International Journal of Pure and Applied Mathematics
Volume 28 No. 3 2006, 339-354

FINITE DIFFERENCE AND ELEMENT METHODS FOR PRICING OPTIONS WITH STOCHASTIC VOLATILITY

Vasile L. Lazar
"Vasile Goldiş" Western University
Mihai Viteazul Str., No. 26,
Satu Mare, 440030, ROMANIA
e-mail: vasilazar@yahoo.com

Abstract: Solving partial differential equations by standard numerical methods as finite difference and element methods is possible for a wide range of option models. We concentrate on stochastic volatility models where as the name suggests the volatility is not constant like in the Black-Scholes model, Black and Scholes [2], but it is a stochastic process itself. We take the example of Heston's model (see Heston [4]), to illustrate how to use this numerical methods.

AMS Subject Classification: 49M25
Key Words: option pricing, stochastic volatility, partial differential equations

1. Introduction

There are several numerical methods available for solving option models.

Solving the partial differential equations by standard numerical methods, like finite difference and element methods, is possible for a wide rage of option models. Also, Monte-Carlo methods are popular, and efficient at least for option models in higher dimensions.

Many pricing models can be cast into continuous time and as a result will naturally lead to partial differential equations. These types of PDEs are usually linear and parabolic.

We investigate the pricing of European put option under stochastic volatility. A variety of alternative random volatility models have appeared in the

Received: April 3, 2006

literature but we will take, only the example of Heston's model.

1.1. The Black-Scholes Formula

In the early 1970s, Fischer Black and Myron Scholes [2] made a major breakthrough by deriving a differential equation that must be satisfied by the price of any derivative security dependent on a non-dividend-paying stock. They solved this equation and obtained the closed-form solution for European call and put options on stock. This formula, known as the Black-Scholes formula, is the most significant tool for pricing financial instruments.

The Black-Scholes partial differential equation is:

$$\frac{\partial V}{\partial t} + \frac{1}{2}v^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \tag{1.1}$$

where V is the price of a derivative security; it is assumed to be a function of the current value of the underlying asset S at time t, v - the volatility and r is the continuously compounded risk-free rate.

The closed form solution for the price of European calls, with exercise price K and expiry T, is:

$$C(t,S) = SN(d_1) - Ke^{-r(T-t)}N(d_2) \tag{1.2}$$

where

$$d_1 = \frac{\log(S/K) + \left(r + \frac{1}{2}v^2\right)(T-t)}{v\sqrt{T-t}} \text{ and } d_2 = \frac{\log(S/K) + \left(r - \frac{1}{2}v^2\right)(T-t)}{v\sqrt{T-t}}$$

$N(x)$ is the *cumulative distribution function* for the standard normal distribution.

For a European put option we can use the *put-call parity formula*: $C - P = S - Ke^{-r(T-t)}$. Thus

$$P(t,S) = -SN(-d_1) - Ke^{-r(T-t)}N(-d_2)\,. \tag{1.3}$$

The simplest generalization of the Black-Scholes model is the model price of options which pay out dividends. We assume that the asset receives a continuous and constant dividend yield, D_0. The Black-Scholes equation is

$$\frac{\partial f}{\partial t} + \frac{1}{2}v^2 S^2 \frac{\partial^2 f}{\partial S^2} + (r - D_0)S\frac{\partial f}{\partial S} - rf = 0.$$

The value of the European put option, when there is continuous dividend yield on the underlying, is:

$$P(t,S) = -Se^{-D_0(T-t)}N(-d_1) + Ke^{-r(T-t)}N(-d_2)\,.$$

1.2. Heston's Stochastic Volatility Model

In the standard Black-Scholes model the volatility is assumed to be constant. Naturally the Black-Scholes assumption is incorrect and in reality volatility is not constant and it is not even predictable for timescales of more than a few months. This fact led to the development of stochastic volatility models, in which volatility itself is assumed to be a stochastic process.

We assume that S satisfies

$$dS = \mu S dt + vS dW_1, \tag{1.4}$$

and, in addition the volatility follows the stochastic process:

$$dv = p(t, v, S)dt + q(t, v, S)dW_2\,, \tag{1.5}$$

where the two increments dW_1 and dW_2 have a correlation of ρ.

In this case the value V is not only a function of S and time t, it is also a function of the variance v, $V(t, v, S)$. The partial differential equation governing the option price is a generalization of Black-Scholes equation:

$$\begin{aligned}\frac{\partial V}{\partial t} + \frac{1}{2}v^2 S^2 \frac{\partial^2 V}{\partial S^2} + \rho v S q \frac{\partial^2 V}{\partial S\,\partial v} + \frac{1}{2}q^2 \frac{\partial^2 V}{\partial v^2}\\ + rS\frac{\partial V}{\partial S} + (p - \lambda q)\frac{\partial V}{\partial v} - rV = 0\,,\end{aligned} \tag{1.6}$$

where λ is the market price of volatility risk.

Examples of these models in continuous-time include Hull and White (1987), Johnson and Shanno (1987), Wiggins (1987), Stein and Stein (1991), Heston (1993), Bates (1996),and examples in discrete-time include Taylor (1986), Amin and Ng (1993) and Heston and Nandi (1993).

Among them, Heston's model is very popular because of its main features:

— the existence of semi-closed form solutions for European calls and puts;
— it does not allow negative volatility;
— it allows the correlation between asset return and volatility.

Heston's option pricing formula is derived under the assumption that the stock price and its volatility follow the stochastic processes:

$$\begin{aligned} dS(t) &= S(t)[\mu dt + \sqrt{v(t)}dW_1(t)]\,,\\ dv(t) &= k(\theta - v(t))dt + \xi\sqrt{v(t)}dW_2(t), \end{aligned} \tag{1.7}$$

where:

$$\mathbf{Cov}[dW_1(t), dW_2(t)] = \rho dt\ . \tag{1.8}$$

Finally, the market price of volatility risk is given by:

$$\lambda(t,v,S) \ = \lambda v. \tag{1.9}$$

According to the pricing equation (1.6) we have the following partial differential equation for the Heston model:

$$\frac{\partial V}{\partial t} + \frac{1}{2} v S^2 \frac{\partial^2 V}{\partial S^2} + \rho \xi v S \frac{\partial^2 V}{\partial S \partial v} + \frac{1}{2} \xi^2 v \frac{\partial^2 V}{\partial v^2} + \ rS \frac{\partial V}{\partial S} + [k(\theta - v) - \lambda v] \frac{\partial V}{\partial v} - rV = 0. \tag{1.10}$$

The details of deriving the above equation and its closed-form solution, for a European call option, can be found in Heston's original work [4].

A European put option with strike price K and maturing at time T satisfies the PDE (1.10) and the following boundary conditions:

$$\begin{aligned}
&V(T,v,S) = \max(0, K - S)\,,\\
&V(T,v,0) = Ke^{-r_d \tau}\,,\\
&\frac{\partial V}{\partial S}(t,v,\infty) = 0\,,\\
&(r_d - r_f) S \frac{\partial V}{\partial S}(t,0,S) + k\theta \frac{\partial V}{\partial S}(t,0,S)\\
&\qquad + \frac{\partial V}{\partial t}(t,0,S) - r_d V(t,0,S) = 0\,,\\
&V(t,\infty,S) = Ke^{-r_d \tau}\,,
\end{aligned} \tag{1.11}$$

where $\tau = T - t$ denotes the time to maturity.

The first thing to do is to set $x = \ln \frac{S}{K}$ and $V(t,v,S) = w(t,v,lnS/K)$.

With this new variable the equation (1.10) becomes:

$$\frac{\partial w}{\partial t} + \frac{1}{2} \xi^2 v \frac{\partial^2 w}{\partial v^2} + \rho \xi v \frac{\partial^2 w}{\partial x \partial v} + \frac{1}{2} v \frac{\partial^2 w}{\partial x^2} + [k(\theta - v) - \lambda v] \frac{\partial w}{\partial v} + (r_d - r_f - \frac{1}{2} v) \frac{\partial w}{\partial x} - r_d w = 0\,, \tag{1.12}$$

with terminal condition

$$w(T,v,x) = g(Ke^x), \tag{1.13}$$

for all $(t,v,x) \in \Omega_\infty \times [0,T)$ with $\Omega_\infty = (-\infty, \infty) \times [0, \infty)$, $g(Ke^x)$ is the payoff-function of the option.

We can rearrange the coefficients of the Heston p.d.e. (1.12) in a convenient way in order to obtain a matrix representation:

$$\frac{\partial w}{\partial t} + \frac{1}{2}\xi^2 v\frac{\partial^2 w}{\partial v^2} + \frac{1}{2}\xi^2\frac{\partial w}{\partial v} + \frac{1}{2}\rho\xi\frac{\partial w}{\partial x} + \frac{1}{2}\rho\xi v\frac{\partial^2 w}{\partial x\partial v} + \frac{1}{2}\rho\xi v\frac{\partial^2 w}{\partial v\partial x} + \frac{1}{2}v\frac{\partial^2 w}{\partial x^2} + [k(\theta - v) - \lambda v]\frac{\partial w}{\partial v} - \frac{1}{2}\xi^2\frac{\partial w}{\partial v} + (r_d - r_f - \frac{1}{2}v)\frac{\partial w}{\partial x} - \frac{1}{2}\rho\xi\frac{\partial w}{\partial x} - r_d w = 0\,,$$

$$\nabla\cdot\begin{bmatrix} \frac{1}{2}v\xi^2\frac{\partial w}{\partial v} + \frac{1}{2}v\rho\xi\frac{\partial w}{\partial x} \\ \frac{1}{2}v\rho\xi\frac{\partial w}{\partial v} + \frac{1}{2}v\frac{\partial w}{\partial x} \end{bmatrix} - \left[(-k(\theta - v) + \lambda v + \frac{1}{2}\xi^2)\frac{\partial w}{\partial v} + (-(r_d - r_f) + \frac{1}{2}v + \frac{1}{2}\xi\rho)\frac{\partial w}{\partial x}\right] - r_d w = \frac{\partial w}{\partial t}\,.$$

In the matrix notation equation (1.12) has the form:

$$0 \;=\; \frac{\partial w}{\partial t} \;+\; \nabla\cdot A\;\nabla w \;-\; b\cdot\nabla w \;-\; r_d\;w\;, \tag{1.14}$$

where

$$A \;:=\; \frac{1}{2}\;v\;\begin{bmatrix} \xi^2 & \rho\;\xi \\ \rho\;\xi & 1 \end{bmatrix}$$

and

$$b \;:=\begin{bmatrix} -\;k\;(\theta\;-\;v)\;+\;\lambda\;v\;+\;\frac{1}{2}\;\xi^2 \\ -\;(r_d\;-\;r_f)\;+\;\frac{1}{2}\;v\;+\;\frac{1}{2}\;\xi\;\rho \end{bmatrix}.$$

This is a convection-diffusion equation, where the matrix A is called diffusion matrix and b the convection vector. In the Heston p.d.e the diffusion term is linear in v and so is the convection term up to an additional constant.

We note that the diffusion matrix A is positive semidefinite for all $v\;>\;0$.

2. Finite Difference Method

Finite difference methods are suitable for solving financial problems with two or three random factors.

Besides the finite element and finite volume method the finite difference method is one method to solve partial differential equations numerically. In its simplest form it requires a structured grid that is why the method is not suitable for regions $\Omega \in \mathbb{R}^d$ with a smooth boundary. Since the Heston p.d.e. (1.10) can be approximated by a rectangular domain that causes no problems.

The big advantage of finite difference method it its simplicity and straight forward implementation where,in short, the derivatives are replaced by differential quotients. Additionally, under certain conditions it achieves second order convergence.

We consider the Heston partial differential equation (1.10). As usual, the first thing to do is to discretize the variables. This means, we solve on a three dimensional grid with

$$S \ = \ i \ \delta S \ , \quad v \ = \ j \ \delta v \text{ and } \quad t \ = \ T \ - \ k \ \delta t \quad i = 0, ..., I \ , \ j = 0, ..., J \ .$$

The contract value is written as:

$$V(S \ , \ v \ , \ t) \ = \ V_{ij}^{k} \ .$$

Whatever the problem to be solved might be, we must impose certain conditions on the solution. First we must specify the final condition, which is the payoff function.

Suppose that we have a European put option. In finite-difference notation, the final condition for this problem is:

$$V_{ij}^{0} = \max(K - i\delta S, 0). \tag{2.15}$$

This final condition will get our finite-difference scheme started. Backward time stepping must then be used when calculating the solution at an earlier time step.

Also, we must impose boundary conditions around our domain, boundary conditions which will depend on the contract. In our case the boundary conditions are given by (1.11).

2.1. The Explicit Method

The definition of the first time-derivative of V is

$$\frac{\partial V}{\partial t} = \lim_{\delta t \to 0} \frac{V(S, v, t + \delta t) - V(S, v, t)}{\delta t}.$$

It follows naturally that we can approximate the time derivative from our grid of values using

$$\frac{\partial V}{\partial t}(S, v, t) \approx \frac{V_{ij}^{k} - V_{ij}^{k+1}}{\delta t} + \mathcal{O}(\delta t) \tag{2.16}$$

The same idea can be used for approximating the first S and v derivatives. But now we have three choices: *forward difference, backward difference* and, the most accurate of the three, *central difference*. The central difference has an error of $O(\delta S^2)$ whereas the error in the forward and backward differences is in both much larger, $O(\delta S)$. So, for the above derivatives we have:

$$\frac{\partial V}{\partial S}(S,v,t) \approx \frac{V_{i+1,j}^k - V_{i-1,j}^k}{2\delta S} + \mathcal{O}(\delta S^2) \tag{2.17}$$

and

$$\frac{\partial V}{\partial v}(S,v,t) \approx \frac{V_{i,j+1}^k - V_{i,j-1}^k}{2\delta v} + \mathcal{O}(\delta v^2). \tag{2.18}$$

The natural approximation for the second derivative of the option with respect to the underlying is:

$$\frac{\partial^2 V}{\partial S^2}(S,v,t) \approx \frac{V_{i+1,j}^k - 2V_{ij}^k + V_{i-1,j}^k}{\delta S^2} + \mathcal{O}(\delta S^2). \tag{2.19}$$

Analogously, we have:

$$\frac{\partial^2 V}{\partial v^2}(S,v,t) \approx \frac{V_{i,j+1}^k - 2V_{ij}^k + V_{i,j-1}^k}{\delta v^2} + \mathcal{O}(\delta v^2). \tag{2.20}$$

Note that all the above approximations come from a Taylor series expansion.

We have seen how to use central differences for all of the terms of the equation (1.10) with the exception of the second derivative with respect to both S and v.

We can approximate this by

$$\frac{\partial\left(\frac{\partial V}{\partial v}\right)}{\partial S} \approx \frac{\frac{\partial V}{\partial v}(S+\delta S,v,t) - \frac{\partial V}{\partial v}(S-\delta S,v,t)}{2\delta S},$$

but

$$\frac{\partial V}{\partial v}(S+\delta S,v,t) \approx \frac{V_{i+1,j+1}^k - V_{i+1,j-1}^k}{2\delta v}.$$

This suggests that a suitable discretization might be

$$\frac{\frac{V_{i+1,j+1}^k - V_{i+1,j-1}^k}{2\delta v} - \frac{V_{i-1,j+1}^k - V_{i-1,j-1}^k}{2\delta v}}{2\delta S} = \frac{V_{i+1,j+1}^k - V_{i+1,j-1}^k - V_{i-1,j+1}^k + V_{i-1,j-1}^k}{4\delta S\delta v}. \tag{2.21}$$

Note that we have used the property that

$$\frac{\partial^2 V}{\partial S \partial v} = \frac{\partial^2 V}{\partial v \partial S}.$$

The resulting explicit difference scheme is

$$\begin{aligned}
&\frac{V_{ij}^{k}-V_{ij}^{k+1}}{\delta t}+\frac{1}{2}\ \sigma^{2} j\delta v^{k}\left(\frac{V_{i,j+1}^{k}-2\ V_{ij}^{k}+V_{i,j-1}^{k}}{\delta v^{2}}\right)\\
&\quad+\rho\ \sigma ij\delta v^{k}\delta S^{k}\left(\frac{V_{i+1,j+1}^{k}-V_{i+1,j-1}^{k}-V_{i-1,j+1}^{k}+V_{i-1,j-1}^{k}}{4\delta S\delta v}\right)\\
&\quad+\frac{1}{2}i^{2} j\delta v^{k}\ (\delta S^{k})^{2}\left(\frac{V_{i+1,j}^{k}-2V_{ij}^{k}+V_{i-1,j}^{k}}{\delta S^{2}}\right)\\
&\quad+(r_{d}-r_{f})i\delta S^{k}\left(\frac{V_{i+1,j}^{k}-V_{i-1,j}^{k}}{2\delta S}\right)\\
&\quad+[k(\theta-j\delta v^{k})-\lambda j\delta v^{k}]\left(\frac{V_{i,j+1}^{k}-V_{i,j-1}^{k}}{2\delta v}\right)\\
&\quad+r_{d}V_{ij}^{k}=\mathcal{O}(\delta t,\delta S^{2},\delta v^{2}).
\end{aligned}\tag{2.22}$$

One of the advantages of the explicit method is that it is easy to program. The main disadvantage comes in stability and speed. The method is only stable for sufficiently small timesteps. Having an upper bound on the timestepsize seriously limits the speed of calculation.

We can analyze the stability of the method by looking for solutions of the difference equation that are oscillatory in both the S and v directions. This means looking for a solution of the form

$$V_{ij}^{k}=\alpha^{k}e^{2\pi\sqrt{-1}\left(\frac{1}{\lambda_S}+\frac{1}{\lambda_v}\right)}$$

and assuming that all the coefficients are slowly varying over the δS, δv length-scales.

If we have a pure diffusion problem with no convection or decay terms, and no correlation between the variables, then only σ, v, S are non-zero. In this case, the stability requirement becomes

$$\frac{1}{2}vS^{2}\frac{\delta t}{\delta S^{2}}+\frac{1}{2}\sigma^{2}v\frac{\delta t}{\delta v^{2}}\leq\frac{1}{2}.$$

2.2. An Implicit Method

There are many implicit methods used for two-factor problems. One of this methods is called *Alternating Direction Implicit* or ADI.

If we want to keep the stability advantage of the implicit method and the ease of solution of the explicit method we could try to solve implicitly in one factor but explicitly in the other. This is the idea behind ADI.

As well as V_{ij}^k, we introduce an "intermediate" value $V_{ij}^{k+1/2}$. We solve from timestep k to the intermediate step $k+\frac{1}{2}$ using explicit differences in S and implicite differences in v. Having found the intermediate value $V_{ij}^{k+1/2}$ step forward to timestep $k+1$ using implicite differences in S and explicite differences in v. The method is stable for all timesteps and the error is $O(\delta t^2, \delta S^2, \delta v^2)$.

We will apply the ADI for the following equation

$$\frac{\partial V}{\partial t} + \frac{1}{2} v S^2 \frac{\partial^2 V}{\partial S^2} + \frac{1}{2}\sigma^2 v \frac{\partial^2 V}{\partial v^2} = 0\,. \tag{2.23}$$

The explicit S, implicit v discretization looks like this

$$\frac{V_{ij}^k - V_{ij}^{k+1/2}}{\frac{1}{2}\delta t} + \frac{1}{2} i^2 j \delta v^k (\delta S^k)^2 \left(\frac{V_{i+1,j}^k - 2V_{ij}^k + V_{i-1,j}^k}{\delta S^2} \right)$$
$$+ \frac{1}{2}\sigma^2 j \delta v^{k+1/2} \left(\frac{V_{i,j+1}^{k+1/2} - 2V_{ij}^{k+1/2} + V_{i,j-1}^{k+1/2}}{\delta v^2} \right) = 0\,.$$

Putting all of the $k+\frac{1}{2}$ timestep terms on the left and all k timestep terms on the right we obtain:

$$V_{ij}^{k+1/2} - \frac{1}{4}\sigma^2 j \delta v^{k+1/2} \left(\frac{V_{i,j+1}^{k+1/2} - 2V_{ij}^{k+1/2} + V_{i,j-1}^{k+1/2}}{\delta v^2} \right) \delta t$$
$$= V_{ij} k + \frac{1}{4} i^2 j \delta v^k (\delta S^k)^2 \left(\frac{V_{i+1,j}^k - 2V_{ij}^k + V_{i-1,j}^k}{\delta S^2} \right) \delta t\,. \tag{2.24}$$

If we know all of the k timestep terms we can find the $V_{ij}^{k+1/2}$ by solving a set of simultaneous equations. This looks like a fully implicit scheme.

Having found $V_{ij}^{k+1/2}$ we step forward to find V_{ij}^{k+1} by reversing the implicit and explicit roles:

$$\frac{V_{ij}^{k+1} - V_{ij}^{k+1/2}}{\frac{1}{2}\delta t} + \frac{1}{2} i^2 j \delta v^{k+1} (\delta S^{k+1})^2 \left(\frac{V_{i+1,j}^{k+1} - 2V_{ij}^{k+1} + V_{i-1,j}^{k+1}}{\delta S^2} \right)$$
$$+ \frac{1}{2}\sigma^2 j \delta v^{k+1/2} \left(\frac{V_{i,j+1}^{k+1/2} - 2V_{ij}^{k+1/2} + V_{i,j-1}^{k+1/2}}{\delta v^2} \right) = 0\,.$$

This can be rewritten as

$$V_{ij}^{k+1} + \frac{1}{4} i^2 j \delta v^{k+1} (\delta S^{k+1})^2 \left(\frac{V_{i+1,j}^{k+1} - 2V_{ij}^{k+1} + V_{i-1,j}^{k+1}}{\delta S^2} \right) \delta t$$

$$= V_{ij}^{k+1/2} - \frac{1}{4} \sigma^2 j \delta v^{k+1/2} \left(\frac{V_{i,j+1}^{k+1/2} - 2V_{ij}^{k+1/2} + V_{i,j-1}^{k+1/2}}{\delta v^2} \right) \delta t\,. \quad (2.25)$$

This is a fully explicit scheme for finding V_{ij}^{k+1} from $V_{ij}^{k+1/2}$.

More about the concepts of convergence and stability of the finite difference method for the Heston p.d.e. can be found in Kluge [5].

3. Finite Element Method

In this section, we focus on how to solve equation (1.10) using Finite Element Method (FE). FE approaches offer some advantages:

— A solution for the entire domain is computed, instead of isolated nodes as in the case with FD codes.

— The boundary conditions involving derivatives,which are particularly important for the treatment of American options, are difficult to handle with FD. Neumann conditions, however, are often easier to obtain than Dirichlet conditions when estimating the behaviour of the option when the price of the underlying goes to infinity. FE techniques can incorporate boundary conditions involving derivatives easily.

— In addition, FE can easily deal with high curvature. In most FE codes this is achieved by adaptive remeshing, a technique well-developed in theory and in practice.

3.1. Boundary Conditions

In order to obtain a solution using finite element method we must limit Ω_∞ to a bounded domain Ω:

$$\Omega = (x_{\min}\,,\ x_{\max}) \times (v_{\min}\,,\ v_{\max})\,,$$
$$\partial\Omega = \Gamma_1 \cup \Gamma_2\,,\ \Gamma_1 \cap \Gamma_2 = \emptyset\,.$$

From the Black-Scholes formulae (1.3) we find that the boundary condition for $v_{\min}$ and $v_{\max}$ are:

$$\Gamma_a \;:\; v \;=\; v_{\min} \;: \qquad w(t, v_{\min}, x) \;=\; K\; e^{-r_d(T-t)}\; \Phi(-d_2) - K\; e^{x-r_f(T-t)} \Phi(-d_1)\,,$$

$$\Gamma_b \;:\; v \;=\; v_{\max} \;:\quad w(t, v_{\max}, x) \;=\; K\; e^{-r_d(T-t)}\,.$$

For $x_{\min}$ and $x_{\max}$ we have the following conditions:

$$\Gamma_c \;:\; x \;=\; x_{\min} \;: \qquad \frac{\partial}{\partial v}\, w(t, v, x_{\min}) \;:=\; A\; \nabla w \cdot \vec{n} \;=\; -\,\frac{1}{2}\; v\; K\; e^{x-r_f(T-t)}\,,$$

$$\Gamma_d \;:\; x \;=\; x_{\max} \;: \qquad w(t, v, x_{\max}) \;=\; \lambda\; w(t, v_{\max}, x_{\max}) \;+\; (1-\lambda)\; w(t, v_{\min}, x_{\max}),$$

$$\lambda \;=\; \frac{v \;-\; v_{\min}}{v_{\max} \;-\; v_{\min}}\,.$$

The Dirichlet boundary $\Gamma_1 \;=\; \Gamma_a \;\cup\; \Gamma_b \;\cup\; \Gamma_d$ consists of three edges and must give the same limits on the corner of the domain, when moving to a corner from two distinct vertices. To guarantee the existence of a continuous solution on the entire domain Ω we will make the following *assumption*: on the boundary Γ_d we interpolate linearly between the values of the right and the left boundary.

3.2. Variational Formulation

Further we choose a semidiscretization in time which yields a series of two-dimensional boundary value problems.

We take times t^k and consider the difference quotient:

$$D_t^+ w(t^k, v, x) = \frac{w(t^{k+1}, v, x) - w(t^k, v, x)}{\tau}\,, \tag{3.1}$$

where $\tau \;=\; t^{k+1} \;-\; t^k \;=\; ct$, for all k.

The computational working backwards in time, the initial value is given for $t^N \;=\; T$ and the solution is sought at $t_0 \;=\; 0$, so in each timestep t^k we can assume we know $w(t^{k+1})$. Using a standard finite difference σ-rule, where for $\sigma \;=\; 1$ we have a purely implicit method, for $\sigma \;=\; 0$ an explicit method and for $\sigma \;=\; \frac{1}{2}$ the Crank-Nicolson scheme, our problem reduces to solving a partial differential equation of the form:

$$Lw(t^k, v, x) = f(t^{k+1}, v, x)\,, \tag{3.2}$$

where

$$Lw(t,v,x) = -\sigma\nabla\cdot A\nabla w + \sigma b\cdot\nabla w + (\sigma r_d + \frac{1}{\tau})w \tag{3.3}$$

and

$$f(t,v,x) = (1-\sigma)\nabla\cdot A\nabla w + (1-\sigma)b\cdot\nabla w - \left((1-\sigma)r_d + \frac{1}{\tau}\right)w\,. \tag{3.4}$$

The next step is to choose appropriate spaces. Using the fact that the matrix A is positive definite (see T. Apel, G. Winkler and U. Wystup [13]), define the spaces in the following way:

Definition 3.1. Let $\Omega\subset\mathbb{R}^d$ be a bounded open domain and u and w be two functions. Let the $d\times d$-matrix $A(x)$ be symmetric and positive definite for all $x\in\Omega$ and let the constant $\hat{c}$ be positive. Then we define:

$$(\,u\,,\,w\,)_A \;=\; \int_\Omega A\,\nabla u\cdot\nabla w \;+\; \int_\Omega \hat{c}\,u\,w\,, \tag{3.5}$$

$$\|\,u\,\|_A^2 \;=\; (\,u\,,\,u\,)_A \;=\; \int_\Omega A\,\nabla u\cdot\nabla u \;+\; \int_\Omega \hat{c}\,u^2\,. \tag{3.6}$$

Lemma 3.1. *The bilinear form* $(\,u\,,\,w\,)_A$ *defines a scalar product and satisfies the Cauchy-Schwarz inequality.* $\|\cdot\|$ *defines a norm.*

Definition 3.2. We define the spaces:

$$\begin{aligned} V &:= \{\,\psi\,/\;\|\,\psi\,\|_A < \infty\,\},\\ V_0 &:= \{\,\psi\in V/\;\psi = 0 \text{ on } \Gamma_1\,\},\\ V_* &:= \{\,\psi\in V/\;\psi \text{ satisfies the boundary conditions on } \Gamma_1\,\}. \end{aligned} \tag{3.7}$$

Remark 1. 1. The constant $\hat{c}$ will be taken to be $c-\frac{1}{2}\nabla\cdot b$.
2. If $\hat{c}=\tilde{c}-\frac{1}{2}\nabla\cdot b$.

In order to solve equation (3.2) we multiply it with a test function ψ taken from the function space V_0 and integrate it over the domain Ω :

$$\int_\Omega Lw\psi = \int_\Omega f\psi. \tag{3.8}$$

The left hand side will be:

$$\int_\Omega Lw\psi = \int_\Omega -\sigma\nabla\cdot A\nabla w\psi + \sigma b\cdot\nabla w\psi + cw\psi\,, \tag{3.9}$$

where $c = \sigma r_d + \frac{1}{\tau}$ and the right hand side is:

$$\int_\Omega f\psi = \int_\Omega (1-\sigma)\nabla \cdot A\nabla w\psi - (1-\sigma)b\cdot\nabla w\psi + \tilde{c}w\psi, \tag{3.10}$$

where $\tilde{c} = (1-\sigma)r_d - \frac{1}{\tau}$.

From (3.8), using integration by parts and the given boundary conditions, we obtain the following weak formulation: We must search for each k a function $w^k = w(t^k, v, x) \in V_*$ such that for all $\psi \in V_0$

$$a(w^k\ ,\ \psi)\ =\ \langle\ F\ ,\ \psi\ \rangle \tag{3.11}$$

holds, where

$$\begin{aligned} a(w^k,\psi) = & \int_\Omega A\nabla w^k \cdot \nabla\psi + \frac{1}{2}\int_\Omega (b\cdot\nabla w^k\psi - w^k b\cdot\nabla\psi) \\ & + \int_\Omega (c - \frac{1}{2}\nabla\cdot b)w^k\psi \\ & + \ \frac{1}{2}\int_{\Gamma_c}\left[\frac{1}{2}v - (r_d - r_f) + \frac{1}{2}\xi\rho\right] w^k\psi, \end{aligned} \tag{3.12}$$

$$\langle\ F^k\ ,\ \psi\ \rangle = f(t^{k+1};\psi) + \int_{\Gamma_c} \underbrace{A\nabla w^k \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{g_2} \psi \tag{3.13}$$

$$f(t^{k+1};\psi) = \int_\Omega A\nabla w^{k+1}\cdot\nabla\psi - \int_\Omega (b\cdot\nabla w^{k+1})\psi - \int_\Omega (\tilde{c} - \frac{1}{2}\nabla\cdot b)w^{k+1}\psi.$$

We have used the fact that $\psi = 0$ along the boundaries Γ_a, Γ_b and Γ_d.

Theorem 3.1. *The equation*

$$a(w,\psi) = \langle\ F\ ,\ \psi\ \rangle\ \ \forall\ \psi\ \in\ V_0 \tag{3.14}$$

along with the bilinear functional a taken from equation (3.12) and the linear functional F taken from equation (3.13) has unique solution $w \in V_*$, if

$$\min\left\{\frac{1}{2}v - (r_d - r_f) + \frac{1}{2}\xi\rho\ \ ,\ \ v\in[v_{\min}, v_{\max}]\right\} > -2 \tag{3.15}$$

and

$$\hat{c} = (c - \frac{1}{2}\nabla\cdot b) = r_d + \frac{1}{\sigma\tau} - \frac{1}{2}(k+\lambda) > 0. \tag{3.16}$$

Proof. i) First we show that $a(\cdot,\cdot)$ is a V_0-elliptic operator.

Let us take $a(\psi,\psi)$ for $\psi \in V_0$,

$$a(\psi,\psi) = \int_\Omega A\nabla\psi\cdot\nabla\psi + \int_\Omega (c - \frac{1}{2}\nabla\cdot b)\psi^2 + \frac{1}{2} + \int_\Omega \left[\frac{1}{2}v - (r_d - r_f) + \frac{1}{2}\xi\rho\right]\psi^2 .$$

If $r_d \le r_f + \frac{1}{2}\xi\rho$, i.e. $-(r_d - r_f) + \frac{1}{2}\xi\rho > 0$, by the condition (3.16) we have that:

$$a(\psi,\psi) \ge \| \psi \|_A .$$

Otherwise, if $r_d > r_f + \frac{1}{2}\xi\rho$ we obtain:

$$\int_{\Gamma_c} g\psi^2 = \int_{\Gamma_c^-} g\psi^2 + \int_{\Gamma_c^+} g\psi^2 \ge -\| g \|_{0,\infty,\Gamma_c^-} \int_{\Gamma_c^-} \psi^2 \ge -\| g \|_{0,\infty,\Gamma_c^-} \| \psi \|_A .$$

Hence

$$a(\psi,\psi) \ge \left(1 - \frac{1}{2}\| g \|_{0,\infty,\Gamma_c^-}\right) \| \psi \|_A .$$

By the condition (3.15) we deduce that $1 - \frac{1}{2}\| g \|_{0,\infty,\Gamma_c^-} > 0$.

Consequently $a(\cdot,\cdot)$ is V_0-elliptic.

ii) We show now that the $a(\cdot,\cdot)$ is V_0-bounded operator:

$$\begin{aligned} a(w,\psi) &= \textstyle\int_\Omega A\nabla w\cdot\nabla\psi + \int_\Omega (c - \frac{1}{2}\nabla\cdot b)w\psi + \int_\Omega (b\cdot\nabla w)\psi \\ &= (w,\psi)_A + \textstyle\int_\Omega (A^{-1/2}b\cdot A^{1/2}\nabla w)\psi \\ &\le \| w \|_A \| \psi \|_A + \left(\textstyle\int_\Omega | A^{-1/2}b\cdot A^{1/2}\nabla w |^2 \int_\Omega \psi^2\right)^{1/2} \\ &\le \| w \|_A \| \psi \|_A + \left(\| A^{-1/2}\ b \|^2_{0,\infty,\Omega} \textstyle\int_\Omega |A^{1/2}\nabla w |^2 \int_\Omega \psi^2\right)^{1/2} \end{aligned}$$

since A is regular for all $v \ge v_{\min} > 0$ and $v > v_{\max}$, the norm is finite. Both the integrals are bounded by the norm $\|\cdot\|_A$. Therefore

$$a(w,\psi) \le C\|w\|_A\|\psi\|_A, \text{ where } C = 1 + C_A\|A^{-1/2}b\|_{0,\infty,\Omega}.$$

Note that the constant C_A depends on c.

iii) Let us consider the right hand side of the equation (3.14)

$$\begin{aligned} \langle F,\psi\rangle &= -(w^{k=1},\psi)_{\tilde{A}} - \int_\Omega (b\cdot\nabla w^{k=1})\psi + \int_{\Gamma_c} g_2\psi \\ &\le \|w^{k+1}\|_{\tilde{A}}\|\psi\|_{\tilde{A}} + \|A^{-1/2}b\|_{0,\infty,\Omega}\|w^{k+1}\|_{\tilde{A}}\|\psi\|_A + \int_{\Gamma_c} g_2\psi \end{aligned}$$

$$\leq \|w^{k+1}\|_{\tilde{A}}\|\psi\|_{\tilde{A}} + \|A^{-1/2}b\|_{0,\infty,\Omega}\|w^{k+1}\|_{\tilde{A}}\|\psi\|_A + \mathcal{C}\|g_2\|_{0,\Gamma_c}\|\psi\|_A$$
$$\leq \mathcal{C}\|\psi\|_A.$$

iv) While choosing the boundary conditions we already made sure the Dirichlet-type conditions are continuous along the corners of the domain. Since the domain is a rectangle, its boundary is piecewise smooth, whence a continuous extension of the boundary values to the entire domain Ω is possible.

From all these properties, i)-iv), by the lemma of Lax and Milgram we deduce that the equation (3.14) has a unique solution.

References

[1] W.F. Ames, *Numerical Methods for Partial Differential Equations*, Boston (1992).

[2] F. Black, M. Scholes, The pricing of options and corporate liabilities, *Journal of Political Economy*, **81** (1973), 637-654.

[3] P.A. Forsyth, K.R. Vetzal, R. Zvan, A finite element approach to the pricing of discrete lookbacks with stochastic volatility, *Applied Mathematical Finance*, **6** (1999), 87-106.

[4] S. Heston, A closed-form solution for options with stochastic volatility with applications to bond and currency options, *The Review of Financial Studies*, **6** (1993), 327-343.

[5] T. Kluge, *Pricing Derivatives in Stochastic Volatility Models Using the Finite Difference Method*, Diploma Thesis, Technische Universität Chemnitz (2002).

[6] R. Korn, E. Korn, *Option Pricing and Portfolio Optimization-Modern Methods of Financial Mathematics*, American Math. Soc. (1999).

[7] K.W. Morton, *Numerical Solution of Convection-Diffusion Problems*, Chapman and Hall (1996).

[8] R. Precup, *Ecuatii cu derivate partiale*, Transilvania Press, Cluj Napoca (1997).

[9] H. Roos, M. Stynes, L. Tobiska, *Numerical Methods for Singularly Perturbed Differential Equations*, Springer (1996).

[10] J.C. Strikwerda, *Finite Difference Schemes and Partial Differential Equations*, Wadsworth and Brooks (1989).

[11] P. Wilmott, *Derivatives: The Theory and Practice of Financial Engineering*, John Wiley and Sons Ltd, New York (1999).

[12] P. Wilmott, J. Dewynne, S. Howison, *Option Pricing: Mathematical Models and Computation*, Oxford Financial Press (1993).

[13] G. Winkler, T. Apel, U. Wystup, *Valuation of Options in Heston's Stochastic Volatility Model Using Finite Element Methods*, Foreign Exchange Risk, Risk Publications, London (2001).

[14] U. Wystup et al, *MathFinance*, URL: http://www.mathfinance.de