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POINTS OF SMOOTH CURVES LINEARLY EQUIVALENT TO DIVISORS WITH PRESCRIBED SUPPORT

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Abstract: Let X be a smooth genus g curve, $P \in X$ and $R \in \mathrm{Pic}(X)$. Here we study the following problem. Under what assumptions there is $D \in |R|$ and a divisor A on X such that $\mathrm{Supp}(A) \subseteq \mathrm{Supp}(D)$ and $P \notin \mathrm{Supp}(D)$?

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1. Introduction

Let X be a smooth and connected genus $g \geq 2$ curve defined over an algebraically closed field $\mathbb{K}$. For any divisor $D = \sum_i^x m_i P_i$ with $m_i \in \mathbb{Z}\backslash\{0\}$ for all i and $P_i \neq P_j$ for all $i \neq j$ set $\mathrm{Supp}(D) := \{P_1, \dots, P_x\}$. Fix any $R \in \mathrm{Pic}(X)$ such that $h^0(X,R) \geq 2$. For any linear subspace $V \subseteq H^0(X,R)$ such that $V \neq \{0\}$. Let $|V|$ denote the projective space of all 1-dimensional linear subspaces of V. Set $|R| := |H^0(X,R)$. For any zero-dimensional scheme $Z \subset X$, set $V(-Z) := V \cap H^0(X, \mathcal{I}_Z \otimes R)$. Motivated by [3], Section 3, we introduce the following definition. A point $P \in X$ is said to be R-good and R is said to be nice to P if there is $D \in |L|$ and a divisor D' such that $\mathcal{O}_X(P) \cong \mathcal{O}_X(D')$, $\mathrm{Supp}(D') \subseteq \mathrm{Supp}(D)$ and $P \notin \mathrm{Supp}(D)$. Let $w_1(X)$ be the first integer x such that there exists $L \in \mathrm{Pic}^x(X)$ which is nice to all $P \in X$. Let $w_2(X)$ be the first integer x such that every $L \in \mathrm{Pic}^x(X)$ is nice to all $P \in X$. Let $w_3(X)$ be the first integer y such that for all integers $x \geq y$ every $L \in \mathrm{Pic}^x(X)$ is nice to all $P \in X$. Obviously, $w_1(X) \leq w_2(X) \leq w_3(X)$. By [3], §3, $w_3(X) \leq 3g+1$

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for all smooth genus g curves and $w_3(C) \le 2g+1$ for every smooth genus g hyperelliptic curve C. Let $\widetilde{w}_2$ (resp. $\widehat{w_2}(X)$, resp. $\bar{w}_2(X)$) be the first integer x such that every spanned (resp. very ample, resp. very ample and projectively normal) $L \in \mathrm{Pic}^x(X)$ is nice to all $P \in X$. Let $w_3(X)$ (resp. $\widehat{w_3}(X)$, resp. $\bar{w}_3(X)$) be the first integer y such that for all integers $x \ge y$ every spanned (resp. very ample, resp. very ample and projectively normal) $L \in \mathrm{Pic}^x(X)$ is nice to all $P \in X$. Notice that every $L \in \mathrm{Pic}^x(X)$ with $x \ge 2g$ (resp. $x \ge 2g+1$) is spanned (resp. very ample). Fix any $P \in X$. Notice that $\omega_X(P)$ is not spanned at P and that $\omega_X(2P)$ is spanned, but not very ample: if either X is not hyperelliptic or P is not a Weierstrass point of X, then the morphism induced by the linear system $|\omega_X(2P)|$ is separable, injective, étale outside P, and its image in $\mathbf{P}^g$ is a curve of arithmetic genus $g+1$ with X as its normalization and with an ordinary cusp as its unique singular point. For any $L \in \mathrm{Pic}(X)$ such that $h^0(X,L) > 0$, let B_L denote the scheme theoretical base locus of the linear system $|L|$, i.e. the effective divisor (or the $\emptyset$) such that $\mathcal{I}_{B_L} \otimes L$ is the image of the evaluation map $H^0(X,L) \otimes \mathcal{O}_X \to L$. Notice that P is not L-good if $P \in \mathrm{Supp}(B_L)$. Hence $w_2(X) \ge 2g-1$. Fix any integer $m \ge 1$. We will say that P is $(R;m)$-good or that R is nice to mP if there is $D \in |L|$ and a divisor D' such that $\mathcal{O}_X(mP) \cong \mathcal{O}_X(D')$, $\mathrm{Supp}(D') \subseteq \mathrm{Supp}(D)$ and $P \notin \mathrm{Supp}(D)$. Hence P is R-good if and only if it is $(R;1)$-good. Notice that if P is $(R;m)$-good, then it is $(R;km)$-good for all integers $k \ge 1$. Fix any linear subspace $V \subseteq H^0(X,R)$ such that $V \ne \{0\}$. A point $P \in X$ is said to be (R,V)-good and R is said to be nice to P if there is $D \in |V|$ and a divisor D' such that $\mathcal{O}_X(P) \cong \mathcal{O}_X(D')$, $\mathrm{Supp}(D') \subseteq \mathrm{Supp}(D)$ and $P \notin \mathrm{Supp}(D)$. Similarly, we define the notion of $(R,V;m)$-goodness and the niceness of (R,V) to mP. Obviuosly, if P is $(R,V;m)$-good, then P is not a base point of $|V|$. Hence, even if $\deg(R) \gg g$ for any $P \in X$ there is a codimension one linear subspace of $H^0(X,L)$ such that P is not $(R,V;m)$-good for any $m > 0$: take as V the inclusion of $H^0(X,R(-P))$ in $H^0(X,R)$ induced by the multiplication by a local equation of P.

To state our results we introduce the following notations.

Notation 1. Fix X, P, R and an integer $n > 0$. Set $G(L,n) := \{V \subseteq H^0(X,R) : \dim(V) = n\}$, $U(R,n,P) := \{V \in G(R,n) : P$ is (R,V)-good $\}$, $B'(R,n,P) := \{V \in B(R,n,P) : P$ is not a base point of $|V|\}$, $B''(R,n,P) := B(R,n,P)\backslash B'(R,n,P)$, $U(R,n,P;m) := \{V \in G(R,n) : P$ is $(R,V;m)$-good $\}$, $B(R,n,P) := G(L,n)\backslash U(R,n,P)$, $B(R,n,P;m) := G(L,n)\backslash U(R,n,P;m)$, $B'(R,n,P;m) := \{V \in B(R,n,P;m) : P$ is not a base point of $|V|\}$, and $B''(R,n,P;m) := B(R,n,P;m)\backslash B'(R,n,P;m)$.

Remark 1. Obviuosly, $G(R,n) = \emptyset$ if $n > h^0(X,L)$, while $\dim(G(R, n)) = n \cdot (h^0(X,R) - n)$ if $0 < n \leq h^0(X,R)$. If P is a base point of $|R|$, then $B(R,n,P,m) = G(R,n)$. If P is not a base point of $|R|$, then the $(h^0(X,R)-1)$-dimensional projective space $G(R,h^0(X,R)-1)$ contains a hyperplane contained in $B(R,h^0(X,R)-1,P,m)$ for all $m > 0$: the image of $H^0(X,R(-P))$ by the linear map induced by the inclusion $L(-P) \to L$ obtained taking the multiplication by the local equation of P.

In section 2 we will prove the following results and state and prove a few other results.

Theorem 1. *Let X be a smooth genus g hyperelliptic curve, $P \in X$ and $R \in \mathrm{Pic}^{g+3}(X)$. Assume R very ample and that P is not a Weierstrass point of X. Then P is R-good.*

Theorem 2. *Fix integers g, d, n such that $g \geq 2$ and $d \geq 3g+1$. Assume either $\mathrm{char}(\mathbb{K}) = 0$ or $\mathrm{char}(\mathbb{K}) > 2g-2$. Let X be a smooth curve of genus g, $P \in X$, and $R \in \mathrm{Pic}^d(X)$. If $n = d+1-g$, then $B(R,n,P) = \emptyset$. If $g+1 \leq n \leq d-2g$, then $B(R,n,P))$ has codimension at least $n(g+2)$ in $U(R,n,P)$. If $g+1 \leq n \leq d-2g$, then every $P \in X$ is (R,V)-good for a general $V \in G(n,R)$.*

Theorem 3. *Let X be a smooth genus $g \geq 3$ curve and $P \in X$. Let c_P be the minimal integer t such that there is a divisor A on X such that $P \notin \mathrm{Supp}(A)$ and $\mathcal{O}_X(P) \cong \mathcal{O}_X(A)$. Fix any such divisor A and set $B := \mathrm{Supp}(A)$ Fix integers d, n such that $d \geq 2g + c_P - 1$ and either $n = d+1-g$ or $c_P + 1 \leq n \leq d + 2 - c_P - g$. Fix $R \in \mathrm{Pic}^d(X)$. If $d = 2g + c_P - 1$, assume $R \ncong \omega_X(P+B)$. Then $U(n,R;P) \neq \emptyset$.*

Theorem 4. *Let X be a smooth genus $g \geq 3$ curve. Let c be the minimal integer t such that for every $P \in X$ there is a divisor A on X such that $P \notin \mathrm{Supp}(A)$ and $\mathcal{O}_X(P) \cong \mathcal{O}_X(A)$. Fix a integers d, n such that $d \geq 2g + c$ and either $n = d+1-g$ or $c+1 \leq n \leq d+2-c-g$. Fix $R \in \mathrm{Pic}^d(X)$. Then $U(n,R) \neq \emptyset$.*

2. Proofs and Further Results

Remark 2. Fix X, P, R, V. The linear space V induces a 2-dimensional cone C whose vertex O is normal if and only if $V = H^0(X,R)$ and R is projectively normal. The point P corresponds to a line L_P of C through P. If P is (R,V)-good, then any $Q \in L_P \backslash \{O\}$ is a complete intersection (hint: copy the proof of [3], Lemma 3).

Remark 3. Fix X, P, R, V. The linear space V induces a 2-dimensional cone C whose vertex O is normal if and only if $V = H^0(X, R)$ and R is projectively normal. The point P corresponds to a line L_P of C through P. Assume that P is $(R, V; m)$-good. As in [3], Lemma 3, we see that any $Q \in L_P \backslash \{O\}$ is a set-theoretically a complete intersection and there are two hypersurfaces of C, one of them an m-power of an affine hyperplane whose intersection is a curvilinear length m zero-dimensional subscheme of C with Q as its support.

Theorem 5. *Let X be a hyperelliptic curve, $P \in X$, $R \in \mathrm{Pic}(X)$ and a linear subspace $V \subseteq H^0(X, R)$. Assume that P is not a Weierstrass point of X and call Q, the only point of X such that $|P + Q|$ is the g^1_2 of X. Assume that P is not a base point of $|V|$ and the existence of a Weierstrass point A of X not in the base locus of $|V(-P-Q)|$. Then P is (R, V)-good.*

Proof. We have $\mathcal{O}_X(P) \cong \mathcal{O}_X(2A - Q)$. By assumption there is $D \in |V|$ such that $A \in \mathrm{Supp}(D)$, $Q \in \mathrm{Supp}(D)$ and $P \notin \mathrm{Supp}(D)$. □

Corollary 1. *Let X be a hyperelliptic curve, $P \in X$, $R \in \mathrm{Pic}(X)$ and a linear subspace $V \subseteq H^0(X, R)$. Assume that P is not a Weierstrass point of X, that V is very ample and that*$\mathrm{length}(E \cap \phi_V(X)) \leq 2$ *for every line $E \subset \mathbf{P}(V^*)$ containing $\phi_V(P)$, where $\phi_V : X \to \mathbf{P}(V^*)$ denote the embedding associated to $|V|$. Then P is (R, V)-good.*

Theorem 6. *Let X be a hyperelliptic curve, $P \in X$, $R \in \mathrm{Pic}(X)$ and a linear subspace $V \subseteq H^0(X, R)$. Assume that P is a Weierstrass point of X, that P is not a base point of $|V|$ and that there is at least one Weierstrass point A of X such that P is not in the base locus $|V(-A)|$. Then P is $(R, V; 2)$-good.*

Proof. We have $\mathcal{O}_X(2P) \cong \mathcal{O}_X(2A)$. By assumption there is $D \in |V|$ such that $A \in \mathrm{Supp}(D)$ and $P \notin \mathrm{Supp}(D)$. □

Remark 4. Let k_P be the first integer t such that $h^0(X, \mathcal{O}_X(tP)) \geq 2$. Notice that $1 \leq k_P \leq g + 1$, that $h^0(X, \mathcal{O}_X(k_P P)) = 2$ and that $|\mathcal{O}_X(k_P P)|$ has no base point.

Proposition 1. *Fix X, P and integers d, n such that $d \geq k_P$ and $2 \leq n \leq \max\{2, d+1-g\}$. Then there exists $R \in \mathrm{Pic}^d(X)$ such that $U(R, n, P; k_P) \neq \emptyset$.*

Proof. Take as $R := \mathcal{O}_X(k_P P + A)$, where A is an effective degree $d - k_P$ divisor whose support does not contain P. □

Proposition 2. *Let X be a hyperelliptic curve, $P \in X$ and $R \in \mathrm{Pic}(X)$. Assume $h^1(X,R) > 0$, $h^0(X,R) \geq 3$, P not a Weierstrass point of X and P not a base point of $|R|$. Then P is R-good.*

Proof. Let $M \in \mathrm{Pic}^2(X)$ denote the hyperelliptic line bundle. By assumption there is an effective divisor A such that $P \notin \mathrm{Supp}(A)$ and $|M^{\otimes 2}|+A \subseteq |R|$. Fix a Weierstrass point Q of X. Let $P' \in X$ be the only point of X such that $|M| = |P+P'|$. By assumption $P \neq P'$ and $P \neq Q$. Use that $\mathcal{O}_X(P) \cong \mathcal{O}_X(2Q-P')$. □

Proof of Theorem 1. Let $M \in \mathrm{Pic}^2(X)$ denote the hyperelliptic line bundle. Since X is hyperelliptic and R is very ample, we have $h^1(X,R) = 0$. Since $h^1(X,R) = 0$ and $\deg(R) = g+3$, we have $h^0(X,R) = 4$. Since R is very ample, it is well-knonw and easy to check that there is a unique $L \in \mathrm{Pic}^{g+1}(X)$ such that $h^1(X,L) = 0$ (i.e. $h^0(X,L) = 2$), $R \cong M \otimes L$, and L is spanned. Fix a Weierstrass point Q of X. Let $P' \in X$ be the only point of X such that $|M| = |P+P'|$. By assumption $P \neq P'$ and $P \neq Q$. Since $h^0(X,L) = 2$, $\deg(L) > 2$ and L is spanned, we have $h^0(X, L \otimes M^*) = 0$. Thus P is not contained in the support of the only effective divisor of $|L(-P')|$. Use that $\mathcal{O}_X(P) \cong \mathcal{O}_X(2Q-P')$. □

Remark 5. Take g, X, R as in the statement either of Proposition 2 or of Theorem 1, but assume that P is a Weierstrass point of X. The same proofs show that P is $(R;2)$-good.

Remark 6. Let X be a smooth genus $g \geq 3$ hyperelliptic curve, $P \in X$ and $R \in \mathrm{Pic}^d(X)$. Assume R very ample and that P is not a Weierstrass point of X. Since X is hyperelliptic and R is very ample, we have $h^1(X,R) = h^1(X, R\otimes M^*) = 0$. Let $M \in \mathrm{Pic}^2(X)$ denote the hyperelliptic line bundle and $P' \in X$ the only point such that $\mathcal{O}_X(P+P') \cong M$. $P \neq P'$ because P is not a Weierstrass point. As in the proof of Theorem 1 we see that P is R-good if P is not a base point of $|R\otimes M^*|$. Since $h^1(X, R\otimes M^*) = 0$, P is a base point of $|R\otimes M^*|$ if and only $h^1(X, R\otimes M^*(-P)) \neq 0$, i.e. if and only there is an integer t such that $1 \leq t \leq g$ and an effective divisor A such that $R \cong M^{\otimes t}(P+A)$, where $\deg(A) \leq g-1-2t$. Thus the set $\widetilde{B}(d,P)$ of all very ample $R \in \mathrm{Pic}^d(X)$ such that P is not R-good has dimension at most $g-3$. Let $V(d,X)$ denote the set of all non-special very ample line bundles on X. Notice that $V(d,X) = \emptyset$ if $d \leq g+2$, while $V(d,X)$ is a non-empty g-dimensional integral variety for all $d \geq g+3$. In summary, we just proved the following result. If Q is a Weierstrass point of X, then we get the same result for $(R;2)$-goodness (Remark 5).

Theorem 7. *Let X be a smooth genus $g \geq 3$ hyperelliptic curve. Let $V(d, X)$ denote the set of all non-special very ample line bundles on X. For all integers $d \geq g + 3$ there is a non-empty open subset $U(d, X)$ of the integral g-dimensional variety $V(d, X)$ such that $V(d, X)\backslash U(d, X)$ has codimension at least two in $U(d, X)$ such that every non-Weierstrass point of X is R-good and every Weierstrass point of X is $(R; 2)$-good for all $R \in U(d, X)$.*

Proof of Theorem 2. Our assumption on char($\mathbb{K}$) implies that $h^0(X, \mathcal{O}_X(gQ)) = 1$ (i.e. $h^1(X, \mathcal{O}_X(gQ)) = 0$) for a general $Q \in X$ ([1], [2]). Thus $h^0(X, \mathcal{O}_X(gQ+P)) = 2$ and $|\mathcal{O}_X(gQ+P)|$ has no base point. Hence $|\mathcal{O}_X(gQ+P)|$ induces a degree $g+1$ morphism $X \to \mathbf{P}^1$ with $gQ+P$ as one of its scheme-theoretic fibers. Our assumption on char($\mathbb{K}$) implies that the classical Riemann-Hurwitz formula applied to h gives the existence of at least another unreduced fiber of h, i.e. there is an effective degree $g+1$ divisor A such that $\mathcal{O}_X(gQ+P) \cong \mathcal{O}_X(A)$, $P \notin \mathrm{Supp}(A)$ and $\sharp(\mathcal{O}_X(A)) \leq g$. Set $B := \{Q\} \cup \mathrm{Supp}(A)$. Since $d \geq 2g-1+\sharp(B)$, we have $h^0(X, R(-B)) = h^0(X, R) - \sharp(B)$ and $h^1(X, R(-B)) = 0$. Since $d \geq 3g+1$, then $h^0(X, R(-B-P)) = h^0(X, R(-B))-1$, i.e. $h^1(X, R(-B-P)) = 0$. The set Σ of all $E \in |R|$ such that $B \subseteq \mathrm{Supp}(E)$ is parametrized by the projective space $|R(-B)|$. With suitable identifications the subset Σ' of all $E \in \Sigma$ such that $P \in \mathrm{Supp}(E)$ is parametrized by the hyperplane $|R(-B-P)|$ of $|R(-B)|$. Since $n \geq g+2$, the projective space associated to every $V \in G(n, R)$ intersects Σ. Take a general $V \in R(n, R)$. By the generality of V we have $|V| \cap \Sigma \nsubseteq \Sigma'$. Hence P is (R, V)-good. Furthermore, P is R-good. Now assume $g+2 \leq n \leq d-2g$. In this case $B(n, R, P) \subseteq \Phi := \{V \in G(n, R) : |V| \cap \Sigma \subseteq \Sigma'$. Since $n \leq d-2g$, then $\Phi = G(n, R(-B-P))$ and hence Φ has codimension $n(g+2)$ in $G(n, R)$. The last assertion follows from the first one and the one-dimensionality of X. □

Proofs of Theorems 3 and 4. Use the proof of Theorem 2. Indeed, in the first part of the proof of Theorem 2 we just checked that $c_P \leq g+1$ and $c \leq g+1$ under our assumptions on char($\mathbb{K}$). We also need $h^1(X, R(-B-P)) = 0$. This is true if either $d \geq 2g + c_P$ or $d = 2g + c_P - 1$ and $R \ncong \omega_X(B+P)$. □

Remark 7. Let X be a smooth genus g curve. For all $d \geq 2g+1$, all $L \in \mathrm{Pic}^x(X)$ and all $P \in X$, L is very ample and $h^0(X, L(-2P)) = h^0(X, L(-P)) - 1 = h^0(X, L) - 2$. We easily get $w_3(X) = w_2(X)$ if $w_2(X) \geq 2g+1$.

Proposition 3. *Fix $P, Q \in X$ and $L \in \mathrm{Pic}(X)$ such that $P \neq Q$ and P is L-good. Then P is $L(Q)$-good.*

Proof. Fix $D \in |L|$ and D' such that $P \notin \mathrm{Supp}(D)$, $\mathrm{Supp}(D') \subseteq \mathrm{Supp}(D)$ and $\mathcal{O}_X(P) \cong \mathcal{O}_X(D')$. Since $P \neq Q$, $P \notin \mathrm{Supp}(D+Q)$. Hence the pair

$(D+Q, D')$ shows that P is $L(Q)$-good. ☐

Proposition 4. *Fix $P \in X$ and $L \in \mathrm{Pic}(X)$.*

(i) If $h^0(X, L) > h^0(X, L(-P)) > h^0(X, L(-2P))$, then P is $L^{\otimes 2}(-P)$-good.

(ii) If $h^0(X, L(P)) > h^0(X, L) > h^0(X, L(-P))$, then P is $L^{\otimes 2}(P)$-good.

Proof. Assume $h^0(X, L) > h^0(X, L(-P)) > h^0(X, L(-2P))$. By assumption there are $A \in |L(-P)|$ such that $P \notin \mathrm{Supp}(A)$ and $B \in |L|$ such that $P \notin \mathrm{Supp}(B)$. Use that $\mathcal{O}_X(P) \cong \mathcal{O}_X(B-A)$, $A+B \in |L^{\otimes 2}(-P)|$ and $P \notin \mathrm{Supp}(A+B)$. To check part (ii) apply part (i) to the line bundle $\widetilde{L} := L(P)$. ☐

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