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GLOBAL EXISTENCE FOR A FREE BOUNDARY PROBLEM MODELLING THE GROWTH OF NECROTIC TUMORS IN THE PRESENCE OF INHIBITORS

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Abstract: In this paper we study a free boundary problem modelling the growth of necrotic tumors in the presence of inhibitors. By transforming this free boundary problem into an initial-boundary value problem in a fixed domain of a coupled system of two parabolic equations and one integro-differential equation, in which all equations involve discontinuous terms, and using the approximation method combined with the Schauder Fixed Point Theorem and the L^p-theory for parabolic equations, we prove that this problem has a global solution in any given time interval $[0, T]$.

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1. Introduction

Since the pioneering work of Greenspan in 1970's [7, 8], an increasing number

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of free boundary problems of partial differential equations have been proposed to model the growth of solid tumors [1-3, 9-14]. Numerical simulations and asymptotic analysis of these tumor-growth free-boundary problems have shown satisfactory coincidence with experimental observations. Rigorous mathematical analysis of these free boundary problems has drawn great interest, and many interesting results have been obtained.

In this paper we study a free boundary problem modelling the growth of necrotic tumors in the presence of inhibitors. The model develops a similar but simpler model of Byrne and Chaplain proposed in [1]. It comprises two reaction-diffusion equations describing the evolution of the nutrient and the inhibitor, whose concentrations are respectively denoted by $\sigma = \sigma(r,t)$ and $\beta = \beta(r,t)$, where r is radial space variable and t is time variable, and an integro-differential equation governing the evolution of the tumor radius $R(t)$. Explicit mathematical formulation of this model reads as follows:

$$\sigma_t = D_1\Delta_r\sigma - f(\sigma,\beta)H(\sigma-\sigma_D), \quad 0 < r < R(t),\, t > 0, \tag{1.1}$$

$$\beta_t = D_2\Delta_r\beta - g(\sigma,\beta)H(\sigma-\sigma_D), \quad 0 < r < R(t),\, t > 0, \tag{1.2}$$

$$\frac{\partial\sigma}{\partial r}(0,t) = 0, \quad \sigma(R(t),t) = \bar{\sigma}, \quad t > 0, \tag{1.3}$$

$$\frac{\partial\beta}{\partial r}(0,t) = 0, \quad \beta(R(t),t) = \bar{\beta}, \quad t > 0, \tag{1.4}$$

$$\frac{dR(t)}{dt} = \frac{\mu}{R^2(t)}\Big\{\int_{\sigma>\sigma_D} S(\sigma,\beta)r^2dr - \nu\int_{\sigma\le\sigma_D} r^2dr\Big\}, \quad t > 0, \tag{1.5}$$

$$\sigma(r,0) = \sigma_0, \quad \beta(r,0) = \beta_0, \quad 0 \le r \le R_0, \tag{1.6}$$

$$R(0) = R_0. \tag{1.7}$$

Here $H(\cdot)$ is the Heaviside function (i.e., $H(x) = 1$ if $x > 0$, and $H(x) = 0$ if $x \le 0$), D_1, D_2 are diffusion coefficients, σ_D is a positive constant representing the threshold value of necrosis (i.e., in the region where $\sigma > \sigma_D$ all tumor cells are alive while in the region where $\sigma \le \sigma_D$ all tumor cells are dead due to starvation), $\bar{\sigma}$, $\bar{\beta}$ are positive constants respectively representing the constant supplies of the nutrient and the inhibitor that the tumor receives from its surface, μ is the mitosis rate coefficient of live tumor cells, ν is the dissolution rate of dead cells, $f(\sigma,\beta)$, $g(\sigma,\beta)$ are respectively the consumption rates of the nutrient and the inhibitor when their concentrations are respectively at the levels σ and β, $S(\sigma,\beta)$ is the proliferation rate of tumor cells when the nutrient and the inhibitor concentrations are respectively at the levels σ and β, $\sigma_0(r)$, $\beta_0(r)$, R_0 are respectively the initial data of σ, β and R, and Δ_r represents the radial

Laplacian, i.e.

$$\Delta_r = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}).$$

The above model develops the tumor growth model of Byrne and Chaplain proposed in [2] in the following aspects: In [2] f, g are explicitly linear functions, S has the form

$$S(\sigma,\beta) = (\sigma - \tilde{\sigma})\Big(1 - \frac{\tilde{\beta}}{\beta}\Big), \tag{1.8}$$

where $\tilde{\sigma}$, $\tilde{\beta}$ are positive constants, and the terms σ_t, β_t in (1.1), (1.2) are omitted. In this paper we consider f, g and S as general nonlinear functions satisfying some very weak biologically reasonable conditions (see the assumptions (A1), (A2) in Section 2), and we do not omit the terms σ_t, β_t in (1.1) and (1.2). The reason of replacing linear functions f, g with general nonlinear functions f, g is as follows: First, linear functions are not the unique choice of consumption rate functions; using linear functions in the model is mainly for mathematical simplicity. Actually, many other authors used nonlinear consumption rate functions in their tumor growth models (cf. [13, 14] for instance). Second and the most importantly, if f, g are linear functions and $f(\sigma,\beta) \neq f(\sigma)$, $g(\sigma,\beta) \neq g(\beta)$, then the solutions σ, β of the equations (1.1) and (1.2) can take negative values, as one can easily verify by a direct computation, which contradicts the fact that σ, β denote concentrations. The reason that we do not use the expression (1.8) as a proliferation rate function but instead merely assume that S is a general nonlinear function satisfying some general conditions is that (1.8) is not a biologically and medically reasonable proliferation rate function in general situations, as was pointed in [4], and, besides, there are also many other different choices of the expression of S (cf. [13, 14]). Finally, equations (1.1), (1.2) involving the terms σ_t, β_t are clearly more realistic than similar equations without these terms.

In [2], Byrne and Chaplain also made numerical simulations of their model in various different special situations. Their numerical results showed very good coincidence with experimental observations. We are interested in rigorous mathematical analysis of the above free boundary problem. Some interesting results in this line have been explored by Cui and Friedman [5] and Cui [6]. In [5], Cui and Friedman considered the above problem in the special linear f, S and inhibitor-free case. Under some very rigid conditions, they established existence of a global solution and proved that the global solution converges to a unique stationary solution as $t \to \infty$. In [6], also in the special linear f, S and inhibitor-free case, Cui proved global existence for the above problem under some general conditions on the initial function which are similar to the conditions (A3) given in the succeeding section. In this paper we extend the result of Cui [6] to the general f, g, S and inhibitor-present case, and prove that the above problem has a global solution under the general conditions (A1)–(A3) given in the succeeding section.

2. The Main Result

For given $T > 0$, we introduce the notation:

$$Q_T = \{(x,t) \in R^3 \times R : |x| < R(t),\ 0 \le t \le T\},$$

$$W_p^{2,1}(Q_T) = \{u, v \in L^p(Q_T) : \nabla u, \nabla v, \nabla^2 u, \nabla^2 v, u_t, v_t \in L^p(Q_T)\}.$$

We make the following assumptions:

(A1) $f(\sigma,\beta) = \sigma f_1(\sigma,\beta)$, $g(\sigma,\beta) = \beta g_1(\sigma,\beta)$, where f_1, g_1 are defined and Lipschitz continuous on $[0,\infty)\times[0,\infty)$, and $f_1(\sigma,\beta) \ge 0$, $g_1(\sigma,\beta) \ge 0$ for $\sigma \ge 0$, $\beta \ge 0$;

(A2) S is defined and Lipschitz continuous on $[0,\infty)\times[0,\infty)$;

(A3) $\sigma_0,\ \beta_0 \in W^{2,\infty}(0,R_0)$, $0 \le \sigma_0(r) \le \bar{\sigma}$, $0 \le \beta_0(r) \le \bar{\beta}$, $\sigma_0'(0) = 0$, $\beta_0'(0) = 0$, $\sigma_0(R_0) = \bar{\sigma}$ and $\beta_0(R_0) = \bar{\beta}$. Here $W^{2,\infty}(0,R_0) = \{\phi \in L^\infty(0,R_0) : \phi', \phi'' \in L^\infty(0,R_0)\}$.

These assumptions are conformable to biological and medical principles. Indeed, biologically and medically meaningful consumption rate functions f, g and proliferation rate function S satisfy the following stronger conditions: For any $\sigma \ge 0$ and $\beta \ge 0$,

$$\frac{\partial f(\sigma,\beta)}{\partial \sigma} \ge 0, \quad \frac{\partial f(\sigma,\beta)}{\partial \beta} \ge 0, \quad f(0,\beta) = 0, \tag{2.1}$$

$$\frac{\partial g(\sigma,\beta)}{\partial \sigma} \ge 0, \quad \frac{\partial g(\sigma,\beta)}{\partial \beta} \ge 0, \quad g(\sigma,0) = 0, \tag{2.2}$$

$$\frac{\partial S(\sigma,\beta)}{\partial \sigma} \ge 0, \quad \frac{\partial S(\sigma,\beta)}{\partial \beta} \le 0, \quad S(0,0) < 0. \tag{2.3}$$

The first inequality in (2.1) reflects the fact that increasing nutrient concentration will enlarge the amount of nutrient consumed by tumor cells per unit time interval, resulting in acceleration of tumor cell mitosis. The second inequality in (2.1) reflects the fact that increasing inhibitor concentration will accelerate the consumption of nutrient, for more amount of inhibitor will react with more amount of nutrient. The last relation in (2.1) reflects the fact that the nutrient consumption rate is zero at points where no nutrient exists, no matter there exists or does not exist inhibitor. Relations in (2.2) can be explained similarly.

The first inequality in (2.3) means that increasing nutrient concentration will increase the proliferation rate of tumor cells, and the second inequality in (2.3) means that increasing inhibitor concentration will decrease the proliferation rate of tumor cells. The relation $S(0,0) < 0$ reflects the fact that if there is no nutrient then tumor cells will undergo negative proliferation, even if there is no inhibitor. It is clear that (2.1)–(2.3) imply the conditions (A1) and (A2). However, in this paper we do not need these stronger conditions; the weaker assumptions (A1), (A2) are sufficient to ensure global existence of a solution.

The main result of this paper is as follows.

Theorem 2.1. *Under the conditions (A1), (A2) and (A3), for any* $T > 0$ *the problem (1.1)-(1.7) has a solution* $(\sigma(r,t), \beta(r,t), R(t))$ *for all* $0 \leq t \leq T$*, satisfying:* $\sigma(|x|,t), \beta(|x|,t) \in W_p^{2,1}(Q_T)$ $(1 < p < \infty)$*, and*

$$0 \leq \sigma(r,t) \leq \bar{\sigma}, \quad 0 \leq \beta(r,t) \leq \bar{\beta} \quad (0 \leq t \leq T).$$

It is well-know that the usual method of establishing existence of solutions for evolutionary equations is using the Banach Fixed Point Theorem to get a local solution, and then extending the local solution to a global solution. Since the equations (1.1) and (1.2) involve discontinuous terms, this method fails to the problem (1.1)–(1.7) because we are unable to get a contraction mapping. One might expect to use the approximation method combined with the usual Banach Fixed Point Method to get a local solution and then extend the solution into a global solution. This idea also does not apply to the problem (1.1)–(1.7), because we are unable to ensure that the solutions of the approximation problems have a common existence (time) interval, so that we are unable to get a convergent sequence of the approximation solutions. This is what the difficulty lies at in establishing the above result. In the following sections we shall use the idea of Cui [6] to prove the above theorem, namely, we shall use the approximation method combined with the Schauder Fixed Point Theorem and L^p-estimates for parabolic equations to prove existence of a solution in any given time interval $[0,T]$.

3. A Preliminary Lemma

In order to solve the free boundary problem (1.1)–(1.7), we first transform it into an initial-boundary value problem in the fixed domain $\{(z,t),\ 0 \leq z \leq 1,\ t \geq 0\}$. For this purpose, we introduce a transformation of variables $(r,t,\sigma,\beta,R) \rightarrow$

(z, t, u, v, R) as follows:

$$z = \frac{r}{R(t)}, \quad t = t, \quad u(z,t) = \sigma(zR(t),t), \quad v(z,t) = \beta(zR(t),t), \\ R(t) = R(t), \quad u_0(z) = \sigma_0(zR_0), \quad v_0(z) = \sigma_0(zR_0). \tag{3.1}$$

Then the problem (1.1)–(1.7) is transformed into the following problem (3.2)–(3.8):

$$u_t = \frac{D_1}{R^2(t)}\Delta_z u + \frac{\dot{R}(t)}{R(t)} z \cdot u_z - f(u,v)H(u-\sigma_D), \quad 0 < z < 1,\ t > 0, \tag{3.2}$$

$$v_t = \frac{D_2}{R^2(t)}\Delta_z v + \frac{\dot{R}(t)}{R(t)} z \cdot v_z - g(u,v)H(u-\sigma_D), \quad 0 < z < 1,\ t > 0, \tag{3.3}$$

$$u_z(0,t) = 0, \quad u(1,t) = \bar{\sigma}, \quad t \geq 0, \tag{3.4}$$

$$v_z(0,t) = 0, \quad v(1,t) = \bar{\beta}, \quad t \geq 0, \tag{3.5}$$

$$\dot{R}(t) = \mu R(t)\Big\{ \int_0^1 S(u(z,t),v(z,t))H(u(z,t)-\sigma_D)z^2 dz \\ -\nu \int_0^1 [1 - H(u(z,t)-\sigma_D)]z^2 dz\Big\}, \quad t \geq 0, \tag{3.6}$$

$$u(z,0) = u_0(z), \quad v(z,0) = v_0(z), \quad 0 \leq r \leq R_0, \tag{3.7}$$

$$R(0) = R_0. \tag{3.8}$$

Let $X_T = C([0,1]\times[0,T])\times C([0,1]\times[0,T])$. We denote by G the functional on $X_T \times [0,T]$ defined by

$$G(u,v,t) = \mu\Big\{ \int_0^1 S(u(z,t),v(z,t))H(u(z,t)-\sigma_D)z^2 dz \\ -\nu \int_0^1 [1 - H(u(z,t)-\sigma_D)]z^2 dz\Big\}, \quad t \geq 0. \tag{3.9}$$

Using this notation, the equation (3.6) can be simply written as follows:

$$\dot{R}(t) = R(t)G(u,v,t), \quad t \geq 0. \tag{3.10}$$

It is clear that $R(t)$ can be expressed by

$$R(t) = R_0 \exp\{\int_0^t G(u,v,\tau)d\tau\}, \quad t \geq 0. \tag{3.11}$$

Hence, the problem (3.2)–(3.8) is reduced to the problem (3.12)–(3.16) below:

$$u_t = D_1 R_0^{-2} \exp\Big\{ -2\int_0^t G(u,v,\tau)d\tau\Big\}\Delta_z u + G(u,v,t)(z\cdot u_z) \\ -f(u,v)H(u-\sigma_D), \quad 0 < z < 1,\ t > 0, \tag{3.12}$$

$$v_t = D_2 R_0^{-2} \exp\Big\{-2\int_0^t G(u,v,\tau)d\tau\Big\}\Delta_z v + G(u,v,t)(z\cdot v_z)$$
$$-g(u,v)H(u-\sigma_D), \quad 0<z<1,\ t>0, \tag{3.13}$$
$$u_z(0,t)=0, \quad u(1,t)=\bar{\sigma}, \quad t\ge 0, \tag{3.14}$$
$$v_z(0,t)=0, \quad v(1,t)=\bar{\beta}, \quad t\ge 0, \tag{3.15}$$
$$u(z,0)=u_0(z), \quad v(z,0)=v_0(z), \quad 0\le z\le 1. \tag{3.16}$$

We summarize the above result in the following lemma.

Lemma 3.1. *Under the variable transformation (3.1), the free boundary problem (1.1)-(1.7) is equivalent to the initial-boundary value problem (3.12)-(3.16).*

4. Proof of Theorem 2.1

Clearly, the functional $G(u,v,t)$ is discontinuous in u and v. It follows that the coefficients of (3.12), (3.13) are discontinuous nonlinear functionals. Hence, we are unable to solve this problem by directly using the Banach Fixed Point Theorem. To overcome this difficulty, we consider the following approximation problem:

$$u_t = D_1 R_0^{-2} \exp\{-2\int_0^t G_\varepsilon(u,v,\tau)d\tau\}\Delta_z u + G_\varepsilon(u,v,t)(z\cdot u_z)$$
$$-f(u,v)H_\varepsilon(u-\sigma_D), \quad 0<z<1,\ t>0, \tag{4.1}$$
$$v_t = D_2 R_0^{-2} \exp\{-2\int_0^t G_\varepsilon(u,v,\tau)d\tau\}\Delta_z v + G_\varepsilon(u,v,t)(z\cdot v_z)$$
$$-g(u,v)H_\varepsilon(u-\sigma_D), \quad 0<z<1,\ t>0, \tag{4.2}$$
$$u_z(0,t)=0, \quad u(1,t)=\bar{\sigma}, \quad t\ge 0, \tag{4.3}$$
$$v_z(0,t)=0, \quad v(1,t)=\bar{\beta}, \quad t\ge 0, \tag{4.4}$$
$$u(z,0)=u_0(z), \quad v(z,0)=v_0(z), \quad 0\le z\le 1, \tag{4.5}$$

where ε is an arbitrary positive number,

$$G_\varepsilon(u,v,t) = \mu\Big\{\int_0^1 S(u(z,t),v(z,t))H_\varepsilon(u(z,t)-\sigma_D)z^2dz$$
$$-\nu\int_0^1 [1-H_\varepsilon(u(z,t)-\sigma_D)]z^2dz\Big\}, \quad t\ge 0, \tag{4.6}$$

and

$$H_\varepsilon(x) = \begin{cases} 1, & \text{for } x \geq \varepsilon, \\ \frac{x}{\varepsilon}, & \text{for } 0 < x < \varepsilon, \\ 0, & \text{for } x \leq 0. \end{cases} \tag{4.7}$$

It is clear that H_ε is a Lipschitz continuous function.

We introduce the following new notation:

$$\overline{B_1(0)} = \{y \in R^3,\ |y| \leq 1\}, \quad \tilde{Q}_T = \overline{B_1(0)} \times [0,T],$$

$$W_p^{2,1}(\tilde{Q}_T) = \{u, v \in L^p(\tilde{Q}_T):\ \nabla u, \nabla v, \nabla^2 u, \nabla^2 v, u_t, v_t \in L^p(\tilde{Q}_T)\}.$$

Recalling $u_0(z) = \sigma_0(zR_0)$, $v_0(z) = \beta_0(zR_0)$, by condition (A3) we have

$$\begin{gathered} u_0,\ v_0 \in W^{2,\infty}(0,1),\ 0 \leq u_0(r) \leq \bar{\sigma},\ 0 \leq v_0(r) \leq \bar{\beta}, \\ u_0'(0) = 0,\ v_0'(0) = 0,\ u_0(R_0) = \bar{\sigma},\ v_0(R_0) = \bar{\beta}. \end{gathered} \tag{4.8}$$

Lemma 4.1. *Under the condition (4.8), for any $T > 0$ the problem (4.1)–(4.5) has a unique solution $(u,v) = (u_\varepsilon, v_\varepsilon)$ on $[0,1] \times [0,T]$, satisfying the following properties:*

(i) $u_\varepsilon(|y|,t),\ v_\varepsilon(|y|,t) \in W_p^{2,1}(\tilde{Q}_T), \forall p \in (1,\infty)$, and for any $\varepsilon > 0$ there holds

$$||u_\varepsilon(|y|,t)||_{W_p^{2,1}(\tilde{Q}_T)} \leq C(T,p,||u_0||_{W^{2,\infty}(0,1)},\bar{\sigma}), \tag{4.9}$$

$$||v_\varepsilon(|y|,t)||_{W_p^{2,1}(\tilde{Q}_T)} \leq C(T,p,||v_0||_{W^{2,\infty}(0,1)},\bar{\beta}), \tag{4.10}$$

where $C(T,p,||u_0||_{W^{2,\infty}(0,1)},\bar{\sigma})$, $C(T,p,||v_0||_{W^{2,\infty}(0,1)},\bar{\beta})$ are constants independent of ε.

(ii) For any $\varepsilon > 0$ there holds:

$$0 \leq u_\varepsilon(z,t) \leq \bar{\sigma}, \quad 0 \leq v_\varepsilon(z,t) \leq \bar{\beta}, \quad 0 \leq z \leq 1,\ 0 \leq t \leq T.$$

Proof. We endow X_T with the following norm:

$$||(u,v)|| = \max_{[0,1]\times[0,T]} |u| + \max_{[0,1]\times[0,T]} |v|.$$

Then X_T is clearly a Banach space. We define a mapping $F: X_T \to X_T$ as follows: For any $(u,v) \in X_T$, define $F(u,v) = (\tilde{u}, \tilde{v})$, where $\tilde{u}(z,t) = \tilde{\tilde{u}}(zy^o,t)$, $\tilde{v} = \tilde{\tilde{v}}(zy^o,t)$, $y^o \in R^3$, $|y^o| = 1$, and $(\tilde{\tilde{u}}, \tilde{\tilde{v}})$ are the solution of the problem (4.11)–(4.15) below:

$$\begin{aligned} \tilde{\tilde{u}}_t = {} & D_1 R_0^{-2} \exp\{-2\int_0^t G_\varepsilon(u,v,\tau)d\tau\}\Delta\tilde{\tilde{u}} + G_\varepsilon(u,v,t)(y \cdot \nabla\tilde{\tilde{u}}) \\ & -\tilde{\tilde{u}} f_1(u,v) H_\varepsilon(u - \sigma_D) \quad \text{on } \tilde{Q}_T, \end{aligned} \tag{4.11}$$

$$\tilde{\tilde{v}}_t = D_2 R_0^{-2} \exp\{-2\int_0^t G_\varepsilon(u,v,\tau)d\tau\}\Delta\tilde{\tilde{v}} + G_\varepsilon(u,v,t)(y \cdot \nabla\tilde{\tilde{v}})$$

$$-\tilde{\tilde{v}}g_1(u,v)H_\varepsilon(u-\sigma_D) \quad \text{on } \tilde{Q}_T, \tag{4.12}$$

$$\tilde{\tilde{u}}(y,t)=\bar{\sigma}, \quad |y|=1,\ t\geq 0, \tag{4.13}$$

$$\tilde{\tilde{v}}(y,t)=\bar{\beta}, \quad |y|=1,\ t\geq 0, \tag{4.14}$$

$$\tilde{\tilde{u}}(y,0)=u_0(|y|), \quad \tilde{\tilde{v}}(y,0)=v_0(|y|), \quad |y|\leq 1. \tag{4.15}$$

In the following section we prove that F is well-defined.

Clearly (4.11)–(4.15) can be decoupled into two independent problems as follows:

$$(I)\ \begin{cases} \tilde{\tilde{u}}_t = D_1R_0^{-2}\exp\{-2\int_0^t G_\varepsilon(u,v,\tau)d\tau\}\Delta\tilde{\tilde{u}} + G_\varepsilon(u,v,t)(y\cdot\nabla\tilde{\tilde{u}}) \\ \qquad -\tilde{\tilde{u}}f_1(u,v)H_\varepsilon(u-\sigma_D) \quad \text{on } \tilde{Q}_T, \\ \tilde{\tilde{u}}(y,t)=\bar{\sigma}, \quad |y|=1,\ t\geq 0, \\ \tilde{\tilde{u}}(y,0)=u_0(|y|), \quad |y|\leq 1, \end{cases}$$

$$(II)\ \begin{cases} \tilde{\tilde{v}}_t = D_2R_0^{-2}\exp\{-2\int_0^t G_\varepsilon(u,v,\tau)d\tau\}\Delta\tilde{\tilde{v}} + G_\varepsilon(u,v,t)(y\cdot\nabla\tilde{\tilde{v}}) \\ \qquad -\tilde{\tilde{v}}g_1(u,v)H_\varepsilon(u-\sigma_D) \quad \text{on } \tilde{Q}_T, \\ \tilde{\tilde{v}}(y,t)=\bar{\beta}, \quad |y|=1,\ t\geq 0, \\ \tilde{\tilde{v}}(y,0)=v_0(|y|), \quad |y|\leq 1. \end{cases}$$

Since all coefficients in the above equations are bounded continuous functions and

$$R_0^{-2}\exp\Big\{-2\int_0^t G_\varepsilon(u,v,\tau)d\tau\Big\} \geq \lambda_\varepsilon(T,||(u,v)||)>0, \quad 0\leq t\leq T,$$

using the L^p theory for linear parabolic equations [15] we infer that the problems (I) and (II) have unique solutions $\tilde{\tilde{u}}(y,t)$, $\tilde{\tilde{v}}(y,t)$, respectively. Since the initial and boundary values are spherically symmetric in the space variable y, it is easy to see that for any orthogonal transformation $O: R^3\to R^3$, $\tilde{\tilde{u}}_1(y,t)=\tilde{\tilde{u}}(Oy,t)$ and $\tilde{\tilde{v}}_1(y,t)=\tilde{\tilde{v}}(Oy,t)$ are also solutions of these problems. This implies, by uniqueness of solutions, that there exist functions $\tilde{u}$, $\tilde{v}\in C([0,1]\times[0,T])$ such that

$$\tilde{\tilde{u}}(y,t)=\tilde{u}(|y|,t), \quad \tilde{\tilde{v}}(y,t)=\tilde{v}(|y|,t), \quad |y|\leq 1,\ 0\leq t\leq T.$$

By L^p-estimates we have, for any $p\in(1,\infty)$,

$$||\tilde{\tilde{u}}(y,t)||_{W_p^{2,1}(\tilde{Q}_T)} \leq C(T,p,||(u,v)||)(||u_0(|y|)||_{W^{2,\infty}(B_1(0))}+\bar{\sigma} \\ + ||\tilde{\tilde{u}}(|y|,t)||_{L^\infty(\tilde{Q}_T)}),$$

$$||\tilde{\tilde{v}}(y,t)||_{W_p^{2,1}(\tilde{Q}_T)} \leq C(T,p,||(u,v)||)(||v_0(|y|)||_{W^{2,\infty}(B_1(0))}+\bar{\beta} \\ + ||\tilde{\tilde{v}}(|y|,t)||_{L^\infty(\tilde{Q}_T)}).$$

By the maximum principle it is clear that

$$0\leq\tilde{\tilde{u}}(y,t)\leq\bar{\sigma}, \quad 0\leq\tilde{\tilde{v}}(y,t)\leq\bar{\beta}, \quad |y|\leq 1,\ 0\leq t\leq T.$$

Hence

$$||\tilde{u}(|y|,t)||_{W_p^{2,1}(\tilde{Q}_T)} \le C(T,p,||(u,v)||)(||u_0||_{W^{2,\infty}(0,1)} + \bar{\sigma}), \qquad (4.16)$$

$$||\tilde{v}(|y|,t)||_{W_p^{2,1}(\tilde{Q}_T)} \le C(T,p,||(u,v)||)(||v_0||_{W^{2,\infty}(0,1)} + \bar{\beta}), \qquad (4.17)$$

and

$$0 \le \tilde{u}(|y|,t) \le \bar{\sigma}, \quad 0 \le \tilde{v}(|y|,t) \le \bar{\beta}, \quad |y| \le 1,\ 0 \le t \le T.$$

We take particularly $p > \frac{5}{2}$. Then by the embedding $W_p^{2,1}(\tilde{Q}_T) \hookrightarrow C(\tilde{Q}_T)$, we infer that $(\tilde{u},\tilde{v}) \in X_T$. In this way we get functions $\tilde{u} = \tilde{u}(z,t)$, $\tilde{v} = \tilde{v}(z,t)$ defined for $(z,t) \in [0,1] \times [0,T]$, belonging to X_T. Hence the mapping F is well-defined.

Let $E = \{(u,v) : u,v \in C([0,1]\times[0,T]),\ 0 \le u \ \le \bar{\sigma},\ 0 \le v \le \bar{\beta}\}$. E is clearly a bounded closed convex subset of X_T. We first prove that $F(E)$ is precompact in X_T. Indeed, since for any $(u,v) \in E$

$$||(u,v)|| \le \bar{\sigma} + \bar{\beta},$$

by (4.16) and (4.17) we infer that for any $1 < p < \infty$ there hold

$$||\tilde{u}||_{W_p^{2,1}(\tilde{Q}_T)} \le C(T,p), \quad ||\tilde{v}||_{W_p^{2,1}(\tilde{Q}_T)} \le C(T,p). \qquad (4.18)$$

Taking $p > \frac{5}{2}$ and using the compact embedding $W_p^{2,1}(\tilde{Q}_T) \hookrightarrow\hookrightarrow C(\tilde{Q}_T)$ $(p > \frac{5}{2})$, we conclude that $F(E)$ is precompact in X_T.

Next we prove that F is continuous. Let $u_1, u_2, v_1, v_2 \in X_T$ and denote $(\tilde{u}_1,\tilde{v}_1) = F(u_1,v_1)$, $(\tilde{u}_2,\tilde{v}_2) = F(u_2,v_2)$, $\tilde{\tilde{u}}^*(y,t) = \tilde{u}_1(|y|,t) - \tilde{u}_2(|y|,t)$, $\tilde{\tilde{v}}^*(y,t) = \tilde{v}_1(|y|,t) - \tilde{v}_2(|y|,t)$. It is obvious that $\tilde{\tilde{u}}^*$, $\tilde{\tilde{v}}^*$ are respectively solutions of the problems (I') and (II') below:

$$(I') \begin{cases} \tilde{\tilde{u}}^*_t = D_1 R_0^{-2} \exp\{-2\int_0^t G_\varepsilon(u_1,v_1,\tau)d\tau\}\Delta\tilde{\tilde{u}}^* + G_\varepsilon(u_1,v_1,t)(y\cdot\nabla\tilde{\tilde{u}}^*) \\ \qquad -\tilde{\tilde{u}}^* f_1(u_1,v_1)H_\varepsilon(u_1-\sigma_D) + h_1(y,t), \quad |y|<1,\ 0<t<T, \\ \tilde{\tilde{u}}^*(y,t) = 0, \quad |y|=1,\ 0\le t\le T, \\ \tilde{\tilde{u}}^*(y,0) = 0, \quad |y|\le 1, \end{cases}$$

$$(II)' \begin{cases} \tilde{\tilde{v}}^*_t = D_2 R_0^{-2} \exp\{-2\int_0^t G_\varepsilon(u_1,v_1,\tau)d\tau\}\Delta\tilde{\tilde{v}}^* + G_\varepsilon(u_1,v_1,t)(y\cdot\nabla\tilde{\tilde{v}}^*) \\ \qquad -\tilde{\tilde{v}}^* g_1(u_1,v_1)H_\varepsilon(u_1-\sigma_D) + h_2(y,t), \quad |y|<1,\ 0<t<T, \\ \tilde{\tilde{v}}^*(y,t) = 0, \quad |y|=1,\ 0\le t\le T, \\ \tilde{\tilde{v}}^*(y,0) = 0, \quad |y|\le 1, \end{cases}$$

where

$$\begin{aligned} h_1(y,t) = D_1 R_0^{-2}[\exp\{-2\int_0^t G_\varepsilon(u_1,v_1,\tau)d\tau\} \\ - \exp\{-2\int_0^t G_\varepsilon(u_2,v_2,\tau)d\tau\}]\tilde{\tilde{u}}_2 + [G_\varepsilon(u_1,v_1,t) \\ - G_\varepsilon(u_2,v_2,t)](y\cdot\nabla\tilde{\tilde{u}}_2) - \tilde{\tilde{u}}_2[f_1(u_1,v_1)H_\varepsilon(u_1-\sigma_D) - f_1(u_2,v_2)H_\varepsilon(u_2-\sigma_D)], \end{aligned}$$

$$h_2(y,t) = D_2 R_0^{-2}[\exp\{-2\int_0^t G_\varepsilon(u_1,v_1,\tau)d\tau\} - \exp\{-2\int_0^t G_\varepsilon(u_2,v_2,\tau)d\tau\}]\Delta\tilde{\tilde{v}}_2 + [G_\varepsilon(u_1,v_1,t) - G_\varepsilon(u_2,v_2,t)](y\cdot\nabla\tilde{\tilde{v}}_2) - \tilde{\tilde{v}}_2[g_1(u_1,v_1)H_\varepsilon(u_1-\sigma_D) - g_1(u_2,v_2)H_\varepsilon(u_2-\sigma_D)].$$

Clearly,

$$|G_\varepsilon(u,v,t)| \leq C(||(u,v)||) \quad \text{for } u,v\in X_T,\ t\in[0,T],$$

where $C(||(u,v)||)$ is independent of ε and T. This further implies that

$$e^{-2TC(||(u,v)||)} \leq \exp\{-2\int_0^y G_\varepsilon(u,v,\tau)d\tau\} \leq e^{2TC(||(u,v)||)} \quad \text{for } u,v\in X_T, t\in[0,T],$$

and

$$|\exp\{-2\int_0^t G_\varepsilon(u,v,\tau)d\tau\} - \exp\{-2\int_0^{t'} G_\varepsilon(u,v,\tau)d\tau\}| \leq C(||(u,v)||,T)|t-t'| \text{ for } u,v\in X_T, t\in[0,T].$$

Besides, it is also clear that

$$0 \leq H_\varepsilon(u-\sigma_D) \leq 1, \quad |y|\leq 1,\ 0\leq t\leq T.$$

Hence, by the L^p-estimate we obtain

$$||\tilde{\tilde{u}}^*||_{W_p^{2,1}(\tilde{Q}_T)} \leq C(T,p,||(u_1,v_1)||)||h_1||_{L^p(\tilde{Q}_T)},$$
$$||\tilde{\tilde{v}}^*||_{W_p^{2,1}(\tilde{Q}_T)} \leq C(T,p,||(u_1,v_1)||)||h_2||_{L^p(\tilde{Q}_T)}. \tag{4.19}$$

Using Lipschitz continuity of H_ε and (4.16) and (4.17), one can easily verify that

$$||h_1||_{L^p(\tilde{Q}_T)} \leq C(T,p,||(u_2,v_2)||,\varepsilon)||(u_1-u_2,v_1-v_2)||,$$
$$||h_2||_{L^p(\tilde{Q}_T)} \leq C(T,p,||(u_2,v_2)||,\varepsilon)||(u_1-u_2,v_1-v_2)||.$$

Substituting these estimates into (4.19), we obtain

$$||\tilde{\tilde{u}}^*||_{W_p^{2,1}(\tilde{Q}_T)} \leq C(T,p,||(u_1,v_1)||,||(u_2,v_2)||,\varepsilon)||(u_1-u_2,v_1-v_2)||, \tag{4.20}$$

$$||\tilde{\tilde{v}}^*||_{W_p^{2,1}(\tilde{Q}_T)} \leq C(T,p,||(u_1,v_1)||,||(u_2,v_2)||,\varepsilon)||(u_1-u_2,v_1-v_2)||. \tag{4.21}$$

Summing up (4.20), (4.21), taking $p>\frac{5}{2}$ and using the embedding $W_p^{2,1}(Q_T) \hookrightarrow C(Q_T)$ $(p>\frac{5}{2})$ we get

$$||(\tilde{u}_1-\tilde{u}_2,\tilde{v}_1-\tilde{v}_2)|| \leq ||\tilde{\tilde{u}}^*||_{W_p^{2,1}(\tilde{Q}_T)} + ||\tilde{\tilde{v}}^*||_{W_p^{2,1}(\tilde{Q}_T)} \leq C(T,p,||(u_1,v_1)||,||(u_2,v_2)||,\varepsilon)||(u_1-u_2,v_1-v_2)||.$$

Hence, F is continuous.

Since we have proved F maps E into itself, using the Schauder Fixed Point Theorem we conclude that F has a fixed point in E. Therefore, the problem (4.1)–(4.5) has a solution $(u, v) = (u_\varepsilon, v_\varepsilon)$ satisfying

$$||u_\varepsilon(|y|, t)||_{W_p^{2,1}(\tilde{Q}_T)} \leq C(T, p)(\bar{\sigma} + (||u_0||_{W^{2,\infty}(0,1)}),$$

$$||v_\varepsilon(|y|, t)||_{W_p^{2,1}(\tilde{Q}_T)} \leq C(T, p)(\bar{\beta} + ||v_0||_{W^{2,\infty}(0,1)}).$$

Hence (4.9), (4.10) are verified.

Next we prove uniqueness. Let $u^* = u_1 - u_2$, $v^* = v_1 - v_2$. Then u^*, v^* satisfy

$$u_t^* = D_1 R_0^{-2} \exp\{-2 \int_0^t G_\varepsilon(u_1, v_1, \tau) d\tau\} \Delta u^* + G_\varepsilon(u_1, v_1, t)(y \cdot u^*) + p(y, t),$$
$$|y| < 1, \quad t > 0, \tag{4.22}$$

$$v^* = D_2 R_0^{-2} \exp\{-2 \int_0^t G_\varepsilon(u_1, v_1, \tau) d\tau\} \Delta v^* + G_\varepsilon(u_1, v_1, t)(y \cdot v^*) + q(y, t),$$
$$|y| < 1, \quad t > 0, \tag{4.23}$$

$$u^*|_{|y|=1} = 0, \quad v^*|_{|y|=1} = 0, \quad 0 \leq t \leq T, \tag{4.24}$$

$$u^*|_{t=0} = 0, \quad v^*|_{t=0} = 0, \quad |y| \leq 1, \tag{4.25}$$

where

$$\begin{aligned} &p(y, t) \\ &= D_1 R_0^{-2} [\exp\{-2 \int_0^t G_\varepsilon(u_1, v_1, \tau) d\tau\} - \exp\{-2 \int_0^t G_\varepsilon(u_2, v_2, \tau) d\tau\}] \Delta u_2 \\ &[G_\varepsilon(u_1, v_1, t) - G_\varepsilon(u_2, v_2, t)](y \cdot \nabla u_2) - [f(u_1, v_1) - f(u_2, v_2)] H_\varepsilon(u_1(|y|, t) - \sigma_D) \\ &\quad - f(u_2, v_2)[H_\varepsilon(u_1(|y|, t) - \sigma_D) - H_\varepsilon(u_2(|y|, t) - \sigma_D)], \end{aligned}$$

$$\begin{aligned} &q(y, t) \\ &= D_2 R_0^{-2} [\exp\{-2 \int_0^t G_\varepsilon(u_1, v_1, \tau) d\tau\} - \exp\{-2 \int_0^t G_\varepsilon(u_2, v_2, \tau) d\tau\}] \Delta v_2 \\ &[G_\varepsilon(u_1, v_1, t) - G_\varepsilon(u_2, v_2, t)](y \cdot \nabla v_2) - [g(u_1, v_1) - g(u_2, v_2)] H_\varepsilon(u_1(|y|, t) - \sigma_D) \\ &\quad - g(u_2, v_2)[H_\varepsilon(u_1(|y|, t) - \sigma_D) - H_\varepsilon(u_2(|y|, t - \sigma_D))]. \end{aligned}$$

Multiplying (4.22) with u^* and integrating in y, we get

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int_{|y| \leq 1} (u^*(y, t))^2 dy \\ &\qquad = -D_1 R_0^{-2} [\exp\{-2 \int_0^t G_\varepsilon(u_1, v_1, \tau) d\tau\} \int_{|y| \leq 1} |\nabla u^*(y, t)|^2 dy \end{aligned}$$

$$
\begin{aligned}
-\frac{3}{2}G_\varepsilon(u_1,v_1,t)\int_{|y|\le 1}(u^*(y,t))^2dy+\int_{|y|\le 1}p(y,t)u^*(y,t)dy \\
\le -D_1R_0^{-2}e^{-2TC(||(u_1,v_1)||)}||\nabla u^*(\cdot,t)||^2_{L^2(B_1(0))} \\
+\frac{3}{2}C(||(u_1,v_1)||)||u^*(\cdot,t)||^2_{L^2(B_1(0))}+I(t), \quad (4.26)
\end{aligned}
$$

where

$$
\begin{aligned}
I(t)=\int_{|y|\le 1}p(y,t)u^*(y,t)dy = D_1R_0^{-2}[\exp\{-2\int_0^t G_\varepsilon(u_1,v_1,\tau)d\tau\} \\
-\exp\{-2\int_0^t G_\varepsilon(u_2,v_2,\tau)d\tau\}]\int_{|y|\le 1}\Delta u_2\cdot\nabla u^*(y,t)dy \\
+[G_\varepsilon(u_1,v_1,t)-G_\varepsilon(u_2,v_2,t)]\int_{|y|\le 1}(y\cdot\nabla u_2(y,t))u^*(y,t)dy \\
-\int_{|y|\le 1}[f(u_1,v_1)-f(u_2,v_2)]H_\varepsilon(u_1(|y|,t)-\sigma_D)u^*(y,t)dy \\
-\int_{|y|\le 1}f(u_2,v_2)[H_\varepsilon(u_1(|y|,t)-\sigma_D)-H_\varepsilon(u_2(|y|,t)-\sigma_D)]u^*(y,t)dy \\
\equiv I_1+I_2+I_3+I_4. \quad (4.27)
\end{aligned}
$$

By the Mean-Value Theorem and the fact that and $f(\alpha,\beta)\ge 0$ we have

$$
I_4=-\int_{|y|\le 1}f(u_2,v_2)\int_0^1 H'_\varepsilon d\theta\cdot(u^*)^2dy\le 0. \quad (4.28)
$$

Using (4.9), (4.10) and the embedding $W_p^{2,1}(\tilde{Q}_T)\hookrightarrow\hookrightarrow C(\tilde{Q}_T)(p>\frac{5}{2})$, we get

$$
\max_{\tilde{Q}_T}|\nabla u_2(|y|,t)|\le C(T), \quad \max_{\tilde{Q}_T}|\nabla v_2(|y|,t)|\le C(T). \quad (4.29)
$$

From (4.6) and Lipschitz continuity of $S(\sigma,\beta)$ and H_ε we have

$$
|G_\varepsilon(u_1,v_1,t)-G_\varepsilon(u_2,v_2,t)|\le C(\varepsilon)(||u^*||_2+||v^*||_2), \quad (4.30)
$$

so that by using (4.29) we get

$$
I_2\le C(\varepsilon)(||u^*||_2+||v^*||_2)\int_{|y|\le 1}|u^*(y,t)|dy. \quad (4.31)
$$

By a direct calculation and (4.30) we have

$$
\begin{aligned}
I_1=\exp\{-2\int_0^t G_\varepsilon(u_1,v_1,\tau)d\tau\} \\
-\exp\{-2\int_0^t G_\varepsilon(u_2,v_2,\tau)d\tau\}\int_{|y|\le 1}|\nabla u^*(y,t)dy \\
\le C(\varepsilon)\int_0^t(||u^*||_2+||v^*||_2)d\tau\int_{|y|\le 1}|\nabla u^*(y,t)dy. \quad (4.32)
\end{aligned}
$$

Lipschitz continuity of $f(\sigma,\beta)$ combined with (4.29) yields:

$$I_3 \le C(\varepsilon,T)\int_{|y|\le 1}\left(|u^*(y,t)|^2+|u^*(y,t)||v^*(y,t)|\right)dy. \tag{4.33}$$

Summing up (4.28), (4.31)–(4.33) into (4.27) we get

$$\begin{aligned} I(t) \le D_1 C(\varepsilon,T)\int_0^t(||u^*||_2+||v^*||_2)d\tau\int_{|y|\le 1}|\nabla u^*(y,t)|dy \\ + C(\varepsilon)(||u^*||_2+||v^*||_2)\int_{|y|\le 1}|u^*(y,t)|dy \\ + C(\varepsilon,T)\int_{|y|\le 1}\left(|u^*(y,t)|^2+|u^*(y,t)||v^*(y,t)|\right)dy. \end{aligned} \tag{4.34}$$

Substituting (4.34) into (4.26) and using the Cauchy inequality, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}||u^*(\cdot,t)||_2^2 \le -C_1(T)||\nabla u^*(\cdot,t)||_2^2 \\ + C_2(T)||u^*(\cdot,t)||_2^2 + C(\varepsilon)(||u^*||_2+||v^*||_2)\int_{|y|\le 1}|u^*(y,t)|dy \\ + C(\varepsilon,T)\int_0^t(||u^*||_2+||v^*||_2)d\tau\int_{|y|\le 1}|\nabla u^*(y,t)|dy \\ + C(\varepsilon,T)\int_{|y|\le 1}\left(|u^*(y,t)|^2+|v^*(y,t)|^2\right)dy. \end{aligned} \tag{4.35}$$

Similarly, from (4.23) we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}||v^*(\cdot,t)||_2^2 \le -C_1(T)||\nabla v^*(\cdot,t)||_2^2 \\ + C_2(T)||v^*(\cdot,t)||_2^2 + C(\varepsilon)(||u^*||_2+||v^*||_2)\int_{|y|\le 1}|v^*(y,t)|dy \\ + C(\varepsilon,T)\int_0^t(||u^*||_2+||v^*||_2)d\tau\int_{|y|\le 1}|\nabla v^*(y,t)|dy \\ + + C(\varepsilon,T)\int_{|y|\le 1}\left(|u^*(y,t)|^2+|v^*(y,t)|^2\right)dy. \end{aligned} \tag{4.36}$$

Summing (4.35) and (4.36), and denoting $w_0(t) = ||u^*(\cdot,t)||_2^2 + ||v^*(\cdot,t)||_2^2$, $w_1(t) = ||\nabla u^*(\cdot,t)||_2^2 + ||\nabla v^*(\cdot,t)||_2^2$, we obtain

$$\frac{1}{2}\frac{d}{dt}w_0(t) \le -C_1(T)w_1(t) + C_2(T)w_0(t) + J_1(t) + J_2(t) + C(\varepsilon,T)w_0(t), \tag{4.37}$$

where

$$J_1(t) = C(\varepsilon,T)\int_0^t\sqrt{w_0(\tau)}d\tau\int_{|y|\le 1}\left(|\nabla u^*(y,t)|+|\nabla v^*(y,t)|\right)dy,$$

$$J_2(t) = C(\varepsilon, T)\sqrt{w_0(t)} \int_{|y|\le 1} (|u^*(y,t)| + |v^*(y,t)|)\, dy.$$

Using the δ-Cauchy inequality, we get

$$J_1(t) \le C(\varepsilon, T)C(\delta) \int_0^t w_0(\tau)d\tau + C(\varepsilon, T)\delta w_1(t), \tag{4.38}$$

where δ is an arbitrary positive number and $C(\delta)$ is a positive constant depending on δ. Similarly,

$$J_2(t) \le C(\varepsilon, T)w_0(t). \tag{4.39}$$

Substituting (4.38) and (4.39) into (4.37), we have

$$\frac{1}{2}\frac{d}{dt}w_0(t) \le -C_1(T)w_1(t) + C_3(T)w_0(t) + C(\varepsilon, T)C(\delta) \int_0^t w_0(\tau)d\tau + C(\varepsilon, T)\delta w_1(t), \tag{4.40}$$

where $C_3(T) = C_2(T) + C(\varepsilon, T)$. Hence, by taking δ sufficiently small such that $C(\varepsilon, T)\delta \le C_1(T)$, we get

$$\frac{1}{2}\frac{d}{dt}w_0(t) \le C_3(T)w_0(t) + C(\varepsilon, T)C(\delta) \int_0^t w_0(\tau)d\tau.$$

Since $w(0) = 0$, by the Gronwall Lemma we conclude that $w_0(t) \equiv 0,\ 0 \le t \le T$. Hence $u_1 = u_2$, $v_1 = v_2$. □

Lemma 4.2. *Under the conditions (A1)–(A3), for any $T > 0$ the problem (3.12)–(3.16) has a solution (u, v) on X_T, satisfying:*

$$u(|y|, t),\ v(|y|, t) \in W_p^{2,1}(\tilde{Q}_T), \quad \forall p \in (1, \infty),$$

and

$$0 \le u(z, t) \le \bar{\sigma}, \quad 0 \le v(z, t) \le \bar{\beta}, \quad 0 \le z \le 1,\ 0 \le t \le T.$$

Proof. By Lemma 4.1, for any $\varepsilon > 0$ the problem (4.1)–(4.5) has a unique solution $(u_\varepsilon, v_\varepsilon) = (u_\varepsilon(z, t), v_\varepsilon(z, t))$ on X_T. Take a number $p > 5$. Then we have the compact embedding $W_p^{2,1}(\tilde{Q}_T) \hookrightarrow\hookrightarrow C^{\alpha, \frac{\alpha}{2}}(\tilde{Q}_T)$ $(0 < \alpha < 2 - \frac{5}{p})$. It follows that we can find a sequence of positive numbers $\varepsilon_k \to 0$ $(k \to \infty)$ and functions $u, v \in W_p^{2,1}(\tilde{Q}_T)$, such that if we denote $u_k = u_{\varepsilon_k},\ v_k = v_{\varepsilon_k}$, then:

$$u_k(|y|, t) \to u(|y|, t), \quad \nabla u_k(|y|, t) \to \nabla u(|y|, t) \ \text{ uniformly for } (y, t) \in \tilde{Q}_T,$$

$$v_k(|y|, t) \to v(|y|, t), \quad \nabla v_k(|y|, t) \to \nabla v(|y|, t) \ \text{ uniformly for } (y, t) \in \tilde{Q}_T,$$

$$u_{kt}(|y|, t) \xrightarrow{w} u_t(|y|, t)\ (L^p(\tilde{Q}_T)), \quad v_{kt}(|y|, t) \xrightarrow{w} v_t(|y|, t)(L^p(\tilde{Q}_T)),$$

$$\frac{\partial^2 u_k(|y|, t)}{\partial y_i \partial y_j} \xrightarrow{w} \frac{\partial^2 u(|y|, t)}{\partial y_i \partial y_j}\ (L^p(\tilde{Q}_T)),\ \frac{\partial^2 v_k(|y|, t)}{\partial y_i \partial y_j} \xrightarrow{w} \frac{\partial^2 v(|y|, t)}{\partial y_i \partial y_j}\ (L^p(\tilde{Q}_T)).$$

Clearly, $\{H_{\varepsilon k}(u - \sigma_D)\}$ is a bounded sequence in $L^\infty(\tilde{Q}_T)$, so that we can find a subsequence of $\{\varepsilon_k\}$, which we still denote as $\{\varepsilon_k\}$, and a function $h \in L^\infty(\tilde{Q}_T)$, such that

$$H_{\varepsilon k}(u_{\varepsilon_k} - \sigma_D) \xrightarrow{w} h(L^\infty(\tilde{Q}_T)), \tag{4.41}$$

$$G_{\varepsilon k}(u_{\varepsilon_k}, v_{\varepsilon_k}, t) \xrightarrow{*w} m(t)(L^\infty(\tilde{Q}_T)), \tag{4.42}$$

where

$$m(t) = \mu\Big\{\int_0^1 S(u(z,t), v(z,t))h(z,t)z^2 dz - \nu \int_0^1 (1 - h(z,t))z^2 dz\Big\}. \tag{4.43}$$

Taking $\varepsilon = \varepsilon_k$ in equations (4.1), (4.2), replacing u, v with u_k, v_k, respectively, and letting $k \to \infty$, we obtain

$$u_t = D_1 R_0^{-2} \exp\{-2\int_0^t m(\tau)d\tau\}\Delta u + m(t)(y \cdot \nabla u) - f(u,v)h, \qquad 0 < z < 1,\ t > 0, \tag{4.44}$$

$$v_t = D_2 R_0^{-2} \exp\{-2\int_0^t m(\tau)d\tau\}\Delta v + m(t)(y \cdot \nabla v) - g(u,v)h, \qquad 0 < z < 1,\ t > 0, \tag{4.45}$$

$$u_z(0,t) = 0, \quad u(1,t) = \bar{\sigma}, \quad t \geq 0, \tag{4.46}$$

$$v_z(0,t) = 0, \quad v(1,t) = \bar{\beta}, \quad t \geq 0, \tag{4.47}$$

$$u(z,0) = u_0(z), \quad v(z,0) = v_0(z), \quad 0 \leq z \leq 1. \tag{4.48}$$

We assert that

$$h(|y|,t) \overset{\text{a.e.}}{=} \begin{cases} 1, & u(|y|,t) > \sigma_D, \\ 0, & u(|y|,t) < \sigma_D. \end{cases}$$

Indeed, it is easy to verify that for any $\delta > 0$, $h(|y|,t) = 1$ for a.e. (y,t) in the set $\{(y,t) \in \tilde{Q}_T : u(|y|,t) \geq \sigma_D + \delta\}$. By the arbitrariness of δ, we infer that $h(|y|,t) = 1$ for a.e. $(y,t) \in \tilde{Q}_T$ such that $u(|y|,t) > \sigma_D$. Similarly we can also prove that $h(|y|,t) = 0$ for a.e. $(y,t) \in \tilde{Q}_T$ such that $u(|y|,t) < \sigma_D$. This proves the assertion. Since

$$u_t = 0, \quad \nabla u = 0, \quad \triangle u = 0 \ \text{ a.e. on the set } \{(y,t) \in \tilde{Q}_T :\ u(|y|,t) = \sigma_D\},$$

by (4.44) it follows that

$$h = 0 \ \text{ a.e. on the set } \{(y,t) \in \tilde{Q}_T :\ (|y|,t) = \sigma_D\}.$$

Hence

$$h(|y|,t) = H(u(|y|,t) - \sigma_D) \quad \text{for a.e. } (y,t) \in \tilde{Q}_T. \tag{4.49}$$

Substituting (4.49) into (4.43), we get

$$m(t) = G(u,v,t). \tag{4.50}$$

By (4.44), (4.45), (4.49), (4.50), we conclude that $u = u(z,t)$, $v = v(z,t)$ is a solution of the problem (3.12)–(3.16). This proves existence of a solution. The rest assertions of the lemma follow immediately from taking the weak limit in (4.1), (4.2). □

By Lemma 3.1 and Lemma 4.2, we see that Theorem 2.1 immediately follows.

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