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CHARACTERIZATIONS ON ORDERED Γ-SEMIGROUPS

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Abstract: In this paper we give some characterizations and properties of $po-\Gamma$-semigroups concerning maximal and minimal ideals and left ideals which are analogues to those for ordered semigroups. We also generalize some properties on $poe-\Gamma$-semigroups analogues to those of totally ordered semigroups containing maximal elements.

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1. Introduction and Preliminaries

In this paper mainly we extend and prove some important results and properties of *po*-semigroups to $po-\Gamma$-semigroups and $poe-\Gamma$-semigroups. In the two first sections we introduce and give some results of po-semigroups concerning minimal and maximal ideals and left ideals to $po-\Gamma$-semigroups and we give some

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characterizations of these concepts of $po-\Gamma$-semigroups which are analogues to those for ordered semigroup ([1], [4], [2]). We prove that in a $po-\Gamma$-semigroup M without zero there is at most one minimal ideal which is the intersection of all ideals of M. We also prove that in a $po-\Gamma$-semigroup M for which there exists $a \in M$ such that the ideal of M generated by a is M, there is at most one maximal ideal which is the union of all proper ideals of M. In $po-\Gamma$-semigroups containing the unit element, there is at most one maximal ideal which is the union of all proper ideals of M. In a commutative $po-\Gamma$-semigroups containing the unit element, each ideal maximal is a prime ideal. Also we introduce and give some characterizations of the maximal left ideal of a $po-\Gamma$-semigroup. The authors ([8], [9], [10], [7]) gave some important results on some properties on totally ordered semigroup containing maximal elements. In Section 3 of this paper we extend and generalize these results on $poe-\Gamma$-semigroups.

Sen [11] defined Γ-semigroup as follows.

Definition 1.1. Let M and Γ be any two non-empty sets. Then M is called a $\Gamma-$ *grupoid* if $a\alpha b \in M$ and $\alpha a\beta \in \Gamma$ for all $a, b \in M$ and for all $\alpha, \beta \in \Gamma$.

Definition 1.2. Let $M = \{a, b, c, ...\}$ and $\Gamma = \{\alpha, \beta, \gamma, ...\}$ be two non-empty sets. Then M is called a Γ-semigroup if there exist mappings $M \times \Gamma \times M \to M$, written as $(a, \alpha, b) \longmapsto a\alpha b$ and $\Gamma \times M \times \Gamma \to \Gamma$ written as $(\alpha, a, \beta) \longmapsto \alpha a\beta$, satisfying the following identities:

$$(a, \alpha, b)\beta c = a(\alpha b\beta)c = a\alpha(b\beta c) \text{ for all } a, b, c \in M \text{ and for all } \alpha, \beta \in \Gamma.$$

Sen and Saha [12] weakened the defining conditions of Γ-semigroup and defined Γ-semigroup by replacing the above conditions by follows:

(1) $a\alpha b \in M$

(2) $a\alpha(b\beta c) = (a\alpha b)\beta c$, for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$.

We call a Γ-semigroup M both sided if it further satisfies the identities

$$\alpha a(\beta b\gamma) = (\alpha a\beta)b\gamma = \alpha(a\beta b)\gamma \text{ for all } a, b \in M \text{ and for all } \alpha, \beta, \gamma \in \Gamma.$$

In this paper we consider only both sided Γ-semigroup and call it simply Γ-semigroup.

Kwon and Lee [5] introduced the concept of a *po*-Γ-semigroup and of ideals in a *po*-Γ-semigroup and obtained some results.

Definition 1.3. A *po*-Γ-grupoid is an ordered set M at the same time Γ-grupoid such that

$$a \le b \Rightarrow a\alpha c \le b\alpha c, \quad c\alpha a \le c\alpha b\,,$$

for all $c \in M$ and for all $\alpha \in \Gamma$.

When M is a Γ-semigroup, M is called a *po*-Γ-semigroup. A *poe*-Γ-semigroup is a *po*-Γ-semigroup M with the greatest elements "e" (i.e., $e \geq a, \forall a \in M$).

Let M be a *po*-Γ-semigroup and A be a nonempty subset of M. Then A is called a *right* (resp. *left*) ideal of M if:

1. $A\Gamma M \subseteq A$ (resp. $M\Gamma A \subseteq A$)
2. $a \in A, b \leq a$ for $b \in M \Rightarrow b \in A$

A is called an *ideal* of M if it is right and left ideal of M. A right, left or ideal A of a $po-\Gamma$-semigroup M is called *proper* if $A \neq M$.

An element a of an Γ-semigroup M is called a *left* (resp. *right*) *zero element* of M if $a\gamma b = a$ (resp. $b\gamma a = a$), $\forall b \in M$ and $\forall \gamma \in \Gamma$. If M is a $po-\Gamma$-semigroup, *zero* of M is an element $a \in M$ which is a left and right zero element of M and $a \leq x, \forall a \in M$.

An element a of an Γ-semigroup M is called *idempotent* if $a = a\gamma a$, $\forall \gamma \in \Gamma$.

We denote $M^1 = M \cup \{1\}$, where 1 is a symbol such that $a\gamma 1 = 1\gamma a = a, \forall a \in M$.

Let M be a *po*-Γ-semigroup. For $\emptyset \neq A \subseteq M$, we denote by $J(A)$ the ideal of M generated by A, by $L(A)$ the left ideal of M generated by A and by $R(A)$ the right ideal of M generated by A.

For $A \subseteq M$ we denote

$$(A] = \{t \in M | t \leq a, \text{ for some } a \in A\}.$$

For $A = \{a\}$, we write $(a]$ instead of $(\{a\}]$. We denote by $L(a)$ the left ideal of M generated by $a \in M$, by $R(a)$ the right ideal of M generated by $a \in M$, by $J(a)$ the ideal of M generated by $a \in M$. One can easily prove (see [6]) that $L(a) = M^1\Gamma a = \{a\} \cup M\Gamma\{a\} = (a \cup M\Gamma a] = (a] \cup (M\Gamma a]$, $R(a) = a\Gamma M^1 = \{a\} \cup \{a\}\Gamma M = (a \cup a\Gamma M] = (a] \cup (a\Gamma M]$, $J(a) = M^1\Gamma a\Gamma M^1 = \{a\} \cup M\Gamma\{a\} \cup \{a\}\Gamma M \cup M\Gamma\{a\}\Gamma M = (a \cup M\Gamma a \cup a\Gamma M \cup M\Gamma a\Gamma M]$.

The authors [6] proved the following lemma.

Lemma 1.4. *Let M be a po-Γ-semigroup. The following holds true:*

1. $A \subseteq (A]$ *for any* $A \subseteq M$.
2. *If* $A \subseteq B \subseteq M$, *then* $(A] \subseteq (B]$.
3. $(A]\Gamma(B] \subseteq (A\Gamma B]$ *for all subsets* A *and* B *of* M.
4. $((A]] \subseteq (A]$ *for all* $A \subseteq M$.
5. *For every left (resp. right, two-sided) ideal* T *of* M, $(T] = T$.

6. *If L is a left ideal and R a right ideal of M, then the set $(L\Gamma R]$ is an ideal of M.*

7. *If A, B are ideals of M, then $(A\Gamma B], (B\Gamma A], A\cup B, A\cap B$ are ideals of M.*

8. *$(M\Gamma a]$ $(resp.(a\Gamma M])$ is a left (resp. right) ideal of M for every $a \in M$.*

9. *$(M\Gamma a\Gamma M]$ is an ideal of M for every $a \in M$.*

10. *$((A]\Gamma(B]] = (A\Gamma B]$, for any $A, B \subseteq M$.*

Proof. We prove the statement 6). We have $\emptyset \neq (L\Gamma R] \subseteq M$ (since $L \neq \emptyset, R \neq \emptyset$).

$$M\Gamma(L\Gamma R] = (M]\Gamma(L\Gamma R] \subseteq ((M\Gamma L)\Gamma R] \subseteq (L\Gamma R],$$

$$(L\Gamma R]\Gamma M = (L\Gamma R]\Gamma(M] \subseteq (L\Gamma(R\Gamma M)] \subseteq (L\Gamma R].$$

If $a \in (L\Gamma R]$ and $M \ni b \leq a$, then $b \in (L\Gamma R]$.

Therefore, as corollary of 6) is the statement 7). It is easily to prove the following lemma.

Lemma 1.5. *Let M be a $po-\Gamma$-semigroup and $\{A_i | i \in I\}$ a non-empty family of ideals (resp. left ideals) of M. If $\cap\{A_i | i \in I\} \neq \emptyset$, then the set $\cap\{A_i | i \in I\}$ is an ideal (resp. left ideal) of M and $\cup\{A_i | i \in I\}$ is also an ideal (resp. left ideal) of M.*

2. On Minimal Ideals and Left Ideals of $po-\Gamma$-semigroups

Let M be a $po-\Gamma$-semigroup without zero. An ideal (resp. left ideal) A of M is called *minimal* if for every ideal (resp. left ideal) T of M such that $T \subseteq A$, we have $T = A$. Let M be a $po-\Gamma$-semigroup with zero. Then the ideal $\{0\}$ satisfies this condition and we have: Let T be an ideal (resp. left ideal) of M such that $T \subseteq \{0\}$. Since T is an ideal (resp. left ideal) of M, we have $0 \in T$, then $\{0\} \subseteq T$, and $T = \{0\}$. So, in the $po-\Gamma$-semigroup M, an ideal (resp. left ideal) A is called *minimal* if: 1) $A\backslash\{0\}$ and 2) for every ideal (resp. left ideal) T of M such that $T \subseteq M$, we have $T = \{0\}$ or $T = M$. This is whatever we often call 0-minimal ideal (resp. left ideal). So, when we speak about minimal ideals in $po-\Gamma$-semigroups we have to distinguish if the $po-\Gamma$-semigroups have or do not have a zero.

A $po-\Gamma$-semigroup M is called *left simple* if it does not contain proper left ideals. A left ideal L of M is called left simple if the $po-\Gamma$-semigroup L is left simple. Let M be a $po-\Gamma$-semigroup with a zero element. If $M=\{0\}$, then M is left simple. If there exists $a\in M, a\neq 0$, then $\{0\}$ is a left ideal of M, $\{0\}\subset M$. Thus a $po-\Gamma$-semigroup M with a zero element such that $M\neq\{0\}$ is not left simple.

Theorem 2.1. *Let M be a $po-\Gamma$-semigroup without zero, and $\mathcal{A}$ be the set of all ideals of M. The following are equivalent:*

1. *M contains minimal ideals.*
2. *The set $\cap\{J|J\in\mathcal{A}\}$ is the only minimal ideal of M.*
3. *$\cap\{J|J\in\mathcal{A}\}\neq\emptyset$.*

Proof. 1) $\Rightarrow$ 2) Let I be a minimal ideal of M. Then $I=\cap\{J|J\in\mathcal{A}\}$. Indeed, let J be an ideal of M. We have $J\Gamma I\subseteq M\Gamma I\subseteq I$, then $(J\Gamma I]\subseteq(I]=I$. Since I is a minimal ideal and $(J\Gamma I]$ an ideal of M, we have $I=(J\Gamma I]$. Since $J\Gamma I\subseteq J\Gamma M\subseteq J$, we have $I=(J\Gamma I]\subseteq(J]=J$. Hence the set $\cap\{J|J\in\mathcal{A}\}$ is a minimal ideal of M.

2) $\Rightarrow$ 3). Since $\subseteq\{J|J\in\mathcal{A}\}$ is an ideal of M, we have $\cap\{J|J\in\mathcal{A}\}\neq\emptyset$.

3) $\Rightarrow$ 1). Since $\cap\{J|J\in\mathcal{A}\}\neq\emptyset$, by Lemma 1.5, the set $\cap\{J|J\in\mathcal{A}\}$ is an ideal of M. The set $\cap\{J|J\in\mathcal{A}\}$ is a minimal ideal of M. Indeed, let K be an ideal of M such that $K\subseteq\cap\{J|J\in\mathcal{A}\}$. Since K is an ideal of M, we have $\cap\{J|J\in\mathcal{A}\}\subseteq K$.

Therefore, if a $po-\Gamma$-semigroup M contains minimal ideals, equivalently, if the intersection of all ideals of M is non-empty, then this minimal ideal is unique, and it is the intersection of all ideals of M. This is called Kernel of M. □

Lemma 2.2. *Let M be a $po-\Gamma$-semigroup. M is left simple if and only if $(M\Gamma a]=M$ for all $a\in M$.*

Proof.$\Rightarrow$ Let M be left simple and $a\in M$. Since $(M\Gamma a]$ is a left ideal of M and by the definition we have $(M\Gamma a]=M$.

$\Leftarrow$ Let $(M\Gamma a]=M,\forall a\in M$. Let A be a left ideal of M. Let $x\in A(A\neq\emptyset)$. Since $M=(M\Gamma a]\subseteq(M\Gamma A]\subseteq(A]=A$. We have $A=M$. It implies that M is left simple. □

Lemma 2.3. *Let L be a left ideal of M and K a left simple sub-Γ-semigroup of M. If $K\cap L\neq\emptyset$, then $K\subseteq L$.*

Proof. Let $a\in K\cap L$. Since K is left simple, $a\in K$, by Lemma 2.2, we have $(K\Gamma a]=K$. Then we have $K=(K\Gamma a]\subseteq(K\Gamma L]\subseteq(M\Gamma L]\subseteq(L]=L$.

Theorem 2.4. *Let M be a $po-\Gamma$-semigroup without zero. A left ideal of M is minimal if and only if it is left simple.*

Proof. $\Rightarrow$ Let L be a minimal left ideal of M. If A is a left ideal of L, then $L\Gamma A \subseteq A$. Let $H = \{h \in A | h \leq k\gamma a$ for some $k \in L, a \in A$ and $\gamma \in \Gamma\}$. Then $H \subseteq A \subseteq L$. Let $x \in M, h \in H$. Since $h \in H$ we have $h \leq k\gamma a$ for some $k \in L, a \in A, \gamma \in \Gamma$. Hence $x\delta h \leq x\delta(k\gamma a) = (x\delta k)\gamma a$. Then, since $x\delta h \in M\Gamma H \subseteq M\Gamma L \subseteq L$, $(x\delta k)\gamma a \in (M\Gamma L)\Gamma A \subseteq L\Gamma A \subseteq A$, and A is a left ideal of L, we have $x\delta h \in A$. Since $x\delta h \leq (x\delta k)\gamma a, x\delta k \in M\Gamma L \subseteq L, a \in A$, we have $x\delta h \in H$. Thus $M\Gamma H \subseteq H$. Let $t \in M$, and $t \leq h$ for some $h \in H$. Since $h \in H$, we have $h \in A$ and $h \leq k\gamma a$ for some $k \in L, a \in A, \gamma \in \Gamma$. From $t \leq k\gamma a \in M\Gamma L \subseteq L$, we obtain $t \in L$. Since $k\gamma a \in L\Gamma A \subseteq A, t \in L, t \leq k\gamma a \in A$ and A is a left ideal of L, we have $t \in A$. Then, since $t \in A, t \leq k\gamma a, k \in L, a \in A$ and $\gamma \in \Gamma$, we have $t \in H$. Thus H is a left ideal of M. Since L is a minimal left ideal of M, we have $H = L$ and so $A = L$. Therefore L is left simple.

$\Leftarrow$ Let K be a left ideal of M and left simple. Let I be a left ideal of M such that $I \subseteq K$. I is a left ideal of K. Since K is a left simple, we have $I = K$. Therefore, K is a minimal left ideal of M. □

Theorem 2.5. *Let M be a $po-\Gamma$-semigroup without zero, and let M have proper left ideals. Then every proper left ideal of M is minimal (left simple), if and only if M contains exactly one proper left ideal or M contains exactly two proper left ideals L_1, L_2 such that $M = L_1 \cup L_2$.*

Proof. $\Rightarrow$ Let I be a proper left ideal of M. By hypothesis, I is a minimal left ideal of M. Then we have the following two cases:

α) $M = L(a), \forall a \in M \backslash I$. Suppose that K is also a proper left ideal of M and $K \neq I$. If $K \backslash I = \emptyset$, then $K \subseteq I$ and so $K \subset I$ which is a contradiction, since I is minimal. If $K \backslash I \neq \emptyset$, then $M = L(a) \subseteq K$ for some $a \in K \backslash I \subseteq M \backslash I$ and so $M = K$ which is impossible. Thus, in this case I is the unique proper left ideal of M.

β) $M \neq L(a)$ for some $a \in M \backslash I$. Then $L(a)$ is a proper left ideal of M. By hypothesis, $L(a0$ is minimal. By Lemma 1.5. $L(a) \cup I$ is a left ideal of M. Suppose that $I \cup L(a) \neq M$, then $I \cup L(a)$ is a proper left ideal of M. By hypothesis, $I \cup L(a)$ is a minimal left ideal of M. On the other hand, $I \subset L(a) \cup I$ which is a contradiction. Thus $M = I \cup L(a)$. Let K be an arbitrary proper left ideal of M. By hypothesis, K is a minimal left ideal of M. Since $K = K \cap M = (K \cap I) \cup (K \cap L(a))$, we have the following two cases:

(i) $K \cap I \neq \emptyset$. Then $K \cap I$ is a left ideal of M by Lemma 1.5. Since $K \cap I \subseteq I, I$ is minimal, we have $K \cap I = I$, i.e., $I \subseteq K$. Since K is minimal, we have $I = K$.

(ii) $K \cap L(a) \neq \emptyset$. A similar argument shows that $K = L(a)$. Therefore, in this case, M contains exactly two proper left ideals I and $L(a)$ such that $M = I \cup L(a)$.

$\Leftarrow$ If M contains exactly one proper left ideal L, it is obvious that L is minimal. Suppose that M contains exactly two proper left ideals L_1, L_2 such that $M = L_1 \cup L_2$. Then $L_1 \not\subset L_2$ and $L_2 \not\subset L_1$, otherwise $M \neq L_1 \cup L_2$. Let A be a left ideal of M such that $A \subseteq L_1$. Then $A \subseteq L_1 \subset M$, and so A is a proper left ideal of M. Since $A \subseteq L_1$ and $L_2 \not\subseteq L_1$, we have $A \neq L_2$. Since M contains exactly two proper left ideals L_1, L_2, we have $A = L_1$. Thus L_1 is minimal. A similar argument shows that L_2 is minimal. □

3. On Maximal Ideals and Left Ideals of $po-\Gamma$-Semigroups

An ideal A of a $po-\Gamma$-semigroup M is called *proper* if $A \neq M$. A proper ideal A of M is called *maximal* if for each ideal T of M such that $T \supseteq A$, we have $T = A$ or $T = M$ i.e., there is no ideal T of M such that $A \subset T \subset M$. A proper left ideal L of M is called *maximal* if for a left ideal T of M such that $L \subset T$, then $T = M$.

Proposition 3.1. *Let M be a $po-\Gamma$-semigroup and A a proper ideal of M. Then A is a maximal ideal of M if and only if for every $a \in M \backslash A$, we have $J(A \cup \{a\}) = M$.*

Proof. $\Rightarrow$ Let $a \in M\backslash A$. Since $A \subseteq A \cup \{a\} \subseteq J(A \cup \{a\})$, $J(A \cup \{a\})$ is an ideal of M and A is a maximal ideal of M, we have $J(A \cup \{a\}) = A$ or $J(A \cup \{a\}) = M$. Since $a \in J(A \cup \{a\})$ and $a \notin A$, we have $J(A \cup \{a\}) \neq A$. So $J(A \cup \{a\}) = M$.

$\Leftarrow$ Let T be an ideal of M, $A \subseteq T$ and $A \neq T$. Let $a \in T, a \notin A$. Since $a \in M\backslash A$, we have $J(A \cup \{a\}) = M$. Since $A \subseteq T$ and $a \in T$, we have $A \cup \{a\} \subseteq T$. Thus we have

$$\begin{aligned} M &= J(A \cup \{a\}) \\ &= ((A \cup \{a\}) \cup M\Gamma(A \cup \{a\}) \cup (A \cup \{a\})\Gamma M \cup M\Gamma(A \cup \{a\})\Gamma M] \\ &\subseteq (T \cup M\Gamma T \cup T\Gamma M \cup M\Gamma T\Gamma M] = (T] = T. \end{aligned}$$

Lemma 3.2. *Let M be a $po-\Gamma$-semigroup, $t \in M$ and $M \subseteq J(t)$. Let A_1 be a proper ideal of M and A_2 a maximal ideal of M. Then $A_1 \subseteq A_2$.*

Proof. Let $a \in A_1$ such that $a \notin A_2$. From Proposition 3.1 we have $J(A_2 \cup \{a\}) = M$. On the other hand,

$$J(A_2 \cup \{a\}) = ((A_2 \cup \{a\}) \cup M\Gamma(A_2 \cup \{a\}) \cup (A_2 \cup \{a\})\Gamma M \cup M\Gamma(A_2 \cup \{a\})\Gamma M] \,.$$

Hence we have

$$a \in ((A_2 \cup \{a\}) \cup M\Gamma(A_2 \cup \{a\}) \cup (A_2 \cup \{a\})\Gamma M \cup M\Gamma(A_2 \cup \{a\})\Gamma M] \,.$$

Then there exist $u, v \in M^1$, $y \in A_2 \cup \{a\}$ and $\gamma, \beta \in \Gamma$ such that $t \leq u\gamma y\beta v$. If $y \in A_2$, then $u\gamma y\beta v \in M^1\Gamma A_2\Gamma M^1 \subseteq A_2, t \in A_2, M \subseteq J(t) \subseteq A_2$, and $A_2 = M$ which is impossible.

If $y = a$, then $t \leq u\gamma y\beta v \in M^1\Gamma A_1\Gamma M^1 \subseteq A_1, t \in A_1, M \subseteq J(a) \subseteq A_1$, and $A_1 = M$, which is again impossible. □

From Lemma 3.2, we obtain the following result.

Lemma 3.3. *Let M be a po$-\Gamma$-semigroup, $a \in M$, and $M \subseteq J(a)$. Then, for any two maximal ideals A_1 and A_2 of M, we have $A_1 = A_2$.*

Proposition 3.4. *Let M be a po $-\,\Gamma$-semigroup, $a \in M, M \subseteq J(a)$. Let $\mathcal{A}$ be the set of all proper ideals of M. If $\mathcal{A} \neq \emptyset$, the the set $\cup\{I|I \in \mathcal{A}\}$ is the unique maximal ideal of M.*

Proof. Since $\mathcal{A} \neq \emptyset$, then by Lemma 1.5 the set $\cup\{I|I \in \mathcal{A}\}$ is an ideal of M. The set $\cup\{I|I \in \mathcal{A}\}$ is a proper ideal of M. Indeed, let $\cup\{I|I \in \mathcal{A}\} = M$. Let $I \in \mathcal{A}$ such that $a \in I$. Since I is an ideal of M, $a \in I$, and $M \subseteq J(a)$, we have $M \subseteq J(a) \subseteq I$, then $I = M$. This is impossible. Let now T be an ideal of M such that $\cup\{I|I \in \mathcal{A}\} \subseteq T$, and let $T \neq M$. Since $T \in \mathcal{A}$, we have $T \subseteq \cup\{I|I \in \mathcal{A}\}$. Then $T = \cup\{I|I \in \mathcal{A}\}$.

If A is a maximal ideal of M then, by Lemma 3.3, we have $A = \cup\{I|I \in \mathcal{A}\}$. So the set $\cup\{I|I \in \mathcal{A}\}$ is the unique maximal ideal of M. Therefore, if an $po-\Gamma$-semigroup M for which there is an element $a \in M$ such that $M \subseteq J(a)$ contains proper ideals, then the union of all proper ideals of M is an ideal, and the union of all proper ideals of M is the unique maximal ideal of M.

If M is a $po-\Gamma$-semigroup and $1 \in M$, then we have $M = M\Gamma 1 \subseteq (M\Gamma 1] \subseteq J(1)$. □

From Lemma 3.2, Lemma 3.3 and Proposition 3.4 it follows the following result.

Lemma 3.5. *Let M be a po $-\,\Gamma$-semigroup containing the unit element. Let A_1 be a proper ideal of M, and A_2 be a maximal ideal of M. Then $A_1 \subseteq A_2$.*

Lemma 3.6. *Let M be a po $-\,\Gamma$-semigroup containing the unit element. Then for any two ideal maximal ideals A_1 and A_2 of M, we have $A_1 = A_2$.*

Proposition 3.7. *Let M be a $po-\Gamma$-semigroup containing unit. If the set $\mathcal{A}$ of all proper ideals of M is non-empty, the the set $\cup\{I|I\in\mathcal{A}\}$ is the unique maximal ideals of M.*

Let M be a $po-\Gamma$-semigroup. An ideal A of M is said [3] to be *prime* if $a\Gamma b\subseteq A$ implies that $a\in A$ or $b\in A$ $(a,b\in M)$.

Theorem 3.8. *Let M be a commutative $po-\Gamma$-semigroup containing unit. If A is a maximal ideal of M, then A is a prime ideal of M.*

Proof. Let $a,b\in A, a\Gamma b\in A, a\notin A$. We have to prove that $b\in A$. By Proposition 3.1, since A is a maximal ideal we have $J(A\cup\{a\})=M$. On the other hand, since M is commutative we have:

$$\begin{aligned}&J(A\cup\{a\})\\&\quad=((A\cup\{a\})\cup M\Gamma(A\cup\{a\})\cup(A\cup\{a\})\Gamma M\cup M\Gamma(A\cup\{a\})\Gamma M]\\&\quad=((A\cup\{a\})\cup M\Gamma(A\cup\{a\})\cup M\Gamma M\Gamma(A\cup\{a\})].\end{aligned}$$

Since $A\cup\{a\}=1\Gamma(A\cup\{a\})\subseteq M\Gamma(A\cup\{a\})$, we have

$$M\Gamma(A\cup\{a\})\subseteq M\Gamma M\Gamma(A\cup\{a\})\subseteq M\Gamma(A\cup\{a\}),$$

then $M\Gamma(A\cup\{a\})=M\Gamma M\Gamma(A\cup\{a\})$. Hence we have $J(A\cup\{a\})=(M\Gamma(A\cup\{a\})]$. It follows that $1\in(M\Gamma(A\cup\{a\})]$. Then there exists $x\in M$, $y\in A\cup\{a\}$ and $\gamma\in\Gamma$ such that $1\leq x\gamma y$. Then $b=e\beta 1\leq x\gamma y\beta b$, $\beta\in\Gamma$. If $y\in A$, then $x\gamma y\beta b\in M\Gamma A\Gamma M\subseteq A$, and $b\in A$. If $y=a$, then $b\leq x\gamma(a\beta b)\in M\Gamma A\subseteq A$, and $b\in A$. □

Theorem 3.9. *Let M be a $po-\Gamma$-semigroup and L a proper left ideal of M. Then L is maximal if and only if*

1. *$M\backslash L=\{a\}$, and $a\gamma a\in L$, $\gamma\in\Gamma$, or*
2. *$M\backslash L\subseteq(M\Gamma a], \forall a\in M\backslash L$.*

Proof. ⇒ Let L be a maximal left ideal of M. We consider the two cases:

α) There exists $a\in M\backslash L$ such that $(M\Gamma a]\subseteq L$. In this case, we prove that the property (1) holds. In fact, $a\gamma a\in M\Gamma a\subseteq(M\Gamma a]\subseteq L$. By $(M\Gamma a]\subseteq L$ and by the fact that $L(a)=(a\cup M\Gamma a]=(a]\cup(M\Gamma a]$ we have

$$L\cup(a]=(L\cup(M\Gamma a])\cup(a]=L\cup((M]\cup(a])=L\cup(a\cup M\Gamma a]=L\cup L(a)$$

Then $L\cup(a]$ is a left ideal of M, by Lemma 1.5. On the other hand, since $a\in M\backslash L$, we have $L\subset L\cup(a]$. Since $L\cup(a]$ is a left ideal of M and L a maximal left ideal of M, we have $L \cup (a] = M$. Thus $M \backslash L \subseteq (a]$. Let $x \in M \backslash L$. Then $x \leq a$ and so $(M\Gamma x] \subseteq (M\Gamma a] \subseteq L$. From $(M\Gamma x] \subseteq L, x \in M \backslash L$, a similar argument shows that $M \backslash L \subseteq (x]$. Consequently $a \in (x]$, i.e. $a \leq x$. Hence $a = x$. Thus $M \backslash L = \{a\}$.

β) $(M\Gamma a] \nsubseteq L, \forall a \in M \backslash L$. In this case, the property (2) holds. Indeed: Let $a \in M \backslash L$. Since $(M\Gamma a]$ is a left ideal of M, by Lemma 1.5, we have that $L \cup (M\Gamma a]$ is also a left ideal of M. On the other hand, since $(M\Gamma a] \nsubseteq L$, we have $L \subset L \cup (M\Gamma a]$. Since $L \cup (M\Gamma a]$ is a left ideal of M and L a maximal left ideal of M, we have $L \cup (M\Gamma a] = M$. Thus $M \backslash L \subseteq (M\Gamma a]$,$\forall a \in M \backslash L$.

$\Leftarrow$ Let L be a proper left ideal of M and T a left ideal of M such that $L \subset T$.

1. Let $M \backslash L = \{a\}$ and $a\gamma a \in L, \gamma \in \Gamma$. Then $M = L \cup \{a\}$. Since $L \subset T$, we have $T \backslash L \neq \emptyset$. On the other hand, $T \backslash L \subseteq M \backslash L = \{a\}$. Then we have $T \backslash L = \{a\}$ and $T = L \cup \{a\} = M$. Thus L is maximal.

2. Let $M \backslash L \subseteq (M\Gamma a], \forall a \in M \backslash L$. Let $t \in T \backslash L$. Since $t \in M \backslash L$, by hypothesis, we have $M \backslash L \subseteq (M\Gamma t] \subseteq (M\Gamma T] \subseteq (T] = T$. Hence $M = L \cup (M \backslash L) \subseteq T \cup T = T$ and so $T = M$. Thus, L is a maximal left ideal of M.

Let denote $\Re = \{I | I \text{ is a proper left ideal of } M\}$, then $U = \cup\{I | I \in \Re\}$. □

Remark. $M = U \Leftrightarrow M \neq L(a), \forall a \in M$.

$\Rightarrow$ Suppose that there exists $a \in M$ such that $M = L(a)$. Since $a \in M = U = \cup\{I | I \in \Re\}$, it follows that $a \in I$ for some proper left ideal I of M and so $M = L(a) \subseteq I$ which is impossible since I is a proper left ideal of M.

$\Leftarrow$ Let $a \in M$. By hypothesis, $L(a) \neq M$. Then $L(a)$ is a proper left ideal of M such that $a \in L(a)$. Since $L(a) \in \Re, a \in L(a)$, we have $a \in U$. Thus $M = U$.

Theorem 3.10. *If M is a $po-\Gamma$-semigroup, then one and only one of the following four conditions is satisfied:*

1. *M is left simple.*
2. *$L(a) \neq M, \forall a \in M$.*
3. *There exists $a \in M$ such that $M = L(a), a \notin (M\Gamma a], a\gamma a \in U = M \backslash \{a\}$ and U is the unique maximal left ideal of M.*
4. *$M \backslash U = \{x \in M | (M\Gamma x] = M\}, M \backslash U$ is a subsemigroup of M, and U is the unique maximal left ideal of M.*

Proof. If M is not left simple, then there exists a proper left ideal I of M. Since $I \neq \emptyset$, we have $U \neq \emptyset$. Then by Lemma 1.5 U is a left ideal of M.

i) Let $U = M$. Then the condition 2) is satisfied.

ii) Let $U \neq M$. In this case, either condition 3) or condition 4) is satisfied. In fact, U is the unique maximal left ideal of M. Indeed: Let T be a left ideal of M such that $T \supset U$. Suppose that $T \neq M$. Then T is a proper left ideal of M and so $T \supseteq \cup\{I | I \in \Re\} = U$ which is impossible. Thus U is a maximal left ideal of M. Let K be a maximal left ideal of M. Suppose that $K \neq U$.

α) Let $K \backslash U = \emptyset$. Then $K \subseteq U$, and so $K \subset U$. Since U is a left ideal of M and K is a maximal left ideal of M, we have $U = M$ which is impossible.

β) Let $K \backslash U \neq \emptyset$. Let $a \in K \backslash U$. Since K is a proper left ideal of M, we have $K \subseteq \cup\{I | I \in \Re\} = U$. Then $a \in K \subseteq U$ which is impossible.

Hence $K = U$. Thus U is the unique maximal left ideal of M. Since U is a maximal left ideal of M, by Theorem 3.9, one and only one of the following two conditions is satisfied:

A) $M \backslash U = \{a\}$ and $a\gamma a \in U$.

B) $M \backslash U \subseteq (M\Gamma a], \forall a \in M \backslash U$.

A) Let $M \backslash U\{a\}$ and $a\gamma a \in U$. In this case, condition 3) is satisfied. We have:

α) $L(a) = M$. Indeed: Let $L(a) \neq M$. Then $L(a)$ is a proper left ideal of M and so $L(a) \subseteq U$. Thus $a \in U$ which is impossible.

β) $a \neq (M\Gamma a]$. Indeed: Let $a \in (M\Gamma a]$. Then

$$(a] \subseteq ((M\Gamma a]] = (M\Gamma a] \text{ and } M = L(a) = (a] \cup (M\Gamma a] = (M\Gamma a].$$

We have $a \leq s\gamma a$ for some $s \in M = (M\Gamma a]$, $\gamma \in \Gamma$ and $s \leq t\delta a$ for some $t \in M$ and $\delta \in \Gamma$. Thus we have $a \leq s\gamma a \leq (t\delta a)\gamma a = t\delta(a\gamma a) \in M\Gamma U \subseteq U$, and so $a \in U$ which is impossible.

γ) $a\gamma a \in U = M \backslash \{a\}$. Indeed: By hypothesis, $a\gamma a \in U$. Since $M \backslash U = \{a\}$, we have $U = M \backslash \{a\}$.

B) Let $M \backslash U \subseteq (M\Gamma a], \forall a \in M \backslash U$. Then the condition 4) is satisfied. We have:

α) $M \backslash U = \{x \in M | (M\Gamma a] = M\}$. Indeed: Let $x \in M \backslash U$. By hypothesis, $x \in M \backslash U \subseteq (M\Gamma x]$. Then $(x] \subseteq ((M\Gamma x]] = (M\Gamma x]$, and $L(x) = (x] \cup (M\Gamma x] = (M\Gamma x]$. On the other hand, $L(x) = M$. Indeed: Let $L(x) \subset M$. Since $L(x)$ is a proper left ideal of M, we have $x \in L(x) \subseteq U$ which is imnpossible. Thus $(M\Gamma x] = L(x) = M$. Let $x \in M, (M\Gamma x] = M$. Suppose that $x \in U$. Since U is a left ideal of M and $x \in U$, we have $L(x) \subseteq U$. On the other hand, $L(x) = (x] \cup (M\Gamma x] = (x] \cup M = M$. Since $U \neq M$, we have $U \subset M = L(x)$ which is impossible. Thus $x \notin U$, and $x \in M \backslash U$.

β) $M \backslash U$ is a sub-Γ-semigroup of M. Indeed: $U \neq M$, so $U \subset M$, and $M \backslash U \neq \emptyset$. Let $a, b \in M \backslash U$. By α), we have $(M\Gamma a] = M, (M\Gamma b] = M$. Then $(M\Gamma a\rho b] = ((M\Gamma a]\Gamma b] = (M\Gamma b] = M$, and so $a\rho b \in M \backslash U$. □

4. Some Properties on $poe-\Gamma$-Semigroups

An ideal A of an Γ-semigroup M such that $a\gamma b \in A$ implies that $a \in A$ or $b \in A$ is called a *completely prime* ideal.

An ideal T of a $po-\Gamma$-semigroup M is said to be *convex*, if $\forall x, z \in T$ and $y \in M$, $x < y < z$ implies $y \in T$.

We define the relations "$\mathcal{L}$, $\mathcal{R}$, $\mathcal{D}$" on M as follows:

$$a\mathcal{L}b \Leftrightarrow L(a) = L(b),\ \ a\mathcal{R}b \Leftrightarrow R(a) = R(b),\ \ \mathcal{D} = \mathcal{L} \circ \mathcal{R}.$$

It is clear that $\mathcal{L}, \mathcal{R}, \mathcal{D}$ are equivalence relations on M. We denote L_e, R_e and D_e the $\mathcal{L}-$, $\mathcal{R}-$ or $\mathcal{D}$-class containing the greatest element e respectively.

Theorem 4.1. *Let M be a $poe-\Gamma$-semigroup. Then the following hold true:*

1. *$e = e\alpha e$ for an $\alpha \in \Gamma$ or $M\backslash\{e\}$ is an ideal, which is convex.*
2. *The sets $T_n = \{a|(e\alpha)^{n-1}e\gamma a\gamma(e\alpha)^{n-1}e < (e\alpha)^{n-1}e, \alpha \in \Gamma\}$ and $A_n = \{x|x < (e\alpha)^{n-1}e, \alpha \in \Gamma\}$ with $(e\alpha)^{n-1}e \neq (e\alpha)^n e$ are convex ideals if they are non-empty.*
3. *$T = \{x|e\alpha x < e, \alpha \in \Gamma\}$ and $T^* = \{x|x\alpha e < e, \alpha \in \Gamma\}$ are convex left and right ideals respectively, if they are non-empty.*

Proof. (1) If $e \in M\Gamma M$, then there exist $x, y \in M$ and $\alpha \in \Gamma$ such that $e = x\alpha y$. Hence $e = x\alpha y \leq e\alpha e$. Since e is the greatest element in M, then $e = e\alpha e$. If $e \notin M\Gamma M$, then $\forall x \in M$, $\forall t \in M\backslash\{e\}$ and $\forall \gamma \in \Gamma$, $x\gamma t \in M\Gamma M$ and $t\gamma x \in M\Gamma M$, which means that $x\gamma t \neq e$ and $t\gamma x \neq e$. Thus $x\gamma t \in M\backslash\{e\}$ and $t\gamma x \in M\backslash\{e\}$. Hence $M\backslash\{e\}$ is an ideal. $\forall b \in M$ and $\forall a, c \in M\backslash\{e\}$ with $a < b < c$, if $b \notin M\backslash\{e\}$, then $e = b < c$, which is impossible. So $b \in M\backslash\{e\}$. Hence $M\backslash\{e\}$ is convex.

(2) $\forall a \in T_n$ and $\forall t \in M$, since e is the greatest element of M, then $e\alpha e \leq e$. Thus, $(e\alpha)^n e \leq (e\alpha)^{n-1}e \leq ... \leq e$ by the definition of $po-\Gamma$-semigroup. So $(e\alpha)^{n-1}e\gamma a\alpha t\gamma(e\alpha)^{n-1}e \leq (e\alpha)^{n-1}e\alpha e\alpha e\alpha(e\alpha)^{n-1}e \leq (e\alpha)^{n-1}e$. If
$(e\alpha)^{n-1}e = (e\alpha)^{n-1}e\gamma a\delta t\gamma(e\alpha)^{n-1}e$, then $(e\alpha)^{n-1}e = (e\alpha)^{n-1}e\gamma a\alpha t\gamma (e\alpha)^{n-1}e \leq (e\alpha)^{n-1}e\gamma a\alpha e\gamma(e\alpha)^{n-1}e \leq (e\alpha)^{n-1}e\gamma a\gamma(e\alpha)^{n-1}e < (e\alpha)^{n-1}e$, which is absurd. Hence $(e\alpha)^{n-1}e\gamma a\alpha t\gamma(e\alpha)^{n-1}e < (e\alpha)^{n-1}e$, i.e., $a\alpha t \in T_n$. Similarly $t\alpha a \in T_n$. This shows that T_n is an ideal. Convexity of T_n is obvious.

$\forall a \in A_n$ and $\forall t \in M$, $a\alpha t \leq (e\alpha)^{n-1}e\alpha e \leq (e\alpha)^{n-1}e$. If $a\alpha t = (e\alpha)^{n-1}e$, then $(e\alpha)^{n-1}e = a\alpha t \leq (e\alpha)^{n-1}e\alpha e = (e\alpha)^n e$. But $(e\alpha)^n e \leq (e\alpha)^{n-1}e$, thus

$(e\alpha)^n e = (e\alpha)^{n-1}e$, which is a contradiction with hypothesis. Therefore $a\alpha t < (e\alpha)^{n-1}e$, i.e., $a\alpha t \in A_n$. Similarly $t\alpha a \in A_n$. Convexity of A_n is evident.

(3) $\forall x \in T$ and $\forall t \in M, e\gamma(t\alpha x) = (e\gamma t)\alpha x \leq e\alpha x < e$. Then $t\alpha x \in T$. Convexity of T is obvious. Hence T is a convex left ideal. Similarly we can proof that T^* is a convex right ideal. □

Theorem 4.2. *Let M be a poe $-\ \Gamma$-semigroup with no proper convex ideals. Then $e = e\alpha e = e\gamma\ a\gamma e$ for every $a \in M$.*

Proof. By Theorem 4.1(1) we know that $e = e\alpha e$. By Theorem 4.1(2), $T_1 = \{a | e\gamma a\gamma e < e\}$ is an empty set or a convex ideal. By the hypothesis of Theorem 4.2, $T_1 = \emptyset$ or $T_1 = M$. Since $e = e\alpha e$, i.e., $e \notin T_1$, then $T_1 = \emptyset$, i.e., $\forall a \in M, a \notin T_1$. Since e is the greatest element of M, then $e = e\gamma a\gamma e$. □

Theorem 4.3. *Let M be a poe $-\ \Gamma$-semigroup. Then the following hold true:*

1. *The $\mathcal{R}$-class R_e ($\mathcal{L}$-class L_e) containing e, if non-trivial, is a right (left) zero (sub-)Γ-semigroup.*

2. *If the $\mathcal{D}$-class D_e containing e, is non-trivial, then $|E \cap D_e| > 1$, where E is the set of all idempotents in M.*

3. *If M is left (right) cancellative and $|L_e| > 1$ ($|R_e| > 1$), then $L_e(R_e)$ and $M\backslash L_e(M\backslash R_e)$ are convex (sub-)Γ-semigroups.*

4. *A non-trivial $\mathcal{L}-(\mathcal{R}-)$ class $L_e(R_e)$ is an ideal iff every element of $L_e(R_e)$ is a left (right) zero.*

Proof. (1) $\forall a \in R_e(a \neq e), a\Gamma M^1 = e\Gamma M^1$. By $e \in a\Gamma M^1$ we know that there exists $x \in M, \alpha \in \Gamma$ such that $e = a\alpha x$. Then $e = a\alpha x \leq e\alpha e$, i.e. $e = e\alpha e$. Since $a \in e\Gamma M^1$, then $\exists y \in M$ such that $a = e\gamma y$. Thus $e\alpha a = e\alpha(e\gamma y) = (e\alpha e)\gamma y = e\gamma y = a$, i.e., $a = e\alpha a$. Also $e = a\alpha x$ implies $e \leq a\alpha e$, i.e., $e = a\alpha e$. Hence $a = e\alpha a = (a\alpha e)\alpha a = a\alpha(e\alpha a) = a\alpha a$. Now if $x, y \in R_e$, then $\exists z_1, z_2 \in M$ such that $x = y\delta z_1 = y\beta y\delta z_1 = y\beta x = e\beta x$, and $y = x\rho z_2 = x\beta x\rho z_2 = x\beta y = e\beta y$, which implies that R_e is a right zero (sub-)Γ-semigroup. Similarly we can prove that L_e is a left zero (sub-)Γ-semigroup.

(2) If D_e is non-trivial, then $\exists x \in D_e, x \neq e$. By definition of D_e, there exists $y \in M$ such that $x\mathcal{L}y\mathcal{R}e$. If $y \neq e$, then $|R_e| > 1$, so that by (i), $e\alpha e = e$ and $y\beta y = y \in E \cap D_e$. If $y = e$, then $x\mathcal{L}e$. Since $x \neq e$, by similarly proof of L_e be a left zero (sub-)Γ-semigroup, we have $e = e\alpha e$ and $x = x\beta x$, i.e., $x, e \in E \cap D_e$. Thus in both cases, $|E \cap D_e| > 1$.

(3) We shall indicate the proof in the left cancellative case, the other being verified in a similar way. By (i), L_e is a left zero (sub-)Γ-semigroup. Suppose $x, y \in M \backslash L_e$ with $x\gamma y \in L_e$, then $e = e\alpha e, x\gamma y = x\gamma y\alpha e$, and $e = e\alpha x\gamma y$. Therefore, $e = e\alpha x\gamma y \leq e\alpha e\gamma y = e\gamma y \in M\Gamma y$. On left cancellation, $x\gamma y = x\gamma y\alpha e$ reduces to $y = y\alpha e \in M\Gamma e$. Thus $y\mathcal{L}e$, i.e., $y \in L_e$, which is a contradiction. Hence $x\gamma y \notin L_e$, which shows that $M\backslash L_e$ is a sub-Γ-semigroup. Let $a, c \in L_e, b \in M$ with $a < b < c$, then $a = a\alpha e, e = e\alpha a, c = c\alpha e, e = e\alpha c$. Now, $e = e\alpha a \leq e\alpha b \leq e\alpha c = e$, i.e., $e\alpha b = e$. But $a = a\alpha e$ implies $a\alpha b = (a\alpha e)\alpha b = a\alpha(e\alpha b) = a\alpha e$ and so by left cancellation, $b = e \in L_e$. Hence L_e is a convex sub-Γ-semigroup. Let $x, z \in M\backslash L_e, y \in M$ with $x < y < z$. If $y \in L_e$, then $y = y\alpha e, e = e\alpha y \leq e\alpha z$. By the fact that e is the greatest element we know $e = e\alpha z$. Now $y\alpha z = (y\alpha e)\alpha z = y\alpha(e\alpha z) = y\alpha e$. So by left cancellation, $z = e \in L_e$, which is a contradiction. Thus $y \notin L_e$, i.e., $y \in M\backslash L_e$. This means that $M\backslash L_e$ is a convex sub-Γ-semigroup.

A similar proof yields that $R_e, M\backslash R_e$ are convex sub-Γ-semigroups.

(4) Suppose L_e be an ideal. $\forall x \in L_e, y \in M, x\gamma y \in L_e$. Then $x\gamma y = x\gamma y\alpha e, e = e\alpha x\gamma y$, $x = x\alpha e$, and $e = e\alpha x$. Thus $e = e\alpha x\gamma y = (e\alpha x)\gamma y = e\gamma y$. Therefore $x = x\alpha e = x\alpha(e\gamma y) = (x\alpha e)\gamma y = x\gamma y$. Hence x is a left zero, i.e., every element of L_e is a left zero. Conversely, suppose that every element of L_e is a left zero. Let $x \in L_e$ and $y \in M$. If $y \in L_e$, then $x\gamma y \in L_e, y\gamma x \in L_e$ by (i). So assume $y \notin L_e$. Now $x = x\alpha e$ and $e = e\alpha x$. By hypothesis, $x\gamma y = x$ and $e\gamma y = e$. Thus $x\gamma y = x = x\alpha e \in M\Gamma e$, and $e = e\alpha x = e\alpha x\gamma y \in M\Gamma x\gamma y$. This implies $x\gamma y\mathcal{L}e$, i.e., $x\gamma y \in L_e$. Also $x = x\alpha e$ implies $y\gamma x = y\gamma x\alpha e \in M\Gamma e$ and $e = e\gamma y$ implies $e = (e\alpha x)\gamma y = e\alpha(x\gamma(y\gamma x)) = (e\alpha x)\gamma(y\gamma x) = e\gamma y\gamma x \in M\Gamma y\gamma x$. Therefore $y\gamma x\mathcal{L}e$, i.e., $y\gamma x \in L_e$. Hence L_e is an ideal.

A similar proof yields the result for R_e. □

Theorem 4.4. *Let M be a poe $-$ Γ-semigroup. Then the following are hold:*

1. *If $|D_e| = 1$ or $e\alpha e \neq e, \alpha \in \Gamma$, then $T = \{x : e\alpha x < e\}$ and $T^* = \{x : x\alpha e < e\}$ are convex completely prime ideals, if non-empty.*

2. *If $|D_e| = 1$ or M is commutative, then $L = \{x : e\alpha x\alpha e < e, \alpha \in \Gamma\}$ is a convex completely prime ideal, if non-empty.*

3. *If $|D_e| = 1$, and $e = e\alpha e, \alpha \in \Gamma$, then $L = T = T^*$.*

Proof. (1) First, we show that $\forall x \in T, e \neq e\alpha x\alpha e$, i.e., $e\alpha x\alpha e < e$. Suppose that $e\alpha x\alpha e = e$. If $|D_e| = 1$, then $|R_e| = 1$. By $e = e\alpha x\alpha e$ we know that $e \in e\alpha x\Gamma M$ and $e\alpha x = e\alpha x\alpha e\alpha x$, i.e., $e\alpha x \in e\Gamma M$. Thus $e\mathcal{R}e\alpha x$. But $|R_e| = 1$, then $e = e\alpha x$, which is a contradiction with $x \in T$; if $e\alpha e \neq e$, then $e = e\alpha x\alpha e$ implies $e \leq e\alpha e$, i.e., $e = e\alpha e$, which is also contradiction. So by hypothesis, $\forall x \in T, e \neq e\alpha x\alpha e$, i.e., $e\alpha x\alpha e < e$.

Now let $x \in T$ and $t \in M$. Then $e\beta(t\alpha x) = (e\beta t)\alpha x \leq e\alpha x < e$, and $e\alpha(x\delta t) \leq e$, i.e., $t\alpha x \in T$ and $x\delta t \in T$. Thus T is an ideal; if $x, z \in T, y \in M$ with $x < y < z$, then $e\gamma y \leq e\gamma z < e$, so that $y \in T$. Hence T is convex; if $x, y \notin T$, then $e\alpha x = e = e\alpha y$ and so $e\alpha x\alpha y = (e\alpha x)\alpha y = e\alpha y = e$. Thus $x\alpha y \notin T$. Hence T is completely prime. Therefore, T is convex completely prime ideal.

A similar argument holds for T^*.

(2) By Theorem 4.1(2) we know that $L = T_1$ is a convex ideal. We only need to show L is completely prime.

Let $x, y \notin L$, then $e\alpha x\alpha e = e = e\alpha y\alpha e$ which implies $e\alpha x\mathcal{R}e\mathcal{R}e\alpha y$ and $x\alpha e\mathcal{L}e\mathcal{L}y\alpha e$. If $|D_e| = 1$, then $|L_e| = 1$, and $|R_e| = 1$. Thus $e\alpha x = e\alpha y = e = x\alpha e = y\alpha e$. Hence $e\alpha x\alpha y\alpha e = (e\alpha x)\alpha y\alpha e = e\alpha y\alpha e = e$, i.e., $x\alpha y \notin L$; if M is commutative, then we have $e\alpha e\alpha x = e\alpha e\alpha y = e$ by $e\alpha x\alpha e = e = e\alpha y\alpha e$. Hence $e\alpha x\alpha y\alpha e = e\alpha e\alpha x\alpha y = e\alpha y \geq e\alpha e\alpha y = e$, i.e., $e\alpha x\alpha y\alpha e = e, x\alpha y \notin L$. This proves that L is completely prime.

(3) $\forall x \in T, e\alpha x < e$. If $x \notin T^*$, i.e., $x\alpha e = e$, then $e\alpha x\alpha e = e\alpha e = e$, which implies $e\mathcal{R}e\alpha x$. By $|D_e| = 1$ we know $|R_e| = 1$, then $e = e\alpha x$, which is a contradiction with $x \in T$. Thus $x \in T^*$. This means that $T \subseteq T^*$. Similarly $T^* \subseteq T$. Hence $T = T^*$.

$\forall x \in L, e\alpha x\alpha e < e$. If $x \notin T^*$, i.e., $x\alpha e = e$, then $e\alpha x\alpha e = e\alpha e = e$, which is a contradiction with $x \in L$. Thus $x \in T^*$. This means that $L \subseteq T^*$; $\forall y \in T^*, y\alpha e < e$. If $y \notin L$, then $e\gamma y\gamma e = e$ which implies $e\mathcal{L}y\gamma e$. By $|D_e| = 1$ we know $|L_e| = 1$, then $e = y\gamma e$, which is a contradiction with $y \in T^*$. Thus $y \in L$. This means that $T^* \subseteq L$. Hence $T^* = L$. Therefore $T = T^* = L$. □

Theorem 4.5. *Let M be a poe $-$ Γ-semigroup. Then the following are equivalent:*

1. *$e\Gamma M^1$ is a minimal right ideal;*
2. *$M^1\Gamma e$ is a minimal left ideal;*
3. *$\forall a \in M, e = e\alpha a\alpha e$.*

The proof is easy and is omitted.

Theorem 4.6. *Let M be a poe $-$ Γ-semigroup, and contain no completely prime convex ideals. Then*

1. *If $|D_e| = 1$ and $e = e\alpha e$, then e is a zero.*

2. *If M is commutative and $e = e\alpha e$, then e is a zero.*

3. *If M is commutative and $\exists x \in M$ satisfying $e\alpha e\alpha x = e$, then e is a zero.*

The proof is also easy and hence omitted here.

Theorem 4.7. *Let M be a poe $-$ Γ-semigroup. Then the following hold true:*

1. *$e\Gamma M^1$ is a minimal right ideal and R_e is trivial iff e is a left zero.*

2. *$M^1\Gamma e$ is a minimal left ideal and L_e is trivial iff e is a right zero.*

Proof. (1) Let $e\Gamma M^1$ be a minimal right ideal and R_e is trivial. $\forall a \in M, e\alpha a \in e\Gamma M^1$. Since $e\Gamma M^1$ is a minimal right ideal, then $e\alpha a\Gamma M^1 = e\Gamma M^1$. Thus $e\alpha a\mathcal{R}e$. But R_e is trivial, i.e., $|R_e| = 1$, then $e\alpha a = e$. Hence e is a left zero.

Conversely, if e is a left zero, then $e\Gamma M^1 = \{e\}$, and it is clearly a minimal right ideal. Also, if $a \in R_e$, then $a = e$ or $\exists y \in M$ such that $a = e\gamma y$. Since e is a left zero, then $a = e\gamma y = e$. This shows $|R_e| = 1$, i.e., R_e is trivial.

(2) The case can be verified in a similar way of (i). □

Theorem 4.8. *Let M be a poe $-$ Γ-semigroup. If $e = e\alpha e$, $\alpha \in \Gamma$ and $x < x\rho x$, $\rho \in \Gamma$, for every $x \neq e$, then e is a zero.*

Proof. We first prove that $M^1\Gamma e$ is a minimal left ideal. If A is a left ideal contained in $M^1\Gamma e$ properly. Taking $a \in A \subset M^1\Gamma e$, then $M^1\Gamma a \subset A \subset M^1\Gamma e = \{e\} \cup M\Gamma e$. If $a \neq e$, then $\exists t \in M, \delta \in \Gamma$ such that $a = t\delta e$. Now $a = t\delta e \geq t\delta e\alpha e = a\alpha e \geq a\alpha a$. Since $a \neq e$, then $a < a\alpha a$. Thus $a = a\alpha a$ which is a contradiction. Hence $a = e$, which means that $M^1\Gamma e \subset A \subset M^1\Gamma e$, i.e., $A = M^1\Gamma e$. Therefore $M^1\Gamma e$ is a minimal left ideal. By Theorem 4.3(2), $L_e = \{e\}$. Since otherwise if $|L_e| > 1$, then $|D_e| > 1$. This implies that there exist idempotents other than e in L_e, which contradicts the hypothesis $x < x\rho x$ for every $x \neq e$. So L_e is trivial. Thus by Theorem 4.7(2), e is a right zero. One can similarly prove that $R_e = \{e\}$ and $e\Gamma M^1$ is a minimal right ideal, which implies by Theorem 4.7(1) that e is a left zero. Thus e is a zero. □

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