International Journal of Pure and Applied Mathematics
Volume 28 No. 2 2006, 207-215

DOUBLE-ACTING SEARCH FOR A RANDOMLY MOVING TARGET

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Abstract: A target is assumed to move randomly according to a stochastic process on a straight line. Two searchers S_1 and S_2 start looking for the target, S_1 starts looking for the target from some point a_o and S_2 starts from some other point b_o on the line to detect the target. Each of the searchers moves continuously along the line in both directions from his starting point. In this paper we show the existence of a search plan such that the expected value of the first meeting time of the lost target is minimum.

AMS Subject Classification: 60K30, 90B40
Key Words: expected value, linear search, optimal search plan, stochastic processes

1. Introduction

The search for lost targets that are either stationary or randomly moving is very important in many civilian and military applications such as:

1. The choice of drilling depths in the search for an underground mineral.
2. Search for a lost person on land or lost submarine and ship on or under the sea.
3. Search for a school of fish.
4. Search for an airplane in the sky.

Received: March 14, 2006

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When the target is located or moves randomly on the real line then we obtain the so-called linear search problem (see [2], [3], [4], [5] and [7]). Some problems of search may impose using more than one searcher; such as when we search for a valuable target (e.g. person) or it is serious (e.g. explosives), also in some cases of search problems the cost of the search will reduce by using more than one searcher. Here we suppose that a target is assumed to move randomly on the line according to the process $\{S(t),\, t \in \Re^+\}$, where $\Re$ is the set of real numbers and $S(t)$ is a stochastic process with drift μ and variance $\sigma^2$and satisfies the following condition. If T is a stopping time for $S(t)$, then $E\left|S(T)\right| \leq \sigma\sqrt{E(T)} + |\mu|\, E(T)$, where E terms to the expectation value.

Assume that we have two searchers S_1 and S_2. The initial position of the target is a random variable with a known distribution. Also, we assume that the speeds of S_1 and S_2 are equal, say v. The searchers S_1 and S_2 follow search paths, which are functions $\phi : \Re^+ \to \Re$ and $\overline{\phi} : \Re^+ \to \Re$, respectively, such that:

$$\mid \phi\,(t_1) - \phi\,(t_2) \mid \leq v \mid t_1 - t_2 \mid , \tag{1}$$

and

$$\mid \bar{\phi}\,(t_1) - \bar{\phi}\,(t_2) \mid \leq v \mid t_1 - t_2 \mid , \quad \forall t_1, t_2 \in \Re^+, \tag{2}$$

where v is a constant in $\Re^+$, $\phi\,(0) = a_o$ and $\bar{\phi}(0) = b_o$. Let the set of all search paths of S_1 , which satisfies condition (1) be represented by Φ and the set of all search paths of S_2, which satisfies condition (2) be represented by $\bar{\Phi}$. Let the search plan of S_1 and S_2 be represented by $(\phi, \bar{\phi}) \in \Phi_o$, where $\Phi_o = \{(\phi, \bar{\phi});\, \phi \in \Phi, \bar{\phi} \in \bar{\Phi}\}$. The first meeting time τ is a random variable valued in $\Re^+$ defined as

$$\tau = \inf\{t : \text{ at least } \ \phi\,(t) = X_o + S(t) \ \text{ or } \ \bar{\phi}\,(t) = X_o + S(t)\},$$

where X_o is a random variable independent with $S(t)$ and represents the initial position of the target. Let ν be a probability measure induced by the intional position of the target on the line. The problem is to find a search plan $(\phi, \bar{\phi}) \in \Phi_o$ such that $E(\tau) < \infty$. In this case we call $(\phi, \bar{\phi})$ is a finite search plan. Let h_1 be the first point at which any searcher meets the searched part which is covered by the other searcher and let h be the point at which the two searchers are met face to face. We define the following cases according to the position of h_1.

Case 1. $h_1 \geq 0$. Let O be the set of odd integers, and let $L = \{1, 3, 5 \ldots, l\}$ be a finite subset of O, we have the following path for S_1

$$h_1 \leq a_j < a_{j-2} < \ldots < a_4 < a_2 < a_o < a_1 < a_3 < \ldots$$

and for S_2

$$\ldots < b_4 < b_2 < b_o < b_1 < b_3 < \ldots < b_l < 0 \leq b_m < \ldots < b_n \leq h_1\,,$$

where j is an even integer, n is an odd integer and

$$m = \begin{cases} 1\,, & \text{if } L \text{ is an empty set}\,, \\ l+2\,, & \text{if } L \text{ is a non empty set}\,. \end{cases}$$

Case 2. $h_1 < 0$. Let E^* be the set of even integers and let $J = \{2, 4, 6, \ldots, j\}$ be a finite subset of E^*, then we have the following path for S_1

$$h_1 \leq a_q < \ldots < a_k < 0 < a_j < \ldots < a_2 < a_o < a_1 < a_3 < \ldots$$

and for S_2

$$\ldots < b_4 < b_2 < b_o < b_1 < b_3 < \ldots < b_l \leq h_1,$$

where $q \in E^*$ and

$$k = \begin{cases} 2,\,, & \text{if } J \text{ is an empty set}\,, \\ j+2\,, & \text{if } J \text{ is a non empty set}\,. \end{cases}$$

2. Existence of a Finite Search Plan

Let θ be a positive integer and v be a rational number such that:

(i) $v > |\mu|$.

(ii) $\theta > 1$ such that $c = v\frac{(\theta-1)}{\theta+1} > |\mu|$.

We assume that $|\,a_i - a_{i-1}| = |\,b_i - b_{i-1}|$. We shall define the sequences $\{G_i\}_{i\geq 0}$, $\{r_i\}_{i\geq 0}$, $\{a_i\}_{i\geq 0}$ and $\{b_i\}_{i\geq 0}$ as:

$$G_i = \theta^i - 1, \quad r_i = (-1)^{i+1} c\,[G_i + 1 + (-1)^{i+1}],$$

$a_i = a_o + r_i$ and $b_i = b_o + r_i$.

Now, we will study the case (1) only, where the other case can be studied by the same manner. The possible positions of h_1 are given as follows:

(I) $h_1 = a_j < a_{j-2} < \ldots < a_4 < a_2 < a_o < a_1 < a_3 < \ldots$ and $\ldots < b_4 < b_2 < b_o < b_1 < b_3 < \ldots < b_l < 0 \leq b_m < \ldots < b_n = h_1$.

(II) $h_1 < a_j < a_{j-2} < \ldots < a_4 < a_2 < a_o < a_1 < a_3 < \ldots$ and $\ldots < b_4 < b_2 < b_o < b_1 < b_3 < \ldots < b_l < 0 \leq b_m < \ldots < b_n = h_1$.

(III) $h_1 = a_j < a_{j-2} < \ldots < a_4 < a_2 < a_o < a_1 < a_3 < \ldots$ and $\ldots < b_4 < b_2 < b_o < b_1 < b_3 < \ldots < b_l < 0 \leq b_m < \ldots < b_n < h_1$.

We define the search paths as:

for any $t \in \Re^+$, if $G_{2i-1} \leq t \leq G_{2i}$, $1 \leq i \leq j/2$ then $\phi(t) = a_{2i-1} - a_o - (t - G_{2i-1})v$,

if $G_{2i} \leq t \leq G_{2i+1}$, $1 \leq i \leq \frac{j-2}{2}$, then $\phi(t) = a_o - a_{2i} + (t - G_{2i})v$,

if $t > G_h$, then $\phi(t) = h - (t - G_h)v$,

if $G_{2i-1} \leq t \leq G_{2i}$, $1 \leq i \leq \frac{l+1}{2}$, then $\bar{\phi}(t) = b_o - b_{2i-1} - (t - G_{2i-1})v$,

if $G_{2i} \leq t \leq G_{2i+1}$, $1 \leq i \leq \frac{l-1}{2}$, then $\bar{\phi}(t) = b_{2i} - b_o + (t - G_i)v$,

if $G_i \leq t \leq G_{i+1}$, $l+1 \leq i \leq n$, then $\bar{\phi}(t) = b_i + (-1)^i (t - G_i)v$,

if $G_{n+1} \leq t \leq G_h$, then $\bar{\phi}(t) = b_{n+1} + (t - G_{n+1})v$, if $t > G_h$, then $\bar{\phi}(t) = h + (t - G_h)v$.

The following notations will be used

$$\varphi_1(t) = S(t) - c(t+2) - a_o, \quad \tilde{\varphi}_1(t) = S(t) + ct - a_o,$$
$$\varphi_2(t) = S(t) - c(t+2) - b_o, \quad \tilde{\varphi}_2(t) = S(t) + ct - b_o,$$

where τ_ϕ is the first meeting time between S_1 and the lost target, and $\tau_{\bar{\phi}}$ is the first meeting time between S_2 and the lost target.

Theorem 2.1. *If $(\phi, \bar{\phi}) \in \Phi_o$ is a search plan defined above, then the expectation $E(\tau)$ is finite.*

Proof. For any $0 \le i \le (j-2)/2$

$$p(\tau > t) = p((\tau_\phi > t) \cap (\tau_{\bar{\phi}} > t)) = p(\tau_\phi > t) \times p(\tau_{\bar{\phi}} > t),$$

$$p(\tau_\phi > G_{2i+1}) \le \int_{h_1}^{a_o} p(X_o + S(G_{2i}) < a_{2i} / X_o = x)\nu(dx)$$
$$+ \int_{a_o}^{\infty} p(X_o + S(G_{2i+1}) > a_{2i+1} / X_o = x)\nu(dx),$$

we get

$$p(\tau_\phi > G_{2i+1}) \le \int_{h_1}^{a_o} p(\tilde{\varphi}_1(G_{2i}) < -x)\nu(dx) + \int_{a_o}^{\infty} p(\varphi_1(G_{2i+1}) > -x)\nu(dx)$$

and

$$p(\tau_{\bar{\phi}} > G_{2i+1}) \le \int_{-\infty}^{b_o} p(\tilde{\varphi}_2(G_{2i}) < -x)\nu(dx) + \int_{b_o}^{h_1} p(\varphi_2(G_{2i+1}) > -x)\nu(dx).$$

Also, for any $0 \le i \le j / 2$

$$p(\tau_\phi > G_{2i}) \le \int_{h_1}^{a_o} p(\tilde{\varphi}_1(G_{2i}) < -x)\nu(dx) + \int_{a_o}^{\infty} p(\varphi_1(G_{2i-1}) > -x)\nu(dx),$$

$$p(\tau_{\bar{\phi}} > G_{2i}) \le \int_{-\infty}^{b_o} p(\tilde{\varphi}_2(G_{2i}) < -x)\nu(dx) + \int_{b_o}^{h_1} p(\varphi_2(G_{2i-1}) > -x)\nu(dx),$$

$$p(\tau_\phi > G_h) \le \int_{h}^{\infty} p(X_o + S(G_h) > h \,/\, X_o = x)\nu(dx)$$

and

$$p(\tau_{\bar{\phi}} > G_h) \le \int_{-\infty}^{h} p(X_o + S(G_h) < h \,/\, X_o = x)\nu(dx).$$

Hence,

$$E(\tau) = \int_0^\infty P(\tau > t)dt \leq \sum_{i=0}^{j+1} \int_{G_i}^{G_{i+1}} P(\tau > t)dt, \quad \text{where} \quad j+1 = h.$$

So, we obtain

$$\begin{aligned}
E(\tau) &\leq \sum_{i=0}^{j+1} (G_{i+1} - G_i) P(\tau > G_i) \leq \sum_{i=0}^{j+1} (\theta - 1)\theta^i \, P(\tau > G_i) \\
&\leq (\theta - 1)[p(\tau > 0) + \theta p(\tau > G_1) + \theta^2 \, P(\tau > G_2) + \theta^3 P(\tau > G_3) + \cdots \\
&\qquad + \theta^j p(\tau > G_j) + \theta^h p(\tau > G_h)] \\
&\leq (\theta - 1)[p(\tau > 0) + \theta p(\tau > G_1) + \theta^2 [\int_{h_1}^{a_o} p(\tilde{\varphi}_1(G_2) < -x)\nu(dx) \\
&\qquad + \int_{a_o}^{\infty} p(\varphi_1(G_1) > -x)\nu(dx)] \\
&\qquad \times [\int_{-\infty}^{b_o} p(\tilde{\varphi}_2(G_2) < -x)\nu(dx) + \int_{b_o}^{h_1} p(\varphi_2(G_1) > -x)\nu(dx)] \\
&\qquad + \theta^3 [\int_{h_1}^{a_o} p(\tilde{\varphi}_1(G_2) < -x)\nu(dx) + \int_{a_o}^{\infty} p(\varphi_1(G_3) > -x)\nu(dx)] \\
&\qquad \times [\int_{-\infty}^{b_o} p(\tilde{\varphi}_2(G_2) < -x)\nu(dx) + \int_{b_o}^{h_1} p(\varphi_2(G_3) > -x)\nu(dx)] \\
&\qquad + \ldots + \theta^j [\int_{h_1}^{a_o} p(\tilde{\varphi}_1(G_j) < -x)\nu(dx) + \int_{a_o}^{\infty} p(\varphi_1(G_{j-1}) > -x)\nu(dx)] \\
&\qquad \times [\int_{-\infty}^{b_o} p(\tilde{\varphi}_2(G_j) < -x)\nu(dx) + \int_{b_o}^{h_1} p(\varphi_2(G_{j-1}) > -x)\nu(dx)] \\
&\qquad + \theta^h \int_h^\infty p(X_o + S(G_h) > h / X_o = x)\nu(dx)
\end{aligned}$$

$$\times \int_{-\infty}^{h} p(X_o + S(G_h) < h/X_o = x)\nu(dx)].$$

Hence,

$$E(\tau) \leq (\theta - 1)(g + w_1 + w_2 + w_3 + \ldots + w_6) < \infty,$$

where

$$g = p(\tau > 0) + \theta p(\tau > G_1) + \theta^2 \int_{a_o}^{\infty} p(\varphi_1(G_1) > -x)\nu(dx)]$$

$$\times \int_{b_o}^{h_1} p(\varphi_2(G_1) > -x)\nu(dx)$$

$$+ \theta^j \int_{h_1}^{a_o} p(\tilde{\varphi}_1(G_j) < -x)\nu(dx) \times \int_{-\infty}^{b_o} p(\tilde{\varphi}_2(G_j) < -x)\nu(dx)$$

$$+ \theta^h \int_{h}^{\infty} p(X_o + S(G_h) > h/X_o = x)\nu(dx)$$

$$\times \int_{-\infty}^{h} p(X_o + S(G_h) < h/X_o = x)\nu(dx),$$

$$w_1 = \sum_{i=1}^{j/2} \theta^{2i} \int_{h_1}^{a_o} p(\tilde{\varphi}_1(G_{2i}) < -x)\nu(dx) \times \int_{b_o}^{h_1} p(\varphi_2(G_{2i-1}) > -x)\nu(dx),$$

$$w_2 = \sum_{i=1}^{j/2} \theta^{2i} \int_{a_o}^{\infty} p(\varphi_1(G_{2i-1}) > -x)\nu(dx) \times \int_{-\infty}^{b_o} p(\tilde{\varphi}_2(G_{2i}) < -x)\nu(dx),$$

$$w_3 = (\theta + 1) \sum_{i=1}^{(j-2)/2} \theta^{2i+1} \int_{a_o}^{\infty} p(\varphi_1(G_{2i+1}) > -x)\nu(dx)$$

$$\times \int_{b_o}^{h_1} p(\varphi_2(G_{2i+1}) > -x)\nu(dx),$$

$$w_4 = \sum_{i=1}^{(j-2)/2} \theta^{2i+1} \int_{h_1}^{a_o} p(\tilde{\varphi}_1(G_{2i}) < -x)\nu(dx) \times \int_{b_o}^{h_1} p(\varphi_2(G_{2i+1}) > -x)\nu(dx),$$

$$w_5 = \sum_{i=1}^{(j-2)/2} \theta^{2i+1} \int_{a_o}^{\infty} p(\varphi_1(G_{2i+1}) > -x)\nu(dx) \times \int_{-\infty}^{b_o} p(\tilde{\varphi}_2(G_{2i}) < -x)\nu(dx)$$

and

$$w_6 = (\theta+1) \sum_{i=1}^{(j-2)/2} \theta^{2i} \int_{h_1}^{a_o} p(\tilde{\varphi}_1(G_{2i}) < -x)\nu(dx)$$

$$\times \int_{-\infty}^{b_o} p(\tilde{\varphi}_2(G_{2i}) < -x)\nu(dx). \quad \square$$

Theorem 2.2. *If there exists a finite search plan $(\phi, \bar{\phi}) \in \Phi_o$ then $E|X_o|$ is finite.*

Proof. If $E(\tau) < \infty$, then $p(\tau$ is finite $) = 1$ and so

$p(\tau_\phi$ is finite only or $\tau_{\bar{\phi}}$ is finite only or τ_ϕ and $\tau_{\bar{\phi}}$ are finite$) = 1$,

then

$p(\tau_\phi$ is finite only$)$ + $p(\tau_{\bar{\phi}}$ is finite only$)$ +$p(\tau_\phi$ and $\tau_{\bar{\phi}}$ are finite$) = 1$.

We must have

$p(\tau_\phi$ is finite only$) = 1$ or $p(\tau_{\bar{\phi}}$ is finite only$) = 1$

or $p(\tau_\phi$ and $\tau_{\bar{\phi}}$ are finite$) = 1$.

If $p(\tau_\phi$ is finite only$) = 1$, then $X_o = \phi(\tau_\phi) - S(\tau_\phi)$ with probability one and hence,

$$|X_o| \le |\phi(\tau_\phi)| + |S(\tau_\phi)| \le v\tau_\phi + |S(\tau_\phi)|\ ,$$
$$E|X_o| \le v \quad E\ (\tau_\phi) + E|S(\tau_\phi)|\,.$$

From the condition of $S(t)$, we get $E|S(\tau_\phi)| \le \sigma\sqrt{E(\tau_\phi)}\ + |\mu|\, E(\tau_\phi)$.

If $E\tau_\phi < \infty$, then $E \quad |S(\tau_\phi)| < \infty$, and $E|X_o|$ is finite.

On the other hand if $p(\tau_{\bar{\phi}}$ is finite only $) = 1$, then $X_o = \bar{\phi}(\tau_{\bar{\phi}}) - S(\tau_{\bar{\phi}})$ with probability one, by the same way we can get $E|X_o|$ is finite.

But if $E\tau_\phi < \infty$ and $\tau = G_h$then $p(\tau_\phi$ and $\tau_{\bar{\phi}}$ are finite$) = 1$ and so

$$p(\tau_\phi = \tau_{\bar{\phi}} = G_h \ \text{ is finite }) = 1,$$

then $X_o = \phi(\tau_\phi) - S(\tau_\phi) = \bar{\phi}(\tau_{\bar{\phi}}) - S(\tau_{\bar{\phi}})$ with probability one and hence,

$$|X_o| \le |\phi(\tau_\phi)| + |S(\tau_\phi)| \le v\tau_\phi + |S(\tau_\phi)|\ ,$$
$$E|X_o| \le v \quad E\ (\tau_\phi) + E|S(\tau_\phi)|\ .$$

If $E\tau_\phi < \infty$, then $E|S(\tau_\phi)| < \infty$, and $E|X_o|$ is finite. $\square$

3. Existence of an Optimal Search Path

Let $\{\phi_n\}_{n\ge1} \in \Phi(t)$ and $\{\bar{\phi}_n\}_{n\ge1} \in \bar{\Phi}(t)$ be two sequences of search plans, we say that ϕ_n (or $\bar{\phi}_n$) converges to ϕ (or $\bar{\phi}$) as n tends to ∞ iff for any

$t \in \Re^+$, $\phi_n(t)$(or $\bar{\phi}_n(t)$) converges to $\phi(t)$ (or $\bar{\phi}(t)$) uniformly on every compact subset.

Theorem 3.1. *Let for any $t \in \Re^+, S(t)$ be a stochastic process with continuous sample paths. The mapping $(\phi, \bar{\phi}) \to E(\tau) \in \Re^+$ is lower semi-continuous on $\Phi_o(t)$.*

Proof. Let ξ be a sample point corresponding to the sample path $\xi(t) = X + S(t)$. Let $\{\phi_n\}_{n\geq 1}$ be a sequence of search paths which converges to $\phi \in \Phi_V(t)$ and $\{\bar{\phi}_n\}_{n\geq 1}$ converges to $\bar{\phi} \in \bar{\Phi}_V(t)$. Given $t \in \Re^+$, we define for any $n \geq 1$

$$B_n(t) = \{\xi : \min_{0\leq x\leq t} |\xi(x) - \phi_n(x)| > 0\},$$
$$B(t) = \{\xi : \min_{0\leq x\leq t} |\xi(x) - \phi(x)| > 0\},$$
$$\bar{B}_n(t) = \{\xi : \min_{0\leq x\leq t} \left|\xi(x) - \bar{\phi}_n(x)\right| > 0\},$$
$$\bar{B}(t) = \{\xi : \min_{0\leq x\leq t} \left|\xi(x) - \bar{\phi}(x)\right| > 0\},$$

Let $\xi \in B(t)$, and since $\{\phi_n\}_{n\geq 1}$converges uniformly on $[0, t]$ to ϕ, then there exists an integer $n(\xi)$ such that for all $n \geq n(\xi)$ and for any $0 \leq x \leq t$

$$|\phi_n(x) - \phi(x)| < \varepsilon = 0.5 \min_{0\leq x\leq t} |\xi(x) - \phi(x)|$$

and

$$|\xi(x) - \phi(x)| \leq |\xi(x) - \phi_n(x)| + |\phi(x) - \phi_n(x)|\,,$$

then

$$|\xi(x) - \phi_n(x)| \geq |\xi(x) - \phi(x)| - |\phi(x) - \phi_n(x)| \geq 2\varepsilon - \varepsilon = \varepsilon > 0.$$

Hence $\xi \in B_n(t)$ for all $n > n(\xi)$ and so $B(t) \subset \lim_{n\to\infty} \inf B_n(t)$. By the same way we can prove that

$$\bar{B}(t) \subset \lim_{n\to\infty} \inf \bar{B}_n(t).$$

Thus,

$$p(B(t)) \times p(\bar{B}(t)) \leq \lim_{n\to\infty} \inf p(B_n(t)) \times p(\bar{B}_n(t)),$$

since the sample paths are continuous then

$$B_n(t) = \{\tau_{\phi_n} > t), \quad B(t) = \{\tau_\phi > t),$$

we obtain

$$\int_0^\infty [p(\tau_{\bar{\phi}} > t) \times p(\tau_\phi > t)]dt \leq \int_0^\infty \lim_{n\to\infty} \inf[p(\tau_{\bar{\phi}_n} > t) \times p(\tau_{\phi_n} > t)]dt,$$

then $\int_0^\infty p(\tau > t)dt \leq \int_0^\infty \lim_{n\to\infty} \inf p(\tau_{\Phi_n} > t)dt$, where $\Phi_n = (\phi_n, \bar{\phi}_n)$ hence, $E(\tau) \leq E(\lim_{n\to\infty} \inf \tau_{\Phi_n})$. By Fatou's Lemma (see [6]), we get

$$E(\tau) \leq E(\lim_{n\to\infty} \inf \tau_{\Phi_n}).$$

Since Φ is sequentially compact (see [1]), then by the same way $\bar{\Phi}$ is sequentially compact. It is known that a lower semi-continuous function over a sequentially compact space attains its minimum. □

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