_n - \sum_{i=0}^{n-1} \left(a_{n,i} + \sum_{l=1}^{n-1} \sum_{\substack{k_1,k_2,\dots,k_l \\ i<k_l<k_{l-1}<\cdots<k_1<n}} a_{n,k_1} \left(\prod_{m=2}^{l} a_{k_{m-1},k_m} \right) a_{k_l,i} \right) y_i$$

$$= x_n - \sum_{i=0}^{n-1} a_{n,i} x_i$$

$$-\sum_{i=0}^{n-1}\left(a_{n,i}+\sum_{l=1}^{n-1}\sum_{\substack{k_1,k_2,\dots,k_l\\ i<k_l<\dots<k_1<n}} a_{n,k_1}\left(\prod_{m=2}^{l} a_{k_{m-1},k_m}\right)a_{k_l,i}\right)\left(-x_i+\sum_{j=0}^{i-1} a_{i,j}x_j\right)$$

$$= x_n-\sum_{j=0}^{n-1}\sum_{i=0}^{j-1} a_{n,j}a_{j,i}x_i+\sum_{i=0}^{n-1}\sum_{l=1}^{n-1}\sum_{\substack{k_1,k_2,\dots,k_l\\ i<k_l<k_{l-1}<\dots<k_1<n}} a_{n,k_1}\left(\prod_{m=2}^{l} a_{k_{m-1},k_m}\right)a_{k_l,i}x_i$$

$$-\sum_{j=0}^{n-1}\sum_{i=0}^{j-1}\sum_{l=1}^{n-1}\sum_{\substack{k_1,k_2,\dots,k_l\\ j<k_l<\dots<k_1<n}} a_{n,k_1}\left(\prod_{m=2}^{l} a_{k_{m-1},k_m}\right)a_{k_l,j}\,a_{j,i}x_i$$

and to prove the implication we only need to show that the addition of the sums, from the previous relation, is zero. Using the formula

$$\sum_{j=0}^{n-1}\sum_{i=0}^{j-1} A_{i,j}=\sum_{i=0}^{n-2}\sum_{j=i+1}^{n-1} A_{i,j} \tag{9}$$

for interchanging the order of summation and Lemma 8 we get the assertion after the tedious simplification. □

Proof of Theorem 2. We use induction on n. (6) clearly holds for $n=0$, 1 and 2. Assume it to be true for $0,1,2,\dots,n-1$, $n\geq 3$, we will prove it for n. By the recurrence relation for $\{x_n\}$ and induction hypotheses (6) we get

$$x_n=\sum_{j=0}^{n-1} a_{n,j}x_j=a_{n,0}\,x_0+\sum_{j=1}^{n-1} a_{n,j}x_j$$

$$=\varepsilon\left(a_{n,0}+\sum_{j=1}^{n-1} a_{n,j}a_{j,0}+\sum_{j=1}^{n-1}\sum_{l=1}^{j-1}\sum_{\substack{k_1,k_2,\dots,k_l\\ 0<k_l<\dots<k_1<j}} a_{n,j}a_{j,k_1}a_{k_l,0}\prod_{i=1}^{l-1} a_{k_i,k_{i+1}}\right).$$

Using formula (9) we have

$$x_n=\varepsilon\left(a_{n,0}+\sum_{j=1}^{n-1} a_{n,j}a_{j,0}+\sum_{l=1}^{n-2}\sum_{j=l+1}^{n-1} a_{n,j}\sum_{\substack{k_1,k_2,\dots,k_l\\ 0<k_l<\dots<k_1<j}} a_{j,k_1}a_{k_l,0}\prod_{i=1}^{l-1} a_{k_i,k_{i+1}}\right).$$

With respect to case (ii) of Lemma 8, further replacing k_i by k_{i+1} (for $i = 1, 2, \dots, l$), j by k_1 and shifting index of summation in the last sum we obtain

$$x_n = \varepsilon \left(a_{n,1} + \sum_{j=1}^{n-1} a_{n,j} a_{j,0} + \sum_{l=2}^{n-1} \sum_{\substack{k_1, k_2, \dots, k_l \\ 0<k_l<k_{l-1}<\cdots<k_1<n}} a_{n,k_1} a_{k_l,0} \prod_{i=1}^{l-1} a_{k_i,k_{i+1}} \right)$$

and finally after combining sums we obtain the assertion. □

Proof of Theorem 3. We first prove case (i). We use inversion formula (15) for any sequences $\{f_n\}$ and $\{g_n\}$ from [9] (see p. 110)

$$g_n = \sum_{k=0}^{n} \binom{n}{k} \alpha_k f_{n-k} \quad \text{iff} \quad f_n = \sum_{k=0}^{n} \binom{n}{k} \gamma_k g_{n-k} \ ,$$

where the exponential generating functions $A_E(x)$, $C_E(x)$ of sequences α_n, γ_n, respectively, have to satisfy the relation $A_E(x) C_E(x) = 1$. Setting $\alpha_n = (-1)^{[n\neq 0]}$ and $\gamma_n = B_n$, where B_n is the n–th ordered Bell number, we obtain

$$g_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{[k\neq 0]} f_{n-k} \quad \text{iff} \quad f_n = \sum_{k=0}^{n} \binom{n}{k} B_k g_{n-k} \ ,$$

as the sequence $\{(-1)^{[n\neq 0]}\}_{n=0}^{\infty}$ has the exponential generating function $A_E(x) = 2 - e^x$. We get (i) after simplifications.

Equivalence (ii) we obtain inverting the left-hand side of (ii) by Theorem 1 and using Lemma 7 by the following way

$$\begin{aligned} f_n = g_n - \sum_{j=0}^{n-1} \Bigg(& 2\binom{s-j}{s-n} \\ & + \sum_{l=1}^{n-j-1} (-1)^l \sum_{\substack{k_1,k_2,\dots,k_l \\ j<k_l<\cdots<k_1<n}} 2\binom{s-k_1}{s-n} 2\binom{s-j}{s-k_l} \prod_{i=1}^{l-1} \left(2\binom{s-k_{i+1}}{s-k_i} \right) \Bigg) g_j \\ = g_n - 2\sum_{j=0}^{n-1} \Bigg(& \binom{s-j}{s-n} \\ & + \sum_{l=1}^{n-j-1} \sum_{\substack{k_1,k_2,\dots,k_l \\ j<k_l<k_{l-1}<\cdots<k_1<n}} (-2)^l \binom{s-k_1}{n-k_1} \binom{s-j}{k_l-j} \prod_{i=1}^{l-1} \binom{s-k_{i+1}}{k_i-k_{i+1}} \Bigg) g_j \end{aligned}$$

$$= g_n - 2\sum_{j=0}^{n-1} h_{i-j}(n-j)\, g_j \ .$$

Thus, using the closed form of $\{h_k(n)\}$, which was found in Lemma 6, the assertion follows. □

Proof of Theorem 4. Case (i) is a solution of recurrence (3) obtaned using Theorem 2 and case (ii) follows from Lemma 6 and Lemma 7. □

5. Solving the Recurrence of a Special Sequence

The purpose of this section is to show how Theorem 1 and Theorem 2 can be used to solve the recurrence

$$y_{k+1} = y_k + 2^k\, P_n(k) \ , \tag{10}$$

where $P_n(k)$ is an arbitrary polynomial in the form $P_n(k) = \sum_{i=0}^{n} p_i\, k^{n-i}$. Thus, we can suppose that the solution can be written in the form

$$y_k = 2^k\, Q_n(k) + K \ , \tag{11}$$

where $Q_n(k) = \sum_{i=0}^{n} q_i\, k^{n-i}$ is a polynomial and K is a real number. We need to seek the relation for the computation of coefficients q_i with respect to coefficients p_i. Substituting (11) into (10) and combining terms we have for $i = 0, 1, 2, \ldots, n$

$$p_i = q_i + 2\sum_{j=0}^{i-1} \binom{n-j}{i-j} q_j \ ,$$

which can be inverted by Theorem 3 to the form

$$q_i = p_i + 2\sum_{j=0}^{i-1} (-1)^{i-j} \binom{n-j}{i-j} B_{i-j}\, p_j \ .$$

Hence the explicit formula for y_k is

$$y_k = 2^k \sum_{i=0}^{n} \left(p_i + 2\sum_{j=0}^{i-1} (-1)^{i-j} \binom{n-j}{i-j} B_{i-j}\, p_j \right) k^{n-i} + K \ .$$

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