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SOME OBSERVATIONS ON THE TWISTED LIMITS OF GROUPS

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Abstract: For a category $\mathfrak{C}$, a pro-$\mathfrak{C}$-group is defined as a profinite group G such that for every open normal subgroup H of G, the group G/H is in the class $\mathfrak{C}$. In this note, we observe two properties of the twisted limits of presheaves of category of finite groups which are also valid for presheaves of the category of pro-$\mathfrak{C}$-groups.

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1. Introduction

Recall (see [1]) that for $(\Lambda \leq)$ a partially ordered set, a subset A of Λ is *directed* if for every $a, b \in A$ there is a $c \in A$ such that $a \leq c$ and $b \leq c$. We define a presheaf over $(\Lambda, \leq)$ with values in a suitable category $\mathfrak{C}$ as a functor $G : \Lambda \to \mathfrak{C}$, where Λ is viewed as a category with objects $\lambda, \mu \in \Lambda$ such that $\hom(\lambda, \mu) = \emptyset$ if $\lambda \not\leq \mu$ and $\hom(\lambda, \mu)$ consists of one arrow if $\lambda \leq \mu$. If $\mathfrak{C}$ is a category of finite groups, then G is given by a family $G_\lambda \in \mathfrak{C}$ of finite groups with morphisms $\Phi_{\lambda\lambda'} : G_\lambda \to G_{\lambda'}$ for all $\lambda \leq \lambda'$ in Λ, such that $\Phi_{\lambda\lambda} = 1_{G_\lambda}$ and $\Phi_{\lambda\lambda''} = \Phi_{\lambda'\lambda''} \circ \Phi_{\lambda\lambda'}$, if $\lambda \leq \lambda' \leq \lambda''$. Suppose G is presheaf over a poset $(\Lambda, \leq)$, and $\mathcal{G} := \bigsqcup_{\lambda} G_\lambda$, ($\sqcup$ denotes disjoint union). Let $\mathcal{G}(\Lambda) := \{g_\lambda \mid \Phi_{\lambda\lambda'}(g_\lambda) = g_{\lambda'} \text{ for all } \lambda \leq \lambda' \text{ in } \Lambda\}$. If Λ_1 is another poset then $\varphi : \Lambda \to \Lambda_1$ is a *morphism* (of posets) if it is order preserving. If Λ and Λ_1 are lattices, then a *lattice* morphism $\psi : \Lambda \to \Lambda_1$ is a map satisfying $\psi(a \vee b) = \psi(a) \vee \psi(b)$, $\psi(a \wedge b) = \psi(a) \wedge \psi(b)$. A bijective lattice morphism is called an *isomorphism*. It is convenient to write $\mathcal{W}$ for

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the data $(\Lambda, \mathcal{G})$. If $\mathcal{W}'$ is another system defined in the same way over some Λ' then a morphism $\alpha : \mathcal{W} \to \mathcal{W}'$ consists of morphisms $\overline{\alpha} : \Lambda \to \Lambda'$ and $\alpha_\lambda : G_\alpha \to G'_{\overline{\alpha}(\lambda)}$ such that for all $\lambda, \lambda' \in \Lambda$ we have that $\alpha_{\lambda'}\Phi_{\lambda\lambda'} = \Phi_{\overline{\alpha}(\lambda)\overline{\alpha}(\lambda')}$. A morphism $\alpha : \mathcal{W} \to \mathcal{W}'$ defines a map (again denoted by α) $\alpha : \mathcal{G} \to \mathcal{G}'$, where $\mathcal{G}'$ is over some Λ' in an obvious way; moreover we also have a morphism $\alpha(\Lambda) : \mathcal{G}(\Lambda) \to \mathcal{G}'(\Lambda') : g_\lambda \to \alpha_\lambda(g_\lambda)$. For a more detailed discussion on the properties of $\alpha(\Lambda)$ and $\mathcal{G}(\Lambda)$ we refer the reader to [1].

From suitable maps $\mathcal{G} \to \mathcal{G}'$ we construct a morphism $\mathcal{W} \to \mathcal{W}'$. To this end, consider the map $\sigma : \mathcal{G} \to \Lambda$ defined by $\sigma(g) = \lambda$ when $g \in G_\lambda$ in $\mathcal{G}$. A map $\alpha : \mathcal{G} \to \mathcal{G}'$ is called *compatible* if the following two properties hold:

(i) for all $g, h \in \mathcal{G}$ with $\sigma(g) = \sigma(h)$ we have $\sigma'(\alpha(g)) = \sigma'(\alpha(h))$, where $\sigma' : \mathcal{G}' \to \Lambda'$ is defined by $\sigma(g') = \lambda'$ exactly when $g' \in G'_\lambda$ is in $\mathcal{G}'$,

(ii) the map $\overline{\alpha}$ is a morphism.

A compatible map is said to be a *morphism* if $\sigma(g) = \sigma(h)$ implies $\alpha(gh) = \alpha(g)\alpha(h)$; it is an *isomorphism* when α is a bijective morphism and α^{-1} is compatible. A morphism $\alpha : \mathcal{G} \to \mathcal{G}$ is said to be an *endomorphism* and an isomorphism $\alpha : \mathcal{G} \to \mathcal{G}$ is said to be an *automorphism*. Note that for an automorphism α, we have that $\overline{\alpha}$ is necessarily an automorphism of $(\Lambda, \leq)$. We write $\mathrm{Aut}_\Lambda(\mathcal{G})$ for the isomorphisms $\alpha : \mathcal{G} \to \mathcal{G}$ such that $\overline{\alpha} = 1_\Lambda$. It is now clear that a morphism $\alpha : \mathcal{G} \to \mathcal{G}'$ in the above sense gives rise to a morphism $\alpha : \mathcal{W} \to \mathcal{W}'$ as defined before.

Now let $\mathfrak{C}$ be the category of pro-$\mathfrak{C}$-groups, that is, the category whose objects are pro-$\mathfrak{C}$-groups and whose morphism are the continuous group homomorphisms. Recall that a pro-$\mathfrak{C}$-groups is by definition a profinite group E such that for every open normal subgroup H of E, the group E/H is in the class $\mathfrak{C}$. Let G be a presheaf of pro-$\mathfrak{C}$-groups over $(\Lambda, \leq)$ and let $\mathcal{G} := \bigsqcup_\lambda G_\lambda$. A given group E is said to act on G if we have a group morphism $\psi : E \to \mathrm{Aut}_\Lambda(\mathcal{G})$. If $\psi(E) \subset \mathrm{Aut}_\Lambda(\mathcal{G})$, we say that the action is *normal*. A *twisted presheaf* of pro-$\mathfrak{C}$-groups (of the category $\mathfrak{C}$) consists of a presheaf of pro-$\mathfrak{C}$-groups G over $(\Lambda, \leq)$, together with an action ψ of G on itself. A *twisted limit* for a twisted presheaf G over $(\Lambda, \leq)$ is a pro-$\mathfrak{C}$-groups $\mathbb{G}$ together with a continuous homomorphism $\pi : \mathcal{G} \to \mathbb{G}$ satisfying:

(i) for every $\lambda' \geq \lambda$, we have that $\pi\Phi_{\lambda\lambda'} = \pi$, and $\pi(gh) = \pi(g)\pi(h)$ whenever g and h are in the same G_λ,

(ii) for every $g, h \in G_\lambda$, and any $\lambda \in \Lambda$, we have that $\pi(\psi_\lambda(g)(h)) = \pi(g)\pi(h)\pi(g)^{-1}$,

(iii) if $\mathbb{G}'$ is any pro-$\mathfrak{C}$-group and $\pi' : \mathcal{G} \to \mathbb{G}'$ is a map satisfying properties (i) and (ii) above, then there exists a (unique) group morphism $\rho : \mathbb{G} \to \mathcal{G}'$ such that $\pi' = \rho\pi$.

Note that the universal property (iii) above guarantees that a twisted limit $\mathbb{G}$ is unique up to isomorphism.

Now to the main gist: we want to show the following facts:

1. *If G is a twisted presheaf of pro-$\mathfrak{C}$-groups over Λ with action as defined before, then G has a unique twisted limit $\mathbb{G}$.*

2. *If E is a pro-$\mathfrak{C}$-groups and G is the twisted presheaf of closed subgroup of E of prime power order, then one has that the canonical homomorphism $\psi : \mathbb{G} \to E$ is an isomorphism.*

Proof of 1. Suppose $\tilde{\mathbb{G}}$ is the twisted limit of G in the category of all groups. Let $\tilde{\pi} : \mathcal{G} \to \tilde{\mathbb{G}}$ be the canonical homomorphism. Consider the inverse systems of groups $\tilde{G}/N \in \mathfrak{C}$. Let $\mathbb{G} = \varprojlim \tilde{\mathbb{G}}/N$, so that $\mathbb{G}$ is a pro-$\mathfrak{C}$-groups. Let $\omega : \tilde{\mathbb{G}} \to \mathbb{G}$ be the canonical homomorphism and $\pi = \omega \circ \tilde{\pi}$. It is easy to show that $\mathbb{G}$ and π satisfy the needed conditions. □

Proof of 2. To prove this, we first recall the following lemma (see ([1]) for the proof).

Lemma 1. *If E is a finite group and G is the twisted presheaf of the subgroups of E of prime power order, then one has that the map $\psi : \mathbb{G} \to E$ is an isomorphism.*

Note that in the case where E is a pro-$\mathfrak{C}$-groups and G is the twisted presheaf of the closed subgroups of E of prime power order, that is, every finite epimorphic image has order a power of a prime, Lemma 1 is still valid. To see this, let $\mathcal{G}(G/H)$ denote the family of subgroups of E/H of prime power order (for all open normal subgroup H of E). Let $\mathbb{G}(E/H)$ denote the twisted limit of $\mathcal{G}(G/H)$. Then one has from Lemma 1 that the canonical homomorphism $\psi_H : \mathbb{G}(E/H) \to E/H$ is an isomorphism. Because $\psi = \varprojlim \psi_H$, one has that $\psi : \mathbb{G} \to E$ is an isomorphism. □

References

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