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A WEAKER VERSION OF THE SHADOWING LEMMA FOR OPERATORS WITH CHAOTIC BEHAVIOUR

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Abstract: We show that it is possible under the same computational cost and weaker hypotheses to find solutions of discrete dynamical systems involving operators with chaotic behaviour.

In particular we weaken the Shadowing Lemma [4] using a weaker version of the Newton-Kantorovich Theorem, see [3], recently reported by us in [1], [2].

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1. Introduction

It is well known that complicated behaviour of dynamical systems can easily be detected via numerical experiments. However, it is very difficult to prove mathematically in general that a given system behaves chaotically.

Several authors have worked on various aspects of this problem, see, e.g., [4]–[6], and the references therein. In particular the Shadowing Lemma [4, p. 1684] proved via the celebrated Newton-Kantorovich Theorem [3] was used in [4] to present a computer-assisted method that allows us to prove that a discrete dynamical system admits the shift operator as a subsystem. Motivated by this work and using a weaker version of the Newton-Kantorovich Theorem reported by us in [1] (see Theorem 1 that follows) we show that it is possible

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to weaken the Shadowing Lemma on on which the work in [4] is based. In particular we show that under weaker hypotheses and the same computational cost a larger upper bound on the crucial norm of operator L^{-1} (see (7)) is found and the information on location of the shadowing orbit is more precise. Other advantages have already been reported in [1]. Clearly this apporach widens the applicability of the Shadowing Lemma.

2. The Shadowing Lemma

We need the definitions: Let $D \subseteq \mathbf{R}^k$ be an open subset of $\mathbf{R}^k$ (k a natural number), and let $f : D \to D$ be an injective operator. Then the pair (D, f) is a discrete dynamical system. Denote by $S = l^\infty\left(\mathbf{Z}, \mathbf{R}^k\right)$ the space of $\mathbf{R}^k$ valued bounded sequences $x = \{x_n\}$ with norm $\|x\| = \sup_{n\in\mathbf{Z}} |x_n|_2$. Here we use the Euclidean norm in $\mathbf{R}^k$ and denote it by $|\cdot|$, omitting the index 2. A δ_0-pseudo-orbit is a sequence $y = \{y_n\} \in D^{\mathbf{Z}}$ with $|y_{n+1} - f(y_n)| \leq \delta_0$ ($n \in \mathbf{Z}$). A r-shadowing orbit $x = \{x_n\}$ of a δ_0-pseudo-orbit y is an orbit of (D, f) with $|y_n - x_n| \leq 2$ ($n \in \mathbf{Z}$).

We need the following Semilocal Convergence Theorem for Newton method [1].

Theorem 1. *Let $F : D \subseteq X \to Y$ be a Fréchet differentiable operator. Assume there exist $x_0 \in D$ and positive constant η, β, L_0 and L such that $F'(x_0)^{-1} \in L(Y, X)$,*

$$\left\|F'(x_0)^{-1}\right\| \leq \beta, \tag{1}$$

$$\left\|F'(x_0)^{-1} F(x_0)\right\| \leq \eta, \tag{2}$$

$$\left\|F'(x) - F'(y)\right\| \leq L\|x-y\|, \quad \text{for all } x, y \in D, \tag{3}$$

$$\left\|F'(x) - F'(x_0)\right\| \leq L_0\|x-x_0\|, \quad \text{for all } x \in D, \tag{4}$$

$$h_A = \beta(L_0 + L)\eta \leq 1 \tag{5}$$

and

$$\bar{U}(x_0, s^*) = \{x \in X : \|x - x_0\| \leq s^*\} \subseteq D,$$

where $s^* = \lim_{n\to\infty} s_n$,

$$s_0 = 0, s_1 = \eta, s_{n+2} = s_{n+1} + \frac{L(s_{n+1} - s_n)}{2(1 - L_0 s_{n+1})} \quad (n \geq 0).$$

Then sequence $\{y_n\}$ $(n \geq 0)$ generated by the Newton method

$$y_{n+1} = y_n - F'(y_n)^{-1} F(y_n) \quad (n \geq 0)$$

is well defined, remains in $\bar{U}(x_0, s^*)$ *for all* $n \geq 0$ *and converges to a unique solution* $y^* \in \bar{U}(x_0, s^*)$*, so that estimates*

$$\|y_{n+1} - y_n\| \leq s_{n+1} - s_n$$

and

$$\|y_n - y^*\| \leq s^* - s_n \leq 2\eta - s_n$$

hold for all $n \geq 0$.

Moreover y^* *is the unique solution of equation* $F(y) = 0$ *in* $U(x_0, R)$ *provided that*

$$L_0 (s^* + R) \leq 2$$

and

$$U(x_0, R) \subseteq D.$$

The advantages of Theorem 1 over the Newton-Kantorovich Theorem [3] have been explained in detail in [1], [2].

From now on we set $X = Y = \mathbf{R}^k$.

Sufficient conditions for a δ_0-pseudo-orbit y to admit a unique r-shadowing orbit are given in the following main result.

Theorem 2. (Weak Version of the Shadowing Lemma) *Let* $D \subseteq \mathbf{R}^k$ *be open,* $f \in C^{1,Lip}(D, D)$ *be injective,* $y = \{y_n\} \in D^{\mathbf{Z}}$ *be a given sequence,* $\{A_n\}$ *be a bounded sequence of* $k \times k$ *matrices and let* $\delta_0, \delta, \ell_0, \ell$ *be positive constants. Assume that for the operator*

$$L : S \to S \text{ with } \{Lz\}_n = z_{n+1} - Az_n \tag{6}$$

is invertible and

$$\left\|L^{-1}\right\| \leq a = \frac{1}{\delta + \sqrt{(\ell + \ell_0)\,\delta_0}}. \tag{7}$$

Then the numbers t^*, R *given by*

$$t^* = \lim_{n \to \infty} t_n \tag{8}$$

and

$$R = \frac{2}{\ell_0} - t^* \tag{9}$$

satisfy $0 < t^* \leq R$*, where sequence* $\{t_n\}$ *is given by*

$$t_0 = 0, t_1 = \eta, t_{n+2} = t_{n+1} + \frac{\ell\,(t_{n+1} - t_n)^2}{2\,(1 - \ell_0 t_{n+1})} \quad (n \geq 0) \tag{10}$$

and

$$\eta = \frac{\delta_0}{\frac{1}{\|L^{-1}\|} - \delta}. \tag{11}$$

Let $r \in [t^*, R]$. Moreover, assume that

$$\overline{\bigcup_{n \in \mathbf{Z}} U(y_n, r)} \subseteq D \tag{12}$$

and for every $n \in \mathbf{Z}$

$$|y_{n+1} - f(y_n)| \le \delta_0, \tag{13}$$

$$|A_n - Df(y_n)| \le \delta, \tag{14}$$

$$\left|F'(u) - F'(0)\right| \le \ell_0 |u| \tag{15}$$

and

$$\left|F'(u) - F'(v)\right| \le \ell |u - v|, \tag{16}$$

for all $u, v \in U(y_n, r)$.

Then there is a unique t^*-shadowing orbit $x^* = \{x_n\}$ of y. Moreover, there is no orbit $\bar{x}$ other than x^* such that

$$\|\bar{x} - y\| \le r. \tag{17}$$

Proof. We shall solve the difference equation

$$x_{n+1} = f(x_n) \quad (n \ge 0) \tag{18}$$

provided that x_n is close to y_n. Setting

$$x_n = y_n + z_n \tag{19}$$

and

$$g_n(z_n) = f(z_n + y_n) - A_n z_n - y_{n+1}, \tag{20}$$

we can have

$$z_{n+1} = A_n z_n + g_n(z_n). \tag{21}$$

Define $D_0 = \{z = \{z_n\} : \|z\| \le 2\}$ and nonlinear operator $G : D_0 \to S$, by

$$(G(z))_n = g_n(z_n). \tag{22}$$

Operator G can naturally be extended to a neighborhood of D_0. Equation (21) can be rewritten as

$$F(x) = Lx - G(x) = 0, \tag{23}$$

where F is an operator from D_0 into S.

We will show the existence and uniqueness of a solution $x^* = \{x_n\}$ $(n \ge 0)$ of equation (23) with $\|x^*\| \le r$ using Theorem 1. Clearly we need to express η, L_0, L and β in terms of $\left\|L^{-1}\right\|, \delta_0, \delta, \ell_0$ and ℓ.

(i) $\left\|F'(0)^{-1}F(0)\right\| \leq \eta$.

Using (13), (14) and (20) we get $\|F(0)\| \leq \delta_0$ and $\|G'(0)\| \leq \delta$, since $[G'(0)(w)]_n = (F'(y_n) - A_n)\, w_n$.

By (7) and the Banach Lemma on invertible operators [3] we get $F'(0)^{-1}$ exists and

$$\left\|F'(0)^{-1}\right\| \leq \left(\frac{1}{\|L^{-1}\|} - \delta\right)^{-1}. \tag{24}$$

That is, η can be given by (11).

(ii) $\left\|F'(0)^{-1}\right\| \leq \beta$.

By (24) we can set

$$\beta = \left(\frac{1}{\|L^{-1}\|} - \delta\right)^{-1}. \tag{25}$$

(iii) $\|F'(u) - F'(v)\| \leq L\,\|u - v\|$.

We can have using (16)

$$\begin{aligned}\left|\left(F'(u) - F'(v)\right)(w)_n\right| &= \left|\left(F'(y_n + u_n) - F'(y_n + v_n)\right) w_n\right| \\ &\leq \ell\,|u_n - v_n|\,|w_n|\,. \end{aligned}\tag{26}$$

Hence we can set $L = \ell$.

(iv) $\|F'(u) - F'(0)\| \leq L_0\,\|u\|$.

By (17) we get

$$\begin{aligned}\left|\left(F'(u) - F'(0)\right)(w)_n\right| &= \left|\left(F'(y_n + u_n) - F'(y_n + 0)\right) w_n\right| \\ &\leq \ell_0\,|u_n|\,|w_n|\,. \end{aligned}\tag{27}$$

That is, we can take $L_0 = \ell_0$.

Crucial condition (5) is satisfied by (7) and with the above choices of η, β, L and L_0.

Therefore the claims of Theorem 2 follow immediately from the conclusions of Theorem 1.

That completes the proof of the theorem. □

Remark 1. In general

$$\ell_0 \leq \ell \tag{28}$$

holds and $\frac{\ell}{\ell_0}$ can be arbitrarily large. If $\ell_0 = \ell$, Theorem 2 reduces to Theorem 1 in [4, p. 1684]. Otherwise our Theorem 2 improves Theorem 1 in [4]. Indeed, the upper bound in [4, p. 1684] is given by

$$\left\|L^{-1}\right\| \leq b = \frac{1}{\delta + \sqrt{2\ell\delta_0}}. \tag{29}$$

By comparing (7) with (29) we deduce

$$b < a$$

(if $\ell_0 < \ell$).

That is, we have justified the claims made in the introduction.

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