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ON THE MINIMAL FREE RESOLUTION OF GENERAL UNIONS OF FAT POINTS IN $\mathbf{P}^2$

E. Ballico

Department of Mathematics
University of Trento
380 50 Povo (Trento) - Via Sommarive, 14, ITALY
e-mail: ballico@science.unitn.it

Abstract: Here we prove the following result. Fix integers $s > 0$, $m_1 > \cdots > m_s > 0$, $a_i > 0$, $1 \le i \le s$. Take $\sum_{i=1}^{s} a_i$ general points $P_{i,j} \in \mathbf{P}^2$, $1 \le i \le s$, $1 \le j \le a_i$ and set $Z := \cup_{i=1}^{s} \cup_{j=1}^{a_i} m_i P_{i,j} \subset \mathbf{P}^2$ and $\delta := \sum_{i=1}^{s} a_i \binom{m_i}{2}$. Let k be the minimal integer such that $\delta \le \binom{k+2}{2}$. Assume $m_s = 1$ and either $a_s \ge km(m_1+2)(m_1+1)/2 + m_1 k$ or $m_{s-1} = 2$, $m_s \ge k$ and $2a_{s-1} + a_s \ge km(m_1+2)(m_1+1)/2 + m_1 k$ or $\mathrm{char}(\mathbb{K}) \ne 2$, $m_{s-1} = 2$, $2a_{s-1} + a_s \ge km(m_1 + 2)(m_1+1)/2 + m_1 k$ and $m_s \ge 2$. Then $h^0(\mathbf{P}^2, \mathcal{I}_Z(t)) = 0$ for all $t < k$, $h^1(\mathbf{P}^2, \mathcal{I}_Z(t)) = 0$ for all $t \ge k$ and the homogeneous ideal of Z is minimally generated by $\binom{k+2}{2} - \delta$ forms of degree k and by $\max\{0, 2\delta - k(k+2)\}$ forms of degree $k+1$.

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1. Introduction

Here we use a key lemma proved in [4], i.e. [4], Proposition 2.6, to prove the following result.

Theorem 1. *Fix integers $s > 0$, $m_1 > \cdots > m_s > 0$, $a_i > 0$, $1 \le i \le s$. Take $\sum_{i=1}^{s} a_i$ general points $P_{i,j} \in \mathbf{P}^2$, $1 \le i \le s$, $1 \le j \le a_i$ and set $Z := \cup_{i=1}^{s} \cup_{j=1}^{a_i} m_i P_{i,j} \subset \mathbf{P}^2$ and $\delta := \sum_{i=1}^{s} a_i \binom{m_i}{2}$. Let k be the minimal integer such that $\delta \le \binom{k+2}{2}$. Assume $m_s = 1$ and either $a_s \ge km(m_1+2)(m_1+1)/2 + m_1 k$*

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or $m_{s-1} = 2$, $m_s \geq k$ *and* $2a_{s-1} + a_s \geq km(m_1+2)(m_1+1)/2 + m_1k$ *or* *char*$(\mathbb{K}) \neq 2$, $m_{s-1} = 2$, $2a_{s-1}+a_s \geq km(m_1+2)(m_1+1)/2+m_1k$ *and* $m_s \geq 2$. *Then* $h^0(\mathbf{P}^2, \mathcal{I}_Z(t)) = 0$ *for all* $t < k$, $h^1(\mathbf{P}^2, \mathcal{I}_Z(t)) = 0$ *for all* $t \geq k$ *and the homogeneous ideal of* Z *is minimally generated by* $\binom{k+2}{2} - \delta$ *forms of degree* k *and by* $\max\{0, 2\delta - k(k+2)\}$ *forms of degree* $k+1$.

The assertions on $\mathcal{I}_Z(t)$ in the statement of Theorem 1 are equivalent to say that Z has maximal rank. The assertion on the homogeneous ideal of Z in the statement of Theorem 1 is equivalent to say that the minimal free resolution of Z is the expected one (see e.g. [2] or [3]).

Remark 1. We work over an algebraically closed field $\mathbb{K}$ such that either char$(\mathbb{K}) = 0$ or char$(\mathbb{K}) = 0$. The differential Horace Lemma for line bundles (see [1], Lemma 2.1) is proved and stated without any restriction on char$(\mathbb{K})$. In [4] there is a blanket char$(\mathbb{K}) = 0$ assumption. In [4] the char$(\mathbb{K}) = 0$ assumption is used in [4] only in the proof of Proposition 2.6 (the key result needed for our paper) making a map $t_i \mapsto t_i^{r_i}$, $1 \leq i \leq n$, between formal power series rings. To get the injectivity of the differential of this map at $(0, \dots, 0)$ it is sufficient to assume char$(\mathbb{K}) > r_i$ for all i. Here we will use it only for double points for the last set of assumptions of Theorem 1 and hence it is sufficient to assume char$(\mathbb{K}) \neq 2$ when we want to use those assumptions.

2. Proof of Theorem 1

Remark 2. From the dual of the Euler's sequence of $T\mathbf{P}^2$ we get $h^0(\mathbf{P}^2, \Omega^1{}_{\mathbf{P}^2}(t+1)) = t(t+2)$ for all $t \geq 2$, $h^0(\mathbf{P}^2, \Omega^1{}_{\mathbf{P}^2}(t+1)) = 0$ for all $t \leq 0$, $h^0(\mathbf{P}^2, \Omega^1{}_{\mathbf{P}^2}(2)) = 3$, $h^1(\mathbf{P}^2, \Omega^1{}_{\mathbf{P}^2}(t+1)) = 0$ for all $t \neq -1$ and $h^1(\mathbf{P}^2, \Omega^1{}_{\mathbf{P}^2}) = 1$.

Remark 3. Let $B \subset \mathbf{P}^2$ be a zero-dimensional scheme and t a positive integer. We recall that $h^1(\mathbf{P}^2, \mathcal{I}_B \otimes \Omega^1{}_{\mathbf{P}^2}(t+1)) = h^1(\mathbf{P}^2, \mathcal{I}_B(t)) = 0$ if and only if the minimal free resolution of B is in degree at most t (see e.g. [3]).

Remark 4. Let $D \subset \mathbf{P}^n$ be a rational normal curve. Then $\Omega^1{}_{\mathbf{P}^n}|D$ is isomorphic to the direct sum of n line bundles of degree $-n-1$ ([2], Lemma 1.3). Now assume $n = 2$. Hence D is a smooth conic. For all integers $m \geq 2$ the vector bundle $\Omega^1{}_{\mathbf{P}^2}(m)|D$ is isomorphic to the direct sum of 2 line bundles of degree $2m-3$. Hence $h^0(D, \mathcal{I}_{B,D} \otimes (\Omega^1{}_{\mathbf{P}^2}(m)|D)) = h^1(D, \mathcal{I}_{B,D} \otimes (\Omega^1{}_{\mathbf{P}^2}(m)|D)) = 0$ for every zero-dimensional scheme $B \subset D$ such that length$(B) = 2m-2$. Notice that $2m-2$ is even for every integer m.

Remark 5. Let X be an integral projective scheme, E a vector bundle on X, $Z \subset X$ a closed subscheme and $D \subset X$ an effective Cartier divisor. The residual scheme $\mathrm{Res}_D(Z)$ of Z with respect to D is the closed subscheme of X with $\mathrm{Hom}(\mathcal{I}_D, \mathcal{I}_Z)$ as its ideal sheaf. For instance, if $P \in D_{reg}$ and $m > 0$, then $\{mP, X\}|D = \{mP, D\}$ and $\mathrm{Res}_D(\{mP, X\}) = \{(m-1)P, X\}$, with the convention $\{0P, X\} := \emptyset$. We have an exact sequence on X:

$$0 \to \mathcal{I}_{\mathrm{Res}_{D(X)}}(-D) \to \mathcal{I}_Z \to \mathcal{I}_{Z\cap D,D} \to 0\,. \tag{1}$$

By tensoring the exact sequence (1) with E we obtain the exact sequence

$$0 \to \mathcal{I}_{\mathrm{Res}_{D(Z)}} \otimes E(-D) \to \mathcal{I}_Z \otimes E \to \mathcal{I}_{Z\cap D,D} \otimes (E|D) \to 0\,. \tag{2}$$

Hence:

$$h^0(X, \mathcal{I}_Z \otimes E) \leq h^0(X, \mathcal{I}_{\mathrm{Res}_{D(Z)}} \otimes E(-D)) + h^0(D, \mathcal{I}_{Z\cap D,D} \otimes (E|D)),$$

$$h^1(X, \mathcal{I}_Z \otimes E) \leq h^1(X, \mathcal{I}_{\mathrm{Res}_{D(Z)}} \otimes E(-D)) + h^1(D, \mathcal{I}_{Z\cap D,D} \otimes (E|D)).$$

Notation 1. Let X be an integral variety, D an effective Cartier divisor of X, $P \in D_{reg}$ and $Z \subset X$ a zero-dimensional scheme such that either $Z = \emptyset$ or $Z_{red} = \{P\}$. The associated sequence $(m_1(Z,D,P), m_2(Z,D,P), \dots$ of Z at P with respect to D is defined inductively by the rules $m_1(Z,D,P) := \mathrm{length}(Z \cap D)$ and $m_i(Z,D,P) := m_{i-1}(\mathrm{Res}_D(Z), D, P)$ for all $i \geq 2$.

Remark 6. Let $C \subset \mathbf{P}^2$ be a smooth conic, $P \in C$, $Z \subset \mathbf{P}^2 X$ a zero-dimensional scheme such that $P \notin Z_{red}$ and m, i integers such that $0 \leq i \leq m$. By [4], Proposition 2.6, to prove $h^1(X, \mathcal{I}_{Z\cup mQ} \otimes \Omega^1{}_{\mathbf{P}^2}(t+1)) = 0$ (resp. $h^1(X, \mathcal{I}_{Z\cup mQ} \otimes \Omega^1{}_{\mathbf{P}^2}(t+1)) = 0$) it is sufficient to prove: $h^1(X, \mathcal{I}_{\mathrm{Res}_{D(Z)\cup A}} \otimes \Omega^1{}_{\mathbf{P}^2}(t+1)(-C)) = h^1(C, \mathcal{I}_{Z\cap C, C\cup\{iP,C\}} \otimes (\Omega^1{}_{\mathbf{P}^2}(t+1)|C)) = 0$ (resp. $h^0(X, \mathcal{I}_{\mathrm{Res}_{C(Z)\cup A}} \otimes \Omega^1{}_{\mathbf{P}^2}(t+1)(-C)) = h^0(C, \mathcal{I}_{Z\cap C,C} \otimes (\Omega^1{}_{\mathbf{P}^2}(t+1)|C)) = 0$), where A is suitable zero-dimensional subscheme of X such that $A_{red} = \{P\}$ and its associated sequence at P with respect to D is the same as the one of $\{mP, X\}$ (i.e. $(m, m-1, \dots)$, except that it omits the term "i". Furthermore, A is vertically graded in the sense of [4], Section 2. Since A is vertically graded, by [4], Proposition 2.6, we may do another step (from $\Omega^1{}_{\mathbf{P}^2}(t+1)(-C)$ ot $\Omega^1{}_{\mathbf{P}^2}(t+1)(-2C)$) to reduce to a statement concerning $E(-2C)$ for a vertically graded scheme A' obtained form A deleting one of its rows. For Theorem 1 we need only the case $m = 2$.

Remark 7. Fix an integer $t > 0$ and a zero-dimensional scheme $Z \subset \mathbb{P}^2$. If $h^1(\mathbb{P}^2, \mathcal{I}_Z(t)) = 0$, then $h^1(\mathbb{P}^2, \mathcal{I}_Z(y)) = 0$ for all integers $y > t$. If $h^0(\mathbb{P}^2, \mathcal{I}_Z(t)) = 0$, then $h^0(\mathbb{P}^2, \mathcal{I}_Z(y)) = 0$ for all integers $y < t$. By the cohomology of $\Omega^1{}_{\mathbf{P}^2}(x)$, $x \in \mathbb{Z}$, we also get that if $h^1(\mathbb{P}^2, \mathcal{I}_Z \otimes \Omega^1{}_{\mathbf{P}^2}(t)) = 0$, then

$h^1(\mathbb{P}^2, \mathcal{I}_Z \otimes \Omega^1{}_{\mathbf{P}^2}(y)) = 0$ for all integers $y > t$, while if $h^0(\mathbb{P}^2, \mathcal{I}_Z \otimes \Omega^1{}_{\mathbf{P}^2}(t)) = 0$, then $h^0(\mathbb{P}^2, \mathcal{I}_Z \otimes \Omega^1{}_{\mathbf{P}^2}(y)) = 0$ for all integers $y < t$. By a twist of the Euler's exact sequence we get that if $h^1(\mathbb{P}^2, \mathcal{I}_Z \otimes \Omega^1{}_{\mathbf{P}^2}(t+1)) = 0$, then $h^1(\mathbb{P}^2, \mathcal{I}_Z(t)) = 0$. Since $\Omega^1{}_{\mathbf{P}^2}(2)$ is spanned, if $h^0(\mathbb{P}^2, \mathcal{I}_Z \otimes \Omega^1{}_{\mathbf{P}^2}(t+1)) = 0$, then $h^0(\mathbb{P}^2, \mathcal{I}_Z(t-1)) = 0$.

Proof of Theorem 1. In parts (a), (b) and (c) we will do the case $k(k+1) < 2\delta \leq k(k+2)$. Under this assumption the maximal rank of Z follows from the assertion on the homogeneous ideal of Z by Remark 7. Hence in these parts we will only prove $h^1(\mathbb{P}^2, \mathcal{I}_Z \otimes \Omega^1{}_{\mathbf{P}^2}(k+1)) = 0$. Anyway, it is easier the check that the maximal rank condition is satisfied for a general Z and by the irreducibility of the product of $a_1 + \cdots + a_s$ copies of $\mathbf{P}^2$ and the semicontinuity theorem these two properties may be checked just for one single Z (different for the two properties) and then we have both for a general Z. In part (d) we will assume $k(k+2) \leq 2\delta \leq (k+2)(k+1)$. Here we will only check $h^0(\mathbb{P}^2, \mathcal{I}_Z \otimes \Omega^1{}_{\mathbf{P}^2}(k+1)) = 0$. Let $C \subset \mathbf{P}^2$ be a smooth conic.

(a) Here we assume $m_s = 1$ and $a_s \geq (m_1+2)(m_1+1)/2 + m_1 k$. We take some of the points $P_{i,j} \in C$, until the sum of the corresponding multiplicity add to $2k$, taking as much as possible points with multiplicity m_1 as possible, then as much as possible points with multiplicity m_2, and so on. We may do so obtaing a scheme Z_1 with length$(Z_1) = 2k$, because $m_s = 1$ and $a_s \geq 2$. We apply Remark 4 and Remark 5 to reduce Theorem 1 for the integer k to a certain assertion for the integer $k-2$. Set $A_1 := Z_1$, $W_0 := \emptyset$ and $W_1 := \mathrm{Res}_C(A_1)$. Let Z_2 be the disjoint union of Z_1 and the union A_2 of as many of the remaining points $m_{i,j}$ as possible, with the condition length$(A_2 \cap C) = 2k - 4 -$ length(W_1) and that we take in A_2 as many points of maximal multiplicity as possible. we may satisfy these conditions by our assumption on a_s. Set $W_2 := \mathrm{Res}_C(W_1 \cup A_2)$. Then we continue at most $\lfloor k/2 \rfloor$ times. Since $m_1 = 1$ and $a_s \geq +++$, we may do all these steps by our assumptions on a_s. Since $a_s \geq (m_1+2)(m_1+1)/2 + m_1 k$, in the last $\lceil m_/2 \rceil$ steps we only use points with multiplicity one.

(b) Here we assume $m_s = 1$, $m_{s-1} = 2$, $m_1 \geq k$ and $2a_{s-1} + a_s \geq km(m_1+2)(m_1+1)/2 + m_1 k$. We made the same construction. Our assumptions on m_{s-1} and a_{s-1} assures that we need to use at most one of the a_s points with multiplicity 1 at each step, until we use all the points with higher multiplicities.

(c) Here we assume $\mathrm{char}(\mathbb{K}) \neq 2$, $m_s = 1$, $m_{s-1} = 2$, $m_1 \geq 2$ and $2a_{s-1} + a_s \geq km(m_1+2)(m_1+1)/2 + m_1 k$. The only difference with respect to part (b) is that now we also use Remark 6.

(d) We repeat parts (a), (b) or (c) (according to the different assumptions on a_s), just looking at h^0-vanishings instead of h^1-vanishings. □

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References

[1] J. Alexander, A. Hirschowitz, An asymptotic vanishing theorem for generic unions of multiple points, *Invent. Math.*, **140**, No. 2 (2000), 303-325.

[2] E. Ballico, Generators for the homogeneous ideal of s general points of $\mathbb{P}^3$, *J. Algebra*, **106**, No. 1 (1987), 46-52.

[3] E. Ballico, A.V. Geramita, The minimal free resolution of the ideal of s general points in $\mathbb{P}^3$, In: *Proceedings of the 1984 Vancouver Conference in Algebraic Geometry*, CMS Conference Proceedings, **6**, Canadian Math. Soc. and Amer. Math. Soc, Providence, RI (1986), 1-11,

[4] A. Gimigliano, M. Idà, The ideal resolution for generic 3-fat points in $\mathbb{P}^2$, *J. Pure Appl. Algebra*, **187**, No. 1-3 (2004), 99-128.