International Journal of Pure and Applied Mathematics

Volume 28 No. 2 2006, 161-167

ON THE ESTIMATES OF PROBLEM *O* FOR NONLINEAR ELLIPTIC EQUATIONS OF SECOND ORDER

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Abstract: In this paper, we use the method of symmetry extension to obtain the same estimates of solutions of problem O for nonlinear elliptic equation of second order as stated in [2].

AMS Subject Classification: 35J60
Key Words: nonlinear elliptic equation, boundary value problems, symmetry extension

1. Introduction

Let Q be a bounded region in $\mathbf{R}^N$ and its bound $\partial Q \in C^2$. We consider the following nonlinear elliptic equation of second order

$$F(x, u, D_x u, D_x^2 u) = 0 \quad \text{in } Q, \tag{1}$$

under certain conditions (see (3)-(7) in the following). Equation (1) can be

Received: September 3, 2005

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written as

$$\sum_{i,j=1}^{N} a_{ij}u_{x_ix_j} + \sum_{i=1}^{N} b_iu_{x_i} + cu = f \quad \text{in } Q, \tag{2}$$

where $D_xu = (u_{x_i}), D_x^2u = (u_{x_ix_j})$, and

$$\begin{aligned}
&a_{ij} = \int_0^1 F_{\tau r_{ij}}(x,u,p,\tau r)d\tau,\ b_i = \int_0^1 F_{\tau p_i}(x,u,\tau p,0)d\tau,\\
&c = \int_0^1 F_{\tau u}(x,\tau u,0,0)d\tau,\ f = -F(x,0,0,0),\\
&r = D_x^2u,\ p = D_xu,\ r_{ij} = \frac{\partial^2 u}{\partial x_i\partial x_j},\ p_i = \frac{\partial u}{\partial x_i}.
\end{aligned}$$

Suppose that (2) satisfies condition C, i.e., for arbitrary functions $u_1(x)$, $u_2(x) \in C_\beta^1(\bar{Q})\bigcap W_2^2(Q)$, $F(x,u,D_xu,D_x^2u)$ satisfies the condition

$$\begin{aligned}
F(x,u_1,D_xu_1,D_x^2u_1) &- F(x,u_2,D_xu_2,D_x^2u_2)\\
&= \sum_{i,j=1}^{N} \tilde{a}_{ij}u_{x_ix_j} + \sum_{i=1}^{N} \tilde{b}_iu_{x_i} + \tilde{c}u,
\end{aligned} \tag{3}$$

where $0 < \beta < 1, u = u_1 - u_2$, and

$$\begin{aligned}
&\tilde{a}_{ij} = \int_0^1 F_{u_{x_ix_j}}(x,\tilde{u},\tilde{p},\tilde{r})d\tau, \quad \tilde{b}_i = \int_0^1 F_{u_{x_i}}(x,\tilde{u},\tilde{p},\tilde{r})d\tau,\\
&\tilde{c} = \int_0^1 F_u(x,\tilde{u},\tilde{p},\tilde{r})d\tau, \quad \tilde{u} = u_2 + \tau(u_1 - u_2),\\
&\tilde{p} = D_x[u_2 + \tau(u_1 - u_2)], \quad \tilde{r} = D_x^2[u_2 + \tau(u_1 - u_2)],
\end{aligned} \tag{4}$$

in which $\tilde{a}_{ij}, \tilde{b}_i, \tilde{c}, f$ are measurable in Q and satisfy the conditions

$$q_0\sum_{j=1}^{N}|\xi_j|^2 \le \sum_{i,j=1}^{N} \tilde{a}_{ij}\xi_i\xi_j \le q_0^{-1}\sum_{j=1}^{N}|\xi_j|^2, \quad 0 < q_0 < 1, \tag{5}$$

$$\sup_Q[\sum_{i,j=1}^{N}\tilde{a}_{ij}^2(x)]/\inf[\tilde{a}_{ii}(x)]^2 \le q_1 < \frac{2N-1}{2N^2-2N-1}, \tag{6}$$

$$\begin{aligned}
&|\tilde{a}_{ij}| \le k_0,\ |\tilde{b}_i| \le k_0, \quad i,j = 1,\cdots,N,\\
&-k_0 \le \inf_Q \tilde{c} < \sup_Q \tilde{c} \le 0 \quad \text{in} \quad Q, \qquad L_p[f,\bar{Q}] \le k_1,
\end{aligned} \tag{7}$$

where $k_0, k_1, q_0, q_1, p(> N+2)$ are non-negative constants and (6) is called the Cordes condition. Moreover, for almost every point $x \in Q$ and $D_x^2 u$, $\tilde{a}_{ij}(x,u,D_xu,D_x^2u)$, $\tilde{b}_i(x,u,D_xu)$, $\tilde{c}(x,u)$ are continuous in $u \in \mathbf{R}, D_xu \in \mathbf{R}^N, D_x^2 \in \mathbf{R}^{N(N+1)/2}$. Besides, if the condition (7) is replaced by

$$|\tilde{a}_{ij}| \le k_0, \quad L_p[\tilde{b}_i, \bar{Q}] \le k_0, \qquad i,j = 1,\cdots,N,$$
$$L_p[\tilde{c}, \bar{Q}] \le k_0, \quad \inf_Q \tilde{c} < \sup_Q \tilde{c} \le 0, \quad L_p[f, \bar{Q}] \le k_1,$$

then we say that equation (2) satisfies condition C'.

The so-called regular oblique derivative problem (problem O), i.e. to find a continuously differentiable solution $u = u(x) \in C^1_\beta(\bar{Q}) \bigcap W^2_2(Q)$ which satisfies the boundary conditions

$$\begin{aligned} lu &= d\frac{\partial u}{\partial \nu} + \sigma u = \tau(x), x \in \partial Q, \text{ i.e.} \\ lu &= \sum_{j=1}^N d_j \frac{\partial u}{\partial x_j} + \sigma u = \tau(x),\ x \in \partial Q, \end{aligned} \tag{8}$$

in which $d(x), d_j(x)(j = 1,\cdots,N), \sigma(x), \tau(x)$ satisfy the conditions

$$\begin{aligned} &C^2_\alpha[\sigma(x), \partial Q] \le k_0, \quad C^1_\alpha[d_j(x), \partial Q] \le k_0, \quad C^1_\alpha[\tau(x), \partial Q] \le k_2, \\ &\cos(\nu, \mathbf{n}) \ge q_0 > 0, \quad d > 0,\ \sigma \ge 0,\ d + \sigma \ge 1,\ x \in \partial Q, \end{aligned} \tag{9}$$

where $\mathbf{n}$ is the unit outward normal on $\partial Q, \alpha, \beta$ $(0 < \beta \le \alpha < 1)$, k_0, k_2, q_0 $(0 < q_0 < 1)$ are non-negative constants. If this problem with conditions $d = 0, \nu = \mathbf{s}, \sigma = 1$ on ∂Q, then it will be the Dirichlet boundary value problem (problem D), and problem O with the condition $\nu = \mathbf{n}, \sigma = 0,$ on ∂Q is called problem N.

We definite here that if Q^* is any closed subdomain in Q, α $(0 < \alpha < 1)$ is non-negative constant, then

$$\begin{aligned} &C^1[a, Q^*] = C[a, Q^*] + \sum_{i=1}^N C[a_{x_i}, Q^*], \\ &C^1_\alpha[a, Q^*] = C_\alpha[a, Q^*] + \sum_{i=1}^N C_\alpha[a_{x_i}, Q^*], \\ &C_\alpha[a, Q^*] = C[a, Q^*] + H_\alpha[a, Q^*] = \max_{Q^*} |a| + \max_{x \ne y \in Q^*} \frac{|a(x) - a(y)|}{|x - y|^\alpha}. \end{aligned} \tag{10}$$

2. Some Lemmas and Main Results

Now we recall some lemmas that we use in proofs of our theorem.

Lemma 1. *Suppose that condition C' holds. Then any solution $u(x)$ of problem D for equation (2) satisfies the estimate*

$$C^1_\beta[u, Q_d] \leq M_2(q, p, k, M_1, Q_d), \tag{11}$$

where $Q_d = x|x \in Q, \operatorname{dist}(x, \partial Q) \geq d$, d is a small positive constant, $\beta(0 < \beta \leq \alpha)$, M_2 are non-negative constants, $q = (q_0, q_1)$, and $k = (k_0, k_1, k_2)$.

Lemma 2. *Suppose Q be a bounded in $\mathbf{R}^N$ and the bounded $\partial Q \in C^2$. Then any solution $u(x)$ of the boundary value problem*

$$\begin{cases} \triangle u = 0 \quad \text{in } \bar{Q}, \\ lu = d\frac{\partial u}{\partial \nu} + \sigma u = \tau(x), \quad x \in S_3, \end{cases} \tag{12}$$

satisfies the estimate

$$C^2[u, \bar{Q}] \leq M_3(q, p, \alpha, k, Q). \tag{13}$$

The main result of this paper is the following theorem.

Theorem. *Under condition C', any solution $u(x)$ of problem O for (2) satisfies the estimate*

$$C^1[u, \bar{Q}] = \|u\|_{C^1(\bar{Q})} = \|u\|_{C(\bar{Q})} + \sum_{i=1}^{N} \|u_{x_i}\|_{C(\bar{Q})} \leq M_4, \tag{14}$$

where M_1 is a non-negative constant and $M_4 = M_4(q, p, \alpha, k, Q)$, $q = (q_0, q_1)$, $k = (k_0, k_1, k_2)$.

Proof. First of all, substitute the solution $u(x)$ of problem O into the equation (2), thus we can only discuss the linear elliptic equation in the form

$$Lu = \sum_{i,j=1}^{N} a_{ij} u_{x_i x_j} + \sum_{i=1}^{N} b_i u_{x_i} + cu = f \quad \text{in } Q. \tag{15}$$

Choosing that x^* is an arbitrary point in Q and ε is a small positive number, we construct a function $J(x) \in C^2_\alpha(\bar{Q})$ such that

$$J(x) = \begin{cases} 1, & x \in Q_\varepsilon\ , \\ 0, & x \in Q \backslash Q_{2\varepsilon}, \end{cases} \quad 0 \leq J(x) \leq 1, \quad x \in Q_{2\varepsilon} \backslash Q_\varepsilon, \tag{16}$$

in which $Q_\varepsilon = \{ \mid x - x^* \mid \leq \varepsilon\} \in Q$ and $J(x)$ satisfies the estimate

$$C^2[J(x), Q] \leq M_5 = M_5(\varepsilon, Q). \tag{17}$$

Denote $U(x) = J(x)u(x)$, obviously $U = U(x)$ is a solution of the following boundary value problem

$$\begin{cases} \sum_{i,j=1}^N a_{ij}U_{x_ix_j} + \sum_{i=1}^N b_iU_{x_i} + cU = f^*, \quad x \in \bar{Q}, \\ \\ U(x) = 0, \quad x \in \partial Q, \end{cases} \tag{18}$$

where $f^* = Jf + \sum_{i,j=1}^N a_{ij}[J_{x_i}u_{x_i} + J_{x_ix_j}u] + \sum_{i=1}^N b_iJ_{x_i}u$. Since $b_i, c, f^* \in L_p(Q), p > N + 2$, by using Lemma 1, we can obtain

$$C^1_\beta[U, \bar{Q}] \leq M_6, \quad C^1_\beta[U, Q_\varepsilon] \leq M_7, \tag{19}$$

and form (10) we can also obtain

$$C^1[U, \bar{Q}] \leq M_6, \quad C^1[U, Q_\varepsilon] \leq M_7, \tag{20}$$

where $M_j = M_j(q, p, \alpha, k, Q)$ $(j = 6, 7)$. Combining (16), (18) and (19), we obtain the estimate

$$C^1_\beta[u, Q_\varepsilon] \leq M_8 = M_8(q, p, \alpha, k, Q) \tag{21}$$

and

$$C^1[u, Q_\varepsilon] \leq M_8 = M_8(q, p, \alpha, k, Q). \tag{22}$$

Next, we choose any point $x^* \in \partial Q$, a small positive number d and denote $S_3 = \partial Q \cap \{|x - x^*| \leq d\}$ and we can find out a solution $\hat{u}(x)$ of the boundary value problem

$$\begin{cases} \triangle \hat{u} = 0, \quad \text{in } \bar{Q}, \\ \\ l\hat{u}(x) = \tau(x), \quad \text{i.e.} \quad d\frac{\partial \hat{u}}{\partial \nu} + \sigma\hat{u} = \tau(x), \quad x \in S_3 \end{cases} \tag{23}$$

satisfying the estimate

$$C^2[\hat{u}, \bar{Q}] < M_9(d, Q). \tag{24}$$

Thus $\tilde{u} = u - \hat{u}$ is a solution of the equation with boundary value problem

$$\begin{cases} \sum_{i,j=1}^N a_{ij}\tilde{u}_{x_ix_j} + \sum_{i=1}^N b_i\tilde{u}_{x_i} + c\tilde{u} = f', \quad f' = f - L\hat{u}, \ x \in Q, \\ \\ l\tilde{u} = 0, \quad \text{i.e. } d\frac{\partial \tilde{u}}{\partial \nu} + \sigma\tilde{u} = 0, \quad x \in S_3. \end{cases} \tag{25}$$

Making a non-singular transformation of second order continuously differentiable function $\zeta = \zeta(x)$, such that S_3 maps onto S_4 on the plane $\zeta_N = 0$, and the domain Q onto the domain $\tilde{Q}$ in the half space $\zeta_N < 0$, and the equation with the boundary value problem (25) is reduced to the equation and boundary condition as follows:

$$\begin{cases} \sum_{i,j=1}^N A_{ij}\tilde{u}_{\zeta_i\zeta_j} + \sum_{i=1}^N B_i\tilde{u}_{\zeta_i} + C\tilde{u} = G, & \text{in } Q, \\ l\tilde{u} = 0, & \text{on } S_4. \end{cases} \tag{26}$$

Now, we extend the function $\tilde{u}$ to a symmetric domain $\tilde{Q}$ of Q about S_4, i.e. we make a function

$$U = \begin{cases} \tilde{u}(\zeta), & \zeta \in Q, \\ \tilde{u}(\zeta^*), & \zeta \in \tilde{Q}, \end{cases} \tag{27}$$

where $\zeta^* = (\zeta_1, \cdots, \zeta_{N-1}, -\zeta_N)$, and $U(\zeta)$ is a solution of the equation

$$\sum_{i,j=1}^N \tilde{A}_{ij}U_{\zeta_i\zeta_j} + \sum_{i=1}^N \tilde{B}_i U_{\zeta_i} + \tilde{C}U = \tilde{D}, \quad \text{in } Q \cup \tilde{Q}, \tag{28}$$

where

$$\tilde{A}_{ij} = \begin{cases} A_{ij}(\zeta), \\ (-1)^k A_{ij}(\zeta^*), \end{cases} \quad k = \begin{cases} 1, & i \neq j, i \text{ or } j = N, \\ 0, & \text{other cases}, \end{cases}$$

$$\tilde{B}_i = \begin{cases} B_i(\zeta), \\ (-1)^k B_i(\zeta^*), \end{cases} \quad k = \begin{cases} 1, & i = N, \\ 0, & i \neq N, \end{cases}$$

$$\tilde{C} = \begin{cases} C(\zeta), \\ C(\zeta^*), \end{cases} \quad \tilde{D} = \begin{cases} D(\zeta), & \zeta \in Q, \\ -D(\zeta^*), & \zeta^* \in \tilde{Q}. \end{cases}$$

By using the similar method as stated in the proof of (21) and (22), we can derive that $U(\zeta)$ and $\tilde{u}(x)$ satisfy the estimate

$$C^1_\beta[U, Q \cup \tilde{Q}] \leq M_7, \quad C^1_\beta[\tilde{u}, \bar{Q}] \leq M_8, \tag{29}$$

and

$$C^1[U, Q \cup \tilde{Q}] \leq M_7, \quad C^1[\tilde{u}, \bar{Q}] \leq M_8, \tag{30}$$

where $M_j = M_j(q, p, \alpha, k, Q)$, $j = 7, 8$. Combining (24) and (30), the estimates (14) are obtained. □

Remark. From the proof we can also obtain

$$\|u\|_{C^1_\beta(\bar{Q})} \leq M_9(q,p,\alpha,k,Q), \|u\|_{W^2_2(Q)} \leq M_{10}(q,p,\alpha,k,Q)\,, \tag{31}$$

where M_9, M_{10} are non-negative constants.

Acknowledgments

Research is supported by National Natural Science Foundation of China (No: 10471149), Mathematics Tian Yuan Youth Foundation (A0324610) and the Doctoral Foundation of the Department of Education of Hebei Province (B2004103).

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