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# ON THE LEVEL STATISTICAL CONVERGENCE OF A SEQUENCE OF FUZZY NUMBERS

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**Abstract:** In this study we introduce the notion of level statistical boundedness of a fuzzy number sequence and give some main results related to level statistically convergent sequences of fuzzy numbers.

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**Key Words:** fuzzy number, level statistical boundedness, level statistically convergent sequence of fuzzy numbers

## 1. Introduction

The notion of statistical convergence was first introduced by Steinhaus [18] as a generalization of classical convergence and developed by Fast [6]. Statistical convergence has been studied by many authors [5, 7, 11, 15, 16, 17, 19] and many of the classical notions such as cluster points, boundedness, monotonicity of a real sequence have been extended to the setting of statistical convergence.

In 1972, Chang and Zadeh [4] introduced the concept of fuzzy number which is commonly used in fuzzy analysis and in many applications. The convergence of sequences of fuzzy numbers was first defined by Matloka [12], and in 1995, Nuray and Savaş [14] integrated the theory of statistical convergence to the theory of fuzzy numbers. Statistical convergence of sequences of fuzzy numbers has been studied by Kwon [10], Aytar and Pehlivan [1, 2].

In this paper, we define the level statistical boundedness of a sequence of fuzzy numbers and investigate some properties of level statistical convergence

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of sequences of fuzzy numbers.

## 2. Preliminaries

In this section we recall some of the basic notions related to statistical convergence and fuzzy numbers, and refer readers to [1, 7, 8, 9, 13, 20] for more details.

**Definition 2.1.** The natural density of a set $K$ of positive integers is defined by

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n} \left|\{k \leq n : k \in K\}\right| , \tag{2.1}$$

where $|\{k \leq n : k \in K\}|$ denotes the number of elements of $K$ not exceeding $n$ (see [13]).

It is clear that for a finite set $K$, we have $\delta(K) = 0$.

**Notation.** We will be particularly concerned with integer sets having natural density zero. Thus, if $x = (x_n)$ is a sequence such that $x$ satisfies property $P$ for all $n$ except a set of natural density zero, then we say that $x$ satisfies $P$ for "almost all $n$" and we abbreviate this by "a.a.n".

**Definition 2.2.** A real number sequence $(x_n)$ is said to be statistically convergent to $l \in \mathbb{R}$ provided that for every $\varepsilon > 0$ , the set

$$K(\varepsilon) = \{n \in \mathbb{N} : |x_n - l| \geq \varepsilon\}$$

has natural density zero. In this case we write $\mathrm{st} - \lim x_n = l$ and $l$ is called the statistical limit of $(x_n)$.

**Definition 2.3.** A real sequence $(x_n)$ is said to be a statistically Cauchy sequence provided that for every $\varepsilon > 0$ there exists a number $N = N(\varepsilon) \in \mathbb{N}$ such that

$$\delta\left(\{n \in \mathbb{N} : |x_n - x_N| \geq \varepsilon\}\right) = 0\,. \tag{2.2}$$

**Definition 2.4.** Let $(x_n)$ be sequence of real numbers. If there exists a real number $A > 0$ such that

$$\delta\left(\{n \in \mathbb{N} : |x_n| > A\}\right) = 0, \tag{2.3}$$

then we say that $(x_n)$ is statistically bounded.

**Definition 2.5.** A compact real interval

$$X = [\underline{X}, \overline{X}] = \left\{x \in \mathbb{R} : \underline{X} \leq x \leq \overline{X}\right\}$$

is called as an interval number, where $\underline{X}$, $\overline{X}$ $\in \mathbb{R}$ .

The Hausdorff distance between two interval numbers $X$ and $Y$ is defined by

$$d(X,Y)=\max\left\{|\underline{X}-\underline{Y}|,\left|\overline{X}-\overline{Y}\right|\right\}. \tag{2.4}$$

Let $I$ denote the set of all interval numbers. Then it turns out that $(I,\ d)$ is a complete metric space. Interval numbers play an essential role throughout the paper.

**Definition 2.6.** If a function $X:\ \mathbb{R}\to[0,1]$ satisfies:

$(i)$ $X$ is normal, i.e. there exists $t_0\in\mathbb{R}$ such that $X(t_0)=1$.

$(ii)$ $X$ is fuzzy convex, i.e. $\forall t_1,\ t_2\in\mathbb{R}$ and $\forall\lambda\in[0,1]$,

$$X\left(\lambda t_1+(1-\lambda)t_2\right)\geq\min\left\{X(t_1),X(t_2)\right\}.$$

$(iii)$ $X$ is upper semi-continuous.

$(iv)$ The set $X^0$, defined by $X^0=cl\ \{t\in\mathbb{R}\ :X(t)>0\}$ is compact (where $cl$ stands for "closure"), then it is called a fuzzy real number or fuzzy number, briefly.

Given $\alpha\in(0,1]$, the set $X^\alpha=\{t\in\mathbb{R}\ :X(t)\geq\alpha\}$ is called the $\alpha-$level set of $X$ and it is an interval number. The case for $\alpha=0$ is defined by the set $X^0$, which is also an interval number.

The set of all fuzzy numbers is denoted by $L(\mathbb{R})$. A partial order on $L(\mathbb{R})$ is defined by

$$X\preccurlyeq Y\quad\text{iff}\quad\forall\ \alpha\in[0,\ 1]\ \ \underline{X^\alpha}\ \leq\ \underline{Y^\alpha}\text{ and }\overline{X^\alpha}\leq\overline{Y^\alpha}, \tag{2.5}$$

where $X^\alpha=[\underline{X^\alpha},\ \overline{X^\alpha}]$ and $Y^\alpha=[\underline{Y^\alpha},\ \overline{Y^\alpha}]$.

**Theorem 2.7.** (Representation) *Let* $[\underline{X^\alpha}\ ,\ \overline{X^\alpha}]\ ,\ (\alpha\in[0,1])$ *be a given family of interval numbers. If*

$(i)$ $[\underline{X^{\alpha_1}}\ ,\ \overline{X^{\alpha_1}}]\supset[\underline{X^{\alpha_2}}\ ,\ \overline{X^{\alpha_2}}]$ *for all* $0\leq\alpha_1\leq\alpha_2$

*and*

$(ii)$ $\underline{X^{\alpha_k}}\to\underline{X^\alpha}$ *and* $\overline{X^{\alpha_k}}\to\overline{X^\alpha}$ *whenever* $(\alpha_k)$ *is an increasing sequence in* $[0,1]$ *converging to* $\alpha$, *then the family* $[\underline{X^\alpha}\ ,\ \overline{X^\alpha}]$ *represents the* $\alpha-$ *level sets of a fuzzy number X. Conversely, if* $[\underline{X^\alpha}\ ,\ \overline{X^\alpha}]\ ,\ 0\leq\alpha\leq1$ *, are the* $\alpha-$ *level sets of a fuzzy number X, then the conditions* $(i)$ *and* $(ii)$ *are satisfied.*

**Definition 2.8.** Let $X=\{X_n\}$ be a sequence of fuzzy numbers. Then $X$ is said to be statistically monotone increasing if there exists a subset $K=\{n_1<n_2<...\}\subset\mathbb{N}$ such that $\delta(K)=1$ and $X_{n_k}\preccurlyeq X_{n_{k+1}}$ for every $k\in\mathbb{N}$. A statistically monotone decreasing sequence is defined similarly. A statistically monotone increasing or statistically monotone decreasing sequence is called as statistically monotonic.

**Definition 2.9.** Let $X = \{X_n\}$ be a sequence of fuzzy numbers and $X_0 \in L(\mathbb{R})$. If for each $\alpha \in [0,1]$ , $\lim_{n\to\infty} d(X_n^\alpha\ ,\ X_0^\alpha) = 0$ then $X$ is said to be level convergent to $X_0$ and we denote this by $\{X_n\} \xrightarrow{l} X_0$.

The definition above says that $\{X_n\} \xrightarrow{l} X_0$ iff for each $\alpha \in [0,1]$ , $\left(\underline{X_n^\alpha}\right) \to \underline{X_0^\alpha}$ and $\left(\overline{X_n^\alpha}\right) \to \overline{X_0^\alpha}$ as $n \to \infty$.

## 3. Level Statistical Convergence

In this section we examine some properties of level statistical convergence and introduce the notion of level statistical boundedness.

**Definition 3.1.** (see [2]) Let $X = \{X_n\}$ be a sequence of fuzzy numbers and $X_0 \in L(\mathbb{R})$. If for each $\varepsilon > 0$ and each $\alpha \in [0,\ 1]$,

$$\delta\left(\{n \in \mathbb{N}\ : d(X_n^\alpha\ ,\ X_0^\alpha) \geq \varepsilon\}\right) = 0 \tag{3.1}$$

holds, then $\{X_n\}$ is said to be level statistically convergent to $X_0$ , and we denote this by $\{X_n\} \xrightarrow{l-st} X_0$ or $\mathrm{st}_l - \lim\ X_n = X_0$.

The above definition may be restated as follows:

$$\{X_n\} \xrightarrow{l-st} X_0 \Leftrightarrow \forall\alpha \in [0,\ 1]\ \ \mathrm{st} - \lim\ \left(\underline{X_n^\alpha}\right) = \underline{X_0^\alpha} \text{ and } \mathrm{st} - \lim \overline{X_n^\alpha} = \overline{X_0^\alpha}. \tag{3.2}$$

Since the natural density of a finite set is zero, every level convergent sequence is level statistically convergent but the converse is not true in general.

**Example 3.2.** Let $X = \{X_n\}$ be defined as

$$X_n(t) = \begin{cases} t & ,\ t \in [0,\ 1]\,, \\ 1 & ,\ t \in [1,\ 2]\,, \\ -\frac{1}{2}(t-4) & ,t \in [2,\ 4]\,, \\ 0 & ,\ \text{otherwise}\,, \end{cases} \qquad \text{if } n = k^2\,,$$

and

$$X_n(t) = \begin{cases} \frac{n}{2n-1}t & ,\ t \in [0,\ \frac{2n-1}{n})\,, \\ 1 & ,\ t \in [\frac{2n-1}{n},\ \frac{2n+1}{n}]\,, \\ -\frac{n}{2n-1}(t-4) & ,t \in (\frac{2n+1}{n},\ 4]\,, \\ 0 & ,\ \text{otherwise}\,, \end{cases} \qquad \text{if } n \neq k^2\,,$$

where $k \in \mathbb{N}$ . Hence, for each $\alpha \in [0,\ 1]$ we have $\mathrm{st} - \lim \ \underline{X_n^\alpha} = \underline{X_0^\alpha}$ and $\mathrm{st} - \lim \ \overline{X_n^\alpha} = \overline{X_0^\alpha}$ , where

$$X_0(t) = \begin{cases} \frac{1}{2}t\ , & t \in [0,\ 2)\,, \\ -\frac{1}{2}(t-4)\ , & t \in [2,\ 4]\,, \\ 0\ , & \text{otherwise} \end{cases} .$$

Thus, $\{X_n\} \overset{l-st}{\longrightarrow} X_0$. However, $\lim \ \underline{X_n^\alpha}$ and $\lim \ \overline{X_n^\alpha}$ do not exist for any $\alpha \in [0,1]$. Therefore, $\{X_n\}$ is not level convergent.

**Definition 3.3.** Let $X = \{X_n\}$ be a sequence of fuzzy numbers. If for each $\alpha \in [0,1]$ there exists $A = A(\alpha) \in \mathbb{R}^+$ such that

$$\delta\left(\left\{n \in \mathbb{N}\ : \max(\ \left|\underline{X_n^\alpha}\right|,\ \left|\overline{X_n^\alpha}\right|) > A\right\}\right) = 0 \tag{3.3}$$

holds, then $X$ is said to be level statistically bounded.

**Theorem 3.4.** *Let $X = \{X_n\}$ be a sequence of fuzzy numbers. Then $X$ is level statistically bounded iff the real sequences $\left(\underline{X_n^\alpha}\right)$ and $\left(\overline{X_n^\alpha}\right)$ are statistically bounded for each $\alpha \in [0,1]$.*

*Proof.* Let $X = \{X_n\}$ be a level statistically bounded sequence. Consider the set

$$M(\alpha) = \left\{n \in \mathbb{N}\ : \max(\ \left|\underline{X_n^\alpha}\right|,\ \left|\overline{X_n^\alpha}\right|\ ) > A\right\},$$

where $\alpha \in [0,1]$ and $A > 0$ . Then we can write

$$M(\alpha) = \left\{n \in \mathbb{N}\ : \left|\underline{X_n^\alpha}\right| > A\right\} \cup \left\{n \in \mathbb{N}\ : \left|\overline{X_n^\alpha}\right| > A\right\}. \tag{3.4}$$

By assumption, we have $\delta\left(M(\alpha)\right) = 0$ for each $\alpha \in [0,\ 1]$. Hence the sets on the right hand side of (3.4) must have natural density zero. Thus, for each $\alpha \in [0,1]$, $\left(\underline{X_n^\alpha}\right)$ and $\left(\overline{X_n^\alpha}\right)$ are statistically bounded. The converse implication is clear from (3.4). □

As can be seen by the following example, the monotone convergence theorem that is valid for statistically convergent sequences of real numbers does not hold generally in the setting of level statistical convergence, i.e. a statistically monotonic and level statistically bounded sequence of fuzzy numbers may not be level statistically convergent.

**Example 3.5.** Define the sequence $\{X_n\}$ of fuzzy numbers as

$$X_n(t) = \begin{cases} t - \sqrt{n} + 1\ , & t \in [\sqrt{n} - 1,\ \sqrt{n}]\,, \\ 0, & \text{otherwise}\,, \end{cases} \qquad \text{if } n = k^2\,,$$

and

$$X_n(t) = \begin{cases} 1 + \frac{t-1}{n}\,, & t \in [0,\ 1]\,, \\ 0, & \text{otherwise}\,, \end{cases} \qquad \text{if } n \neq k^2\,,$$

where $k \in \mathbb{N}$.

It is easy to see that $\{X_n\}$ is statistically monotone decreasing and level statistically bounded but is not level statistically convergent since $\mathrm{st-lim}\ \underline{X_n^1} = 1 \neq \underline{X_0^1}$.

**Definition 3.6.** (see [2]) Let $X = \{X_n\}$ be a sequence of fuzzy numbers. If for each $\varepsilon > 0$ and for each $\alpha \in [0,1]$ there exists an $N = N(\varepsilon) \in \mathbb{N}$ such that

$$\delta\left(\{n \in \mathbb{N}\ : d(X_n^\alpha\ ,\ X_N^\alpha) \geq \varepsilon\}\right) = 0 \tag{3.5}$$

holds, then $\{X_n\}$ is said to be a level statistically Cauchy sequence.

It can easily be shown that $X$ is level statistically Cauchy iff for each $\alpha \in [0,1]$ the real sequences $\left(\underline{X_n^\alpha}\right)$ and $\left(\overline{X_n^\alpha}\right)$ are statistically Cauchy.

**Theorem 3.7.** *Let $X = \{X_n\}$ be a sequence of fuzzy numbers. If $X$ is level statistically convergent then it is level statistically Cauchy.*

*Proof.* Let $\{X_n\} \xrightarrow{l-st} X_0$ and $\varepsilon > 0$. Then for each $\alpha \in [0,1]$ and for almost all $n$ , we can write $d(X_n^\alpha\ ,\ X_0^\alpha) < \varepsilon/2$. Also we can choose an $N = N(\varepsilon) \in \mathbb{N}$ which is large enough such that $d(X_N^\alpha,\ X_0^\alpha) < \varepsilon/2$. Hence for almost all $n$ and each $\alpha \in [0,1]$ we get

$$d(X_n^\alpha\ ,\ X_N^\alpha) \leq d(X_n^\alpha\ ,\ X_0^\alpha) + d(X_0^\alpha\ ,\ X_N^\alpha) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

This shows that $\{X_n\}$ is level statistically Cauchy. □

**Remark.** The converse of this theorem does not hold in general. Let us consider the sequence $\{X_n\}$ given in Example 3.5. We see that for each $\alpha \in [0,1]$, $\left(\underline{X_n^\alpha}\right)$ and $\left(\overline{X_n^\alpha}\right)$ are statistically Cauchy, hence $\{X_n\}$ is level statistically Cauchy but is not level statistically convergent.

**Theorem 3.8.** *If a sequence $\{X_n\}$ of fuzzy numbers is level statistically Cauchy, then it is level statistically bounded.*

*Proof.* Let $\{X_n\}$ be a level statistically Cauchy sequence of fuzzy numbers. Then for every $\varepsilon > 0$ and each $\alpha \in [0,\ 1]$ there exists $N = N(\varepsilon) \in \mathbb{N}$ such that $d(X_n^\alpha\ ,\ X_N^\alpha) < \varepsilon$ for a.a.n., i.e. $\forall \varepsilon > 0$ and $\forall \alpha \in [0,\ 1]$ $\exists N = N(\varepsilon) \in \mathbb{N}$ such that $\left|\underline{X_n^\alpha} - \underline{X_N^\alpha}\right| < \varepsilon$ and $\left|\overline{X_n^\alpha} - \overline{X_N^\alpha}\right| < \varepsilon$ for a.a.n. We can write

$$\left|\underline{X_n^\alpha}\right| \leq \left|\underline{X_n^\alpha} - \underline{X_N^\alpha}\right| + \left|\underline{X_N^\alpha}\right| \tag{3.6}$$

and

$$\left|\overline{X_n^\alpha}\right| \leq \left|\overline{X_n^\alpha} - \overline{X_N^\alpha}\right| + \left|\overline{X_N^\alpha}\right|\ , \tag{3.7}$$

$\forall n \in \mathbb{N}$ and $\forall \alpha \in [0,\ 1]$ . Similarly, there exists a $B \in \mathbb{R}^+$ such that $\max\left\{\left|\underline{X_N^\alpha}\right|,\ \left|\overline{X_N^\alpha}\right|\right\} \leq B$. Thus, we get for each $\alpha \in [0,\ 1]$ ,

$$\max\left\{\left|\underline{X_n^\alpha}\right|,\ \left|\overline{X_n^\alpha}\right|\right\} \leq \max\left\{\left|\underline{X_n^\alpha} - \underline{X_N^\alpha}\right|,\ \left|\overline{X_n^\alpha} - \overline{X_N^\alpha}\right|\right\} + \max\left\{\left|\underline{X_N^\alpha}\right|,\ \left|\overline{X_N^\alpha}\right|\right\} < \varepsilon + B\,,$$

for a.a.n., which implies that $\left(\underline{X_n^\alpha}\right)$ and $\left(\overline{X_n^\alpha}\right)$ are statistically bounded sequences of real numbers. Hence by Theorem 3.4 we get $\{X_n\}$ is level statistically bounded. □

**Corollary 3.9.** *If a sequence* $\{X_n\}$ *of fuzzy numbers is level statistically convergent, then it is level statistically bounded.*

## References


[1] S. Aytar, S. Pehlivan, Statistically monotonic and statistically bounded sequences of fuzzy numbers, *Information Sciences*, **176** (2006), 734-744.

[2] S. Aytar, S. Pehlivan, Statistical convergence of sequences of fuzzy numbers and sequences of $\alpha$-cuts, *Int. J. General Systems*, To Appear.

[3] R.C. Buck, Generalized asymptotic density, *Amer. J. Math.*, **75** (1953), 335-346.

[4] S.L. Chang, L.A. Zadeh, On fuzzy mapping and control, *IEEE Trans. Syst. Man Cybern*, **2** (1972), 30-34.

[5] J.S. Connor, The statistical and strong p-Cesaro convergence of sequences, *Analysis*, **8** (1988), 47-63.

[6] H. Fast, Sur la convergence statistique, *Coll. Math.*, **2** (1951), 241-244.

[7] J.A. Fridy, On statistical convergence, *Analysis*, **5** (1985), 301-313.

[8] J.A. Fridy, C. Orhan, Statistical limit superior and limit inferior, *Proc. Amer. Math. Soc.*, **125** (1997),3625.

[9] O. Kaleva, On the convergence of fuzzy sets, *Fuzzy Sets and Systems*, **17** (1985), 53-65.

[10] J.S. Kwon, On statistical and p-Cesaro convergence of fuzzy numbers, *Korean J.C.A.M.*, **3** (2000), 95-203.

[11] I.J. Maddox, Statistical convergence in a locally convex sequence space, *Math. Proc. Camp. Phil. Soc.* **104** (1988), 141.

[12] M. Matloka, Sequences of fuzzy numbers, *BUSEFAL*, **28** (1986), 28-37.

[13] I. Niven, H.S. Zuckerman, H.L. Montgomery, *An Introduction to the Theory of Numbers*, John Wiley, New York(1991).

[14] F. Nuray, E. Savaş, Statistical convergence of sequences of fuzzy numbers, *Math. Slovaca*, **3** (1995), 269-273.

[15] S. Pehlivan, M.A. Mamedov, Statistical cluster point and turnpike, *Optimization*, **48** (2000), 93-106.

[16] T. Salat, On statistically convergent sequences of real numbers, *Math. Slovaca*, **30** (1980), 139-150.

[17] I.J. Schoenberg, The integrability of certain functions and related summability methods, *Amer. Math. Monthly*, **66** (1959), 361-375.

[18] H. Steinhaus, Sur la convergence ordinarie et la convergence asymptotique, *Colloq. Math.* ,**2** (1951), 73-74.

[19] B.C. Tripathy, On statistically convergent and statistically bounded sequences, *Bull. Malays. Math. Soc.*, **20** (1997), 31-33.

[20] C. Wu, G. Wang, Convergence of sequences of fuzzy numbers and fixed point theorems for increasing fuzzy mappings and application, *Fuzzy Sets and Systems*, **130** (2002), 383-390.