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TWISTED ENDOMORPHISMS AND STABLE HOLOMORPHIC TRIPLES

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Abstract: Let X be a smooth projective curve, E a vector bundle on X, $L \in \mathrm{Pic}(X)$ and $\sigma > 0$. Here we study the σ-stability of the holomorphic triple $f : E \to E \otimes L$ when f is general in $H^0(X, \mathrm{Hom}(\mathcal{O}_X^{\oplus r}, L^{\oplus r}))$.

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1. Twisted Triples of Vector Bundles

Let X be a smooth and connected projective curve of genus $g \geq 0$. A holomorphic triple on X is a triple $T = (F, E, \phi)$ such that E and F are vector bundles on X and $\phi : E \to F$. A subtriple of T is a triple (E', F', ϕ') on X such that there are inclusions (as sheaves) $i : E' \to E$, $j : F' \to F$ and $\phi \circ i = j \circ \phi'$. If we drop the assumption that i and j are injective, we get the definition of morphism between two triples. Set $\mathrm{rank}(T) := \mathrm{rank}(E) + \mathrm{rank}(F)$. Fix any $\sigma \in \mathbb{R}$. Define the σ-degree of T by the formula $\deg_\sigma(T) := \deg(E) + \deg(F) + \sigma \cdot \mathrm{rank}(E)$ and the σ-slope of T by the formula $\mu_\sigma(T) := \deg_\sigma(T)/\mathrm{rank}(T)$. The σ-slope allows us to define the notions of σ-semistability and σ-stability for triples on X. For the theory of holomorphic triples, see [1], [2] and [3]). Warning: in the terminology of [2] we are using the σ-stability, not the τ-stability. As clear from [2], Proposition 3.18, the cases $\mathrm{rank}(E) = \mathrm{rank}(F)$ and $\mathrm{rank}(E) \neq \mathrm{rank}(F)$ are very different. Here we fix $L \in \mathrm{Pic}(X)$ and take $F := E \otimes L$. Hence $\mathrm{rank}(E) = \mathrm{rank}(F)$. By [2], Proposition 3.17, to have σ-stability for at least

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one $\sigma \in \mathbb{R}$ we need to require $t := \deg(L) \geq 0$. Similarly, to have σ-stability for at least one $\sigma \in \mathbb{R}$ we need to require $t \geq 0$. From now on, we always assume $t \geq 0$. We will write (E, f, L) for any triple $f : E \to E \otimes L$. If $d := \deg(E)$, $r := \mathrm{rank}(E)$ and $t := \deg(L)$, then:

$$\mu(E, f, L) = \mu(E) + t/2 = d/r + t/2\,, \tag{1}$$

$$\mu_\sigma(E, f, L)) = \mu(E) + t/2 + \sigma/2 = d/r + t/2 + \sigma/2\,. \tag{2}$$

First, we will consider one example: $E \cong \mathcal{O}_X^{\otimes r}$ (see Theorem 1 and Proposition 1).

Theorem 1. *Fix any integer $r \geq 2$ and fix any $L \in \mathrm{Pic}^t(X)$ such that $h^0(X, L) \geq 2$. Take a general $f :\in H^0(X, \mathrm{Hom}(\mathcal{O}_X^{\oplus r}, L^{\oplus r})) \cong H^0(X, L)^{r^2}$. Then the twisted triple $(\mathcal{O}_X^{\oplus r}, f, L)$ is σ-stable for all $\sigma > t$.*

Then we will adapt the proof of Proposition 1 to prove the following result.

Theorem 2. *Assume $g > 0$ and fix any rank $r \geq 2$ stable vector bundle E on X and any $L \in \mathrm{Pic}^t(X)$ such that $h^0(X, L) \geq 2$. Take a general f : $H^0(X, \mathrm{Hom}(\mathcal{O}_X^{\oplus r}, L^{\oplus r})) \cong H^0(X, L)^{r^2}$. Then the twisted triple (E, f, L) is σ-stable for all $\sigma > t$.*

Notice that by [2], Proposition 3.17, the assumption $\sigma > t$ in the statements of Theorem 1 and Theorem 2 is sharp.

Proposition 1. *Fix an integer $r \geq 2$ and take any $L \in \mathrm{Pic}^t(X)$ such that $h^0(X, L) \neq 0$. Fix any injective $f : \mathcal{O}_X^{\oplus r} \to L^{\oplus r}$ (as a map of sheaves). Then the twisted triple $(\mathcal{O}_X^{\oplus r}, f, L)$ is σ-semistable for all $\sigma \geq t$. Furthermore, if $(\mathcal{O}_X^{\oplus r}, f, L)$ is not σ-stable for some $\sigma > t$, then any destabilizing proper subtriple is of the form $(\mathcal{O}_X^{\oplus s}, L^{\oplus s}, *)$ for some integer s with $1 \leq s < r$.*

Proof. Fix σ and assume the existence of an integer s such that $1 \leq s < r$, a rank s subsheaf F of $\mathcal{O}_X^{\oplus r}$, a subsheaf G of $L^{\otimes r}$ such that $f(F) \subseteq G$ and $\mu_\sigma(G, F, f|F) > \mu_\sigma(\mathcal{O}_X^{\oplus r}, f, L) = t/2 + \sigma/2$. Since $L^{\oplus r}$ and $\mathcal{O}_X^{\oplus r}$ are semistable, we have $\mu(G) \leq t$ and $\mu(F) \leq 0$. Hence the result is obvious if $\mathrm{rank}(G) = s$. Since f is injective, we have $\mathrm{rank}(f) \geq s$. Assume $y := \mathrm{rank}(f) > s$. Hence $\mu_\sigma(G, F, f|F) \leq yt/(s+y) + s\sigma/(s+y)$. Consider the function $\psi : [s, +\infty) \to \mathbb{R}$ defined by the formula $\psi(x) = xt/(s+x) + s\sigma/(s+x)$. Hence $\psi'(x) = s(t - \alpha)/(s+x)^2$. Thus $\psi' \equiv 0$ if $\sigma = t$ and $\psi'(x) < 0$ for all $x \in (s, +\infty)$ if σ. Hence $\psi(y) \leq \psi'(s)$ and we have strict inequality if $y > s$ and $\sigma > t$. $\square$

Lemma 1. *Assyme $h^0(X, L) \neq 0$. Then a general $f : E \to E \otimes L$ is injective.*

Proof. Fix $s \in H^0(X,L)\backslash\{0\}$ and let $D \subset X$ be its zero-locus. Fix $Q \in X\backslash D_{red}$. Let $h : E \to E \otimes L$ be the map induced twisting the map $s : \mathcal{O}_X \to L$ with E. Notice that h has rank rank(E) at Q. By semicontinuity the same is true for a a general $f : E \to E \otimes L$. □

Proof of Theorem 1. By Lemma 1 the map f is injective. Since $h^0(X,L) \geq 2$, it is easy to show that none of the exceptional cases listed in the statement of Proposition 1 may occur. □

Proof of Theorem 2. With the set-up of the proof of Proposition 1 we have $\mu(F) < \mu(E)$ and $\mu(G) \leq \mu(E \otimes L)$ with equality if and only if $G = E \otimes L$. Hence the proof of Proposition 1 works. □

Remark 1. Fix an integer $r \geq 2$ and take any $L \in \mathrm{Pic}(X)$ such that $h^0(X,L) \neq 0$. There are many non-injctive $f : \mathcal{O}_X^{\oplus r} \to L^{\oplus r}$ such that for all $\sigma \in \mathbb{R}$ the twisted triple $(\mathcal{O}_X^{\oplus r}, f, L)$ is not σ-semistable.

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