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ON SOME SUBCLASSES OF A NEW CLASS OF ANALYTIC FUNCTIONS

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Abstract: In this present investigation, the authors obtain certain argument estimate properties and Fekete-Szegö inequality for a new class of analytic functions. The motivation of this paper is to give a generalization of the Fekete-Szegö inequalities obtained by Dinggong.

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1. Introduction

Let $\mathcal{A}$ denote the class of all *analytic* functions $f(z)$ of the form

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$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in \Delta := \{z \in C \,|\, |z| < 1\}), \tag{1}$$

and $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent functions.

Let $S^*(\alpha, \beta)$ be the class of all functions $f(z) \in \mathcal{A}$ satisfying

$$-\frac{\beta\pi}{2} < \arg \frac{zf'(z)}{f(z)} < \frac{\alpha\pi}{2} \qquad (z \in \Delta; 0 < \alpha \leq 1, 0 < \beta \leq 1).$$

This class was introduced by Takahashi and Nunokawa [10]. We note that $S^*(\alpha, \alpha) =: SS^*(\alpha)$ is the familiar class of *strongly starlike functions of order* α.

Let $\phi(z)$ be an analytic function with positive real part on Δ with $\phi(0) = 1$, $\phi'(0) > 0$ which maps the unit disk Δ onto a region starlike with respect to 1 which is symmetric with respect to the real axis. Let $S^*(\phi)$ be the class of functions in $f \in \mathcal{S}$ for which

$$\frac{zf'(z)}{f(z)} \prec \phi(z) \qquad (z \in \Delta),$$

and $C(\phi)$ be the class of functions in $f \in \mathcal{S}$ for which

$$1 + \frac{zf''(z)}{f'(z)} \prec \phi(z) \qquad (z \in \Delta),$$

where $\prec$ denotes the subordination between analytic functions. These classes were introduced and studied by Ma and Minda [2]. They have obtained the Fekete-Szegö inequality for the functions in the class $C(\phi)$. Since $f \in C(\phi)$ if and only if $zf'(z) \in S^*(\phi)$, we get the Fekete-Szegö inequality for functions in the class $S^*(\phi)$. Dinggong introduced and studied many subordination results and also obtained Fekete-Szegö inequality for the class $H(\alpha, A, B)$, where $-1 \leq B < A < 1$ defined by

$$f'(z) + \alpha z f''(z) \prec \frac{1 + Az}{1 + Bz} \qquad (\alpha \geq 0).$$

Motivated by the following works we define the following classes.

Definition 1. Let $M(\alpha, \beta, \gamma)$ be the class of all functions $f \in \mathcal{A}$ such that

$$-\frac{\beta\pi}{2} < \arg\left(f'(z) + \alpha z f''(z)\right) < \frac{\gamma\pi}{2}$$
$$(z \in \Delta;\ \alpha \geq 0;\ 0 < \gamma \leq 1;\ 0 < \beta \leq 1)$$

implies

$$-\frac{\beta\pi}{2} < \arg\left(f'(z)\right) < \frac{\gamma\pi}{2} \qquad (z \in \Delta;\ 0 < \gamma \leq 1;\ 0 < \beta \leq 1).$$

Definition 2. Let $\phi(z)$ be a univalent starlike function with respect to 1 which maps the unit disk Δ onto a region in the right half plane which is symmetric with respect to the real axis, $\phi(0) = 1$ and $\phi'(0) > 0$. A function $f \in \mathcal{A}$ is in the class $M_\alpha(\phi)$ if

$$f'(z) + \alpha z f''(z) \prec \phi(z) \quad (\alpha \geq 0).$$

We need the following result of Nunokawa, Owa, Saitoh, Cho and Takahashi [3] (see [10]) to prove our main result.

Lemma 3. *Let $p(z)$ be analytic in Δ with $p(0) = 1$ and $p(z) \neq 0$. If there exists two points $z_1, z_2 \in \Delta$ such that*

$$-\frac{\beta\pi}{2} = \arg p(z_1) < \arg p(z) < \arg p(z_2) = \frac{\alpha\pi}{2}\,, \tag{2}$$

for $\alpha > 0$, $\beta > 0$, and for $|z| < |z_1| = |z_2|$, then we have

$$\frac{z_1 p'(z_1)}{p(z_1)} = -i\frac{\alpha + \beta}{2} m\,, \tag{3}$$

and

$$\frac{z_2 p'(z_2)}{p(z_2)} = i\frac{\alpha + \beta}{2} m\,, \tag{4}$$

where

$$m \geq \frac{1 - |a|}{1 + |a|} \quad \text{and} \quad a := i \tan\left(\frac{\pi}{4}\frac{\alpha - \beta}{\alpha + \beta}\right).$$

In the present paper, an interesting result using the argument estimates for the class $M(\alpha, \beta, \gamma)$ is obtained. Also, we obtain the Fekete-Szegö inequality for functions in a more general class $M_\alpha(\phi)$ of functions. Applications of our results to certain functions defined through convolution (or Hadamard product) and in particular functions defined by fractional derivatives is also obtained. The motivation of this paper is to give a generalization of the Fekete-Szegö inequalities of Dinggong [1].

2. Sufficient Condition

By making use of Lemma 3, we first prove the following theorem.

Theorem 4. *If*

$$-\frac{\beta\pi}{2} - \tan^{-1}\left(\frac{1-|a|}{1+|a|}\frac{(\gamma+\beta)}{2}\right) < \arg\left(f'(z)+\alpha z f''(z)\right) < \frac{\gamma\pi}{2} + \tan^{-1}\left(\frac{1-|a|}{1+|a|}\frac{(\gamma+\beta)}{2}\right), \tag{5}$$

for $\alpha, \beta > 0$, then $f(z) \in M(\alpha,\beta,\gamma)$.

Proof. Define the function $p(z)$ by

$$p(z) := f'(z). \tag{6}$$

Then a computation shows that

$$p(z) + \alpha z p'(z) = f'(z) + \alpha f''(z).$$

Assume that there exists points z_1 and z_2 such that the inequality (2) holds. Then by Lemma 3, (3) and (4) holds where m and a are as in Lemma 3. By making use of (3) and (4), we have

$$\begin{aligned}\arg(p(z_1)+\alpha z_1 p'(z_1)) &= \arg p(z_1) + \arg\left(1+\alpha\frac{z_1 p'(z_1)}{p(z_1)}\right)\\ &= -\frac{\beta\pi}{2} - \tan^{-1}\left(\frac{(\gamma+\beta)\alpha}{2}m\right) \le -\frac{\beta\pi}{2} - \tan^{-1}\left(\frac{1-|a|}{1+|a|}\frac{(\gamma+\beta)\alpha}{2}\right),\end{aligned}$$

and

$$\arg(p(z_2)+\alpha z_2 p'(z_2)) \ge -\frac{\gamma\pi}{2} - \tan^{-1}\left(\frac{1-|a|}{1+|a|}\frac{(\gamma+\beta)\alpha}{2}\right),$$

which contradicts (5). This completes the proof of the theorem. □

3. Fekete-Szegö Problem

To prove our main result in this section, we need the following lemma due to Ravichandran et al [6].

Lemma 5. *If $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is a function with positive real part, then*

$$|c_2 - \mu c_1^2| \le 2\max\{1, |2v-1|\},$$

and the result is sharp for the functions given by

$$p(z) = \frac{1+z^2}{1-z^2}, \quad p(z) = \frac{1+z}{1-z}.$$

Theorem 6. *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$. If $f(z)$ given by (1) belongs to $M_\alpha(\phi)$, then*

$$|a_3 - \mu a_2^2| \leq \frac{B_1}{3(1+2\alpha)} \max\left\{1, \left|-\frac{B_2}{B_1} + \frac{3B_1\mu(1+2\alpha)}{4(1+\alpha)^2}\right|\right\}.$$

The result is sharp.

Proof. If $f(z) \in M_\alpha(\phi)$, then there is a Schwarz function $w(z)$, analytic in Δ with $w(0) = 0$ and $|w(z)| < 1$ in Δ such that

$$f'(z) + \alpha z f''(z) = \phi(w(z)). \tag{7}$$

Define the function $p_1(z)$ by

$$p_1(z) := \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + \cdots. \tag{8}$$

Since $w(z)$ is a Schwarz function, we see that $\Re p_1(z) > 0$ and $p_1(0) = 1$. Define the function $p(z)$ by

$$p(z) := f'(z) + \alpha z f''(z) = 1 + b_1 z + b_2 z^2 + \cdots. \tag{9}$$

In view of the equations (7), (8), (9), we have

$$p(z) = \phi\left(\frac{p_1(z)-1}{p_1(z)+1}\right). \tag{10}$$

Since

$$\frac{p_1(z)-1}{p_1(z)+1} = \frac{1}{2}\left[c_1 z + (c_2 - \frac{c_1^2}{2})z^2 + (c_3 + \frac{c_1^3}{4} - c_1 c_2)z^3 + \cdots\right],$$

and therefore

$$\phi\left(\frac{p_1(z)-1}{p_1(z)+1}\right) = 1 + \frac{1}{2}B_1 c_1 z + \left[\frac{1}{2}B_1(c_2 - \frac{1}{2}c_1^2) + \frac{1}{4}B_2 c_1^2\right] z^2 + \cdots,$$

from this equation (10), we obtain

$$b_1 = \frac{1}{2}B_1 c_1,$$

and

$$b_2 = \frac{1}{2}B_1(c_2 - \frac{1}{2}c_1^2) + \frac{1}{4}B_2 c_1^2.$$

Since $f'(z) + \alpha z f''(z) = 1 + 2a_2(1+\alpha)z + 3a_3(1+2\alpha)z^2 + \cdots$, from the equation

(9), we see that

$$b_1 = 2a_2(1+\alpha)\,, \tag{11}$$

$$b_2 = 3a_3(1+2\alpha)\,, \tag{12}$$

or equivalently we have

$$a_2 = \frac{b_1}{2(1+\alpha)} = \frac{B_1 c_1}{4(1+\alpha)},$$

$$a_3 = \frac{b_2}{3(1+2\alpha)} = \frac{B_1}{6(1+2\alpha)}\left[c_2 - c_1^2\left\{\frac{1}{2} - \frac{B_2}{2B_1}\right\}\right].$$

Therefore we have

$$a_3 - \mu a_2^2 = \frac{bB_1}{4}\left\{c_2 - vc_1^2\right\}, \tag{13}$$

where

$$v := \frac{1}{2}\left[1 - \frac{B_2}{B_1} + (2\mu - 1)bB_1\right].$$

Our result now follows by an application of Lemma 5. □

Corollary 7. *Taking* $\phi(z) = \frac{1+Az}{1+Bz}$, $-1 \le B < A < 1$, *we get the results obtained by Dinggong [1].*

4. Applications to Functions Defined by Fractional Derivatives

In order to introduce the class $M_\alpha^\lambda(\phi)$, we need the following definition.

Definition 8. (see [5], [4], see also [8], [9]) Let f be analytic in a simply connected region of the z-plane containing the origin. The *fractional derivative* of f of order λ is defined by

$$D_z^\lambda f(z) := \frac{1}{\Gamma(1-\lambda)}\frac{d}{dz}\int_0^z \frac{f(\zeta)}{(z-\zeta)^\lambda}d\zeta \qquad (0 \le \lambda < 1)\,,$$

where the multiplicity of $(z-\zeta)^\lambda$ is removed by requiring that $\log(z-\zeta)$ is real for $z - \zeta > 0$.

Using Definition 8 and its known extensions involving fractional derivatives and fractional integrals, Owa and Srivastava [5] introduced the operator $\Omega^\lambda : \mathcal{A} \to \mathcal{A}$ defined by

$$(\Omega^\lambda f)(z) = \Gamma(2-\lambda)z^\lambda D_z^\lambda f(z) \qquad (\lambda \ne 2, 3, 4, \ldots).$$

The class $M_{\alpha}^{\lambda}(\phi)$ consists of functions $f \in \mathcal{A}$ for which $\Omega^{\lambda} f \in M_{\alpha}(\phi)$. Note that $M_{\alpha}^{\lambda}(\phi)$ is the special case of the class $M_{\alpha}^{g}(\phi)$ when

$$g(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} z^{n}. \tag{14}$$

Let $g(z) = z + \sum_{n=2}^{\infty} g_n z^n$ $(g_n > 0)$. Since $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in M_{\alpha}^{g}(\phi)$ if and only if $(f * g) = z + \sum_{n=2}^{\infty} g_n a_n z^n \in M_{\alpha}(\phi)$, we obtain the coefficient estimate for functions in the class $M_{\alpha}^{g}(\phi)$, from the corresponding estimate for functions in the class $M_{\alpha,\beta}(\phi)$. Applying Theorem 6 for the function $(f*g)(z) = z + g_2 a_2 z^2 + g_3 a_3 z^3 + \cdots$, we get the following Theorem 9 after an obvious change of the parameter μ.

Theorem 9. *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$. If $f(z)$ given by (1) belongs to $M_{\alpha}^{g}(\phi)$, then*

$$|a_3 - \mu a_2^2| \leq \frac{B_1}{3 g_3 (1+2\alpha)} \max\left\{1, \left| -\frac{B_2}{B_1} + \frac{3 B_1 \mu g_3 (1+2\ \alpha)}{g_2^2 4 (1+\alpha)^2} \right| \right\}.$$

The result is sharp.

Since

$$(\Omega^{\lambda} f)(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} a_n z^n,$$

we have

$$g_2 := \frac{\Gamma(3)\Gamma(2-\lambda)}{\Gamma(3-\lambda)} = \frac{2}{2-\lambda}, \tag{15}$$

and

$$g_3 := \frac{\Gamma(4)\Gamma(2-\lambda)}{\Gamma(4-\lambda)} = \frac{6}{(2-\lambda)(3-\lambda)}. \tag{16}$$

For g_2 and g_3 given by (15) and (16), Theorem 9 reduces to the following theorem.

Theorem 10. *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$. If $f(z)$ given by (1) belongs to $M_{\alpha}^{g}(\phi)$, then*

$$|a_3 - \mu a_2^2| \leq \frac{(2-\lambda)(3-\lambda)}{18} \frac{B_1}{(1+2\alpha)} \max\left\{1, \left| -\frac{B_2}{B_1} + \frac{9 B_1 \mu (2-\lambda)(1+2\alpha)}{8(3-\lambda)(1+\alpha)^2} \right| \right\}.$$

The result is sharp.

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