International Journal of Pure and Applied Mathematics

Volume 28 No. 2 2006, 197-201

SOME NOTES ON ASYMPTOTIC BEHAVIOR OF RUDIN-SHAPIRO POLYNOMIALS

M. Taghavi[1 §], H. Azadi Kenary[2]

[1,2]Department of Mathematics
College of Science
Shiraz University
Shiraz, 71454, IRAN
[1]e-mail: taghavi@math.susc.ac.ir
[2]e-mail: hazadi@shirazu.ac.ir

Abstract: We study the asymptotic behavior of Rudin-Shapiro polynomials defines as the ratio of $2m$-norm with 2-norm of the Rudin-Shapiro polynomials. In 1980 Saffari stated a conjecture about asymptotic behavior of Rudin-Shapiro. In this article, we prove this conjecture for $m = 2, 3, 5, 7$.

AMS Subject Classification: 47A25
Key Words: Rudin-Shapiro polynomials, asymptotic behavior

1. Introduction

From the references cited in [2, 3], we obtain that the Golay polynomial pairs $(A_n(x), B_n(x))$ of degree n are of the form $A_n(x) = \sum_{k=0}^{n} a_k x^k$ and $B_n(x) = \sum_{k=0}^{n} b_k x^k$, where $a_k = \pm 1, b_k = \pm 1$, and they satisfy for all $t \in \mathbf{R}$

$$|A_n(e^{it})|^2 + |B_n(e^{it})|^2 = 2n + 2.$$

A special case of such polynomials are Rudin-Shapiro polynomials. That is defined recursively by the following formulas

$$P_{n+1}(x) = P_n(x) + x^{2^n} Q_n(x)\,, \quad Q_{n+1}(x) = P_n(x) - x^{2^n} Q_n(x)\,,$$

where $P_0(x) = Q_0(x) = 1$ and $n \geq 0$. In the remaining x is restricted to satisfy $|x| = 1$. It is clear from the definition of Rudin-Shapiro polynomials that $P_n(x)$

Received: March 4, 2006

§Correspondence author

has coefficients ± 1, without gaps, and that the first 2^n coefficients of $P_{n+1}(x)$ are identical with those of $P_n(x)$. It follows then that these coefficients do not depend on n, so we can write

$$P_n(x) = \sum_{k=0}^{2^n-1} \epsilon_k x^k\,, \quad Q_n(x) = \sum_{k=0}^{2^n-1} \delta_k x^k\,,$$

where $\epsilon_k = \pm 1$, $\delta_k = \pm 1$ and $n \geq 0$. According to Parseval's Theorem the former property gives $||P_n||_2^2 = \sum_0^{2^n-1} (\pm 1)^2 = 2^n$ (as each P is the Fourier transform of a ± 1 sequence), while the latter property gives

$$|P_{n+1}(x)|^2 + |Q_{n+1}(x)|^2 = 2(|P_n(x)|^2 + |Q_n(x)|^2) = 2^{n+2}\,. \tag{1}$$

Since $|x^{2^n}| = 1$, this leads to

$$|P_n(x)| \leq \sqrt{2} \times 2^{\frac{n}{2}}\,. \tag{2}$$

This is a uniform bound for P_n. Now, combining the two properties (1) and (2), we obtain:

$$\frac{||P_n||_\infty^2}{||P_n||_2^2} \leq 2\,.$$

This means that the max$|P_n(e^{it})|^2$ is equal to or less than two times the energy of P. This guarantee the polynomial to be somewhat flat.

Now, we state a definition and some useful theorems concerning basic properties of Rudin-Shapiro polynomials in this matter as a base for our main result.

Definition 1.1. For a complex polynomial P and a positive real number α, define $||P||_\alpha$ by

$$||P||_\alpha = (\frac{1}{2\pi} \int_0^{2\pi} |P(e^{it})|^\alpha dt)^{1/\alpha}.$$

The relation between $P_n(x)$ and $Q_n(x)$ has an important role in this article. So, we state the following theorem that J. Brillhart and L. Carlitz have proved in 1969.

Theorem 1.2. $Q_n(x) = (-1)^n x^{2^n-1} P_n(-1/x)$, $n \geq 0$.

Proof. By induction, the theorem holds for $n = 0, 1$. Assume the relation for $n, n \geq 1$. Then

$$\begin{aligned}
&(-1)^{n+1} x^{2^{n+1}-1} P_{n+1}(-1/x) \\
&\qquad = (-1)^{n+1} x^{2^{n+1}-1} [P_n(-1/x) + (-1/x)^{2^n} Q_n(-1/x)] \\
&\quad = (-1)^{n+1} x^{2^{n+1}-1} [(-1)^n Q_n(x)/x^{2^n-1} + x^{-2^n} (-1)^n (-1/x)^{2^n-1} P_n(x)]
\end{aligned}$$

$$= P_n(x) - x^{2^n} Q_n(x) = Q_{n+1}(x). \quad \square$$

Theorem 1.2 implies a corollary that is states the relation between norms of P_n's and Q_n's.

Corollary 1.4. *$||P_n||_m = ||Q_n||_m$ holds for all $m \in N \cup \{0\}$.*

2. Asymptotic Behavior

In 1980 Saffari stated a conjecture about asymptotic behavior of Rudin-Shapiro sequence as follows:

Conjecture. *For all $m \in N$: $\lim_{n\to\infty} \frac{||P_n||_{2m}}{||P_n||_2} = \sqrt[2m]{\frac{2^m}{m+1}}$.*

In this article, The main idea is prove of this conjecture for $m = 1, 2, 3, 5, 7$. Therefore, first of all we prove it for $m = 2$.

Theorem 2.1. *Asymptotic behavior holds for $m = 2$.*

Proof. We show that $\lim_{n\to\infty} \frac{||P_n||_4}{||P_n||_2} = \sqrt[4]{\frac{4}{3}}$. Simple calculation tend to:

$$|P_n|^2|Q_n|^2 = |P_{n-1}|^4 + |Q_{n-1}|^4 - 2Re(e^{2^n it}\overline{P_{n-1}}^2 Q_{n-1}^2). \tag{3}$$

Now assume that

$$x_n = \frac{1}{2\pi}\int_0^{2\pi} [|P_n(e^{it})|^4 + |Q_n(e^{it})|^4]dt = 2||P_n||_4^4. \tag{4}$$

Since polynomials P_{n-1}^2 and Q_{n-1}^2 have degree $2^n - 1$; hence integral of $Re(e^{2^n it}\overline{P_{n-1}}^2 Q_{n-1}^2)$ will be zero. Integrating both sides of equality (3) tend to the following relation.

$$x_{n-1} = \frac{1}{2\pi}\int_0^{2\pi} |P_n|^2|Q_n|^2 dt.$$

By Theorem 1.2 we have

$$|P_n|^4 + |Q_n|^4 + 2|P_n|^2|Q_n|^2 = 2^{2n+2}.$$

Thus the following recursive formula is concluded

$$x_n + 2x_{n-1} = 2^{2n+2}\,. \tag{5}$$

On the other hand

$$\frac{2}{3}(2^{2n+2}) - 2^n \le x_n \le \frac{2}{3}(2^{2n+2}) + 2^n\,, \quad n \in N \cup \{0\}\,, \tag{6}$$

since $P_0 = Q_0 = 1$, so $x_0 = 2$ and this implies that (6) holds for $n = 0$. Let (6) hold for $n-1$ then by (5), $x_{n-1} = 2^{2n+1} - \frac{x_n}{2}$ and

$$\frac{2}{3}(2^n) - 2^{n-1} \leq 2^{2n+1} - \frac{x_n}{2} \leq \frac{2}{3}(2^n) + 2^{n-1}.$$

This is (6). Therefore we have:

$$\frac{4}{3} - 2^{-n-1} \leq \frac{x_n}{2^{2n+1}} \leq \frac{4}{3} + 2^{-n-1}$$

since $\frac{x_n}{2^{2n+1}} = \frac{||P_n||_4^4}{||P_n||_2^4}$. This proves the theorem. □

Now, it is time to prove that the asymptotic behavior holds for $m = 3$.

Theorem 2.2. *Asymptotic behavior holds for $m = 3$.*

Proof.

$$\begin{aligned} ||P_n||_6^6 &= \frac{1}{2\pi}\int_0^{2\pi} |P_n(e^{it})|^6 dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} (2^{n+1} - |Q_n(e^{it})|^2)^3 dt \\ &= 2^{3(n+1)} + 3 \times 2^{n+1}||Q_n||_4^4 - 3 \times 2^{2(n+1)}||Q_n||_2^2 - ||Q_n||_6^6, \end{aligned}$$

since $||P_n||_m = ||Q_n||_m$ for all $m \in N \cup \{0\}$, we have

$$2||P_n||_6^6 = 2^{3(n+1)} + 3 \times 2^{n+1}||P_n||_4^4 - 3 \times 2^{2(n+1)}||P_n||_2^2.$$

On the other hand, since $||P_n||_2^2 = ||Q_n||_2^2 = \sum_0^{2^n-1}(\pm 1) = 2^n$ we have

$$\begin{aligned} \lim_{n\to\infty} \frac{||P_n||_6^6}{||P_n||_2^6} = \lim_{n\to\infty} \frac{2^{3n+2}}{2^{3n}} + \lim_{n\to\infty} \frac{3 \times 2^n||P_n||_4^4}{2^n||P_n||_2^4} - \lim_{n\to\infty} \frac{3 \times 2^{2n+1}||P_n||_2^2}{2^{2n}||P_n||_2^2} \\ = 4 + 3(4/3) - 6 = 2. \end{aligned}$$

This implies that

$$\lim_{h\to\infty} \frac{||P_n||_6}{||P_n||_2} = \sqrt[6]{2}.$$

Thus, the theorem is proved. □

Up to now, we show that the asymptotic behavior holds for $m = 1, 2, 3$. If we want to prove that it holds for $m = 5$, we need to assume that it holds for $m = 2, 4$. Since we proved that asymptotic behavior holds for $m = 2$, so we only assume that it holds for $m = 4$.

Theorem 2.3. *If asymptotic behavior holds for $m = 4$ then it holds for $m = 5$.*

Proof. We show that $\lim_{n\to\infty} \frac{||P_n||_{10}}{||P_n||_2} = \sqrt[10]{\frac{16}{3}}$. For this, let us consider

$$\begin{aligned} ||P_n||_{10}^{10} &= \frac{1}{2\pi}\int_0^{2\pi} |P_n(e^{it})|^{10} dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} (2^{n+1} - |Q_n(e^{it})|^2)^5 dt \\ &= \sum_0^5 (-1)^{5-j} C_j^5 2^{(n+1)j} (\frac{1}{2\pi}\int_0^{2\pi} |Q_n(e^{it})|^{2(5-j)} dt) \\ &= -||Q_n||_{10}^{10} + \sum_1^5 (-1)^{5-j} C_j^5 2^{(n+1)j} ||Q_n||_{2(5-j)}^{2(5-j)}. \end{aligned}$$

Since $||P_n||_m = ||Q_n||_m$ for all $m \in N \cup \{0\}$. So

$$\lim_{n\to\infty} \frac{||P_n||_{10}^{10}}{||P_n||_2^{10}} = \sum_1^5 (-1)^{(5-j)} C_j^5 2^{j-1} \lim_{n\to\infty} \frac{2^{nj}||P_n||_{2(5-j)}^{2(5-j)}}{||P_n||_2^{2j}||P_n||_2^{2(5-j)}}.$$

Now, Theorems 2.1, 2.2 and hypothesis of this theorem implies that

$$\lim_{n\to\infty} \frac{||P_n||_{10}^{10}}{||P_n||_2^{10}} = 16 - 40 + \frac{160}{3} - 40 + 16 = \frac{16}{3}. \qquad \square$$

References

[1] J.P. Allouche, B. Saffari, Upper bound theorems for the autocorrelation coefficients of the Rudin-Shapiro polynomials, In: *Proceeding of the Paris Academy of Science* (1991).

[2] J. Beck, Flat polynomilas on the unit circle-note on a problem of littelwood, *Bull. London Math. Soc.*, **23** (1991), 269-277.

[3] G. Benke, Generalized Rudin-Shapiro systems, *J. Fourier Anal. Appl.*, **1**, No. 1 (1994), 87-101.

[4] P. Borwein, *Computational Excursions in Analysis and Number Theory*, Springer-Verlag (2002).