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NEAR FRATTINI SUBGROUPS OF CERTAIN GENERALIZED FREE PRODUCTS OF GROUPS

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Abstract: Let $G = A *_H B$ be the generalized free product of the groups A and B with the amalgamated subgroup H. Also, let $\lambda(G)$ and $\psi(G)$ represent the lower near Frattini subgroup of G and the near Frattini subgroup of G respectively. We show that G is $\psi-$free provided: (i) G is any ordinary free product of groups; (ii) $G = A *_H B$ and there exists an element c in $G\backslash H$ such that$H^c \cap H = 1$; (iii) $G = A *_H B$ and $\lambda(G) \cap H = \mu(G) \cap H = 1$; (iv) $G = A *_H B$, where A and B are finitely generated and $\lambda-$free, and $H = C(\infty)$; (v) $G = A *_H B$, and $H \neq 1$ is malnormal in at least one of A or B; (vi) G is a surface group; (vii) G is the group of an unknotted circle in $\mathbb{R}^3$; $(viii)$ G is a group of $F-$type with only odd torsion where neither U nor V is a proper power; (ix) G is a non-elementary planar discontinuous group with only odd torsion. Furthermore, we show that if $G = A *_H B$, then: (i) $\lambda(G) \leq H$, provided both A and B are nilpotent; (ii) $\psi(G) \leq H$, provided both A and B are finitely generated and nilpotent.

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1. Introduction

G. Higman and B.H. Neumann have shown that the Frattini subgroup of a free product of (nontrivial) groups is the trivial group [24, Theorem 2, p. 87]. This was a positive response to a question raised by N. Itô regarding the existence of maximal subgroups in ordinary free products of groups. G. Higman and B.H. Neumann asked whether a generalized free product of groups necessarily has maximal subgroups. They asked whether or not the Frattini subgroup of a generalized free product of groups is contained in the amalgamated subgroup. In a series of papers mainly by R.B.J.T. Allenby and C.Y. Tang [4, 5, 6, 7], R.B.J. T. Allenby, C.Y. Tang, S.Y. Tang [8], D.Z. Djokovic and C.Y. Tang [19], C.Y. Tang [29, 30], and A. Whittemore [31], the questions raised by Higman and Neumann have been answered for some certain classes of generalized free products of groups. Motivated by the above papers, the author in [9, 10, 11, 12, 13, 14, 15, 16, 17] produced similar results for the (lower) near Frattini subgroups of such generalized free product of groups. Most of the results produced by the author are achieved in dealing with non-near generators of generalized free products of groups, and hence most of the results are obtained for the lower near Frattini subgroups of such groups. However, R.B.J.T. Allenby [1, 2, 3] has been able to deal successfully with nearly maximal subgroups of generalized free products of groups to produce results for the upper near Frattini subgroups.

In this paper, with the exception of the first part of Theorem 3.9, we have obtained results for the near Frattini subgroups. In fact, aside from Theorem 3.9, we have shown that G is $\psi-$free. In Section 3 we show that G is $\psi-$free provided: (i) G is any ordinary free product of groups; (ii) $G = A *_H B$ and there exists an element c in $G \backslash H$ such that$H^c \cap H = 1$; (iii) $G = A *_H B$ and $\lambda(G) \cap H = \mu(G) \cap H = 1$; (iv) $G = A *_H B$, where A and B are finitely generated and $\lambda-$free, and $H = C(\infty)$; (v) $G = A *_H B$, and $H \neq 1$ is malnormal in at least one of A or B; (vi) G is a surface group; (vii) G is the group of an unknotted circle in $\mathbb{R}^3$; $(viii)$ G is a group of $F-$type with only odd torsion where neither U nor V is a proper power; (ix) G is a non-elementary planar discontinuous group with only odd torsion. Furthermore, we show that if $G = A *_H B$, then: (i) $\lambda(G) \leq H$, provided both A and B are nilpotent; (ii) $\psi(G) \leq H$, provided both A and B are finitely generated and nilpotent.

2. Definitions and Notation

We use standard notation throughout the paper. We use $C(\infty)$ to represent an infinite cyclic group, and by $A \times B$ we mean the direct product of the groups A and B. We use $G = A *_H B$ as in B.H. Neumann's paper [27] to represent the generalized free product of A and B with the amalgamated subgroup H. As before the *core* of the subgroup H in G is represented by $K(G, H)$, and a subgroup H of a group G is *malnormal* if $g^{-1}Hg \cap H = 1$, for every $g \in G \backslash H$. Recall from Fine and Rosenberger (see [20] or [21]) that a group *of* $F-$*type* is a group G with a presentation of the form $G =< a_1, ..., a_n : a_i^{n_i} = 1$, $i = 1, ..., n$, $U(a_1, ..., a_k)V(a_{k+1}, ..., a_n) = 1 >$, where $1 \leq k < n$, $n_i = 0$, or $n_i = 2$ for all i and U, V are cyclically reduced words in the free products on $\{a_1, ..., a_k\}$, and $\{a_{k+1}, ..., a_n\}$ respectively which are of infinite order. Groups of $F-$type are a natural algebraic generalization of Fuchsian groups [20, p. 2175]. For definitions of *surface groups, non-elementary planar discontinuous group with only odd torsion,* and *the group of an unknotted circle in* $\mathbb{R}^3$, see [20, 23, 26].

An element g of a group G is a *near generator* of G if there exists a subset S of G such that $|G :< S > | = \infty$, but $|G :< g, S > | < \infty$. Hence, an element g of G is a *non-near generator* of G if for every subset S of G for which $|G :< g, S > | < \infty$ it follows that $|G :< S > | < \infty$. In any group the identity element is a non-near generator and if G is finite, then every element of G is a non-near generator. Also, Breaz and Calugareanu [18, Lemma 5.1, p. 403] have shown that if G is Abelian and g is an element of infinite order in G, then g is a non-near generator if and only if for every subgroup H of G for which $< g > \cap H = 1$ it follows that $|G :< g, H > | < \infty$. A subgroup M of a group G is *nearly maximal* in G if it is maximal with respect to being of infinite index in G. That is, M is nearly maximal in G if $|G : M| = \infty$, but $|G : N| < \infty$, whenever $M < N \leq G$. If G is Abelian, then M is nearly maximal in G if and only if $G/H \simeq C(\infty)$, yet another characterization of nearly maximal subgroups given by Breaz and Calugareanu [18, p. 397]. The intersection of all nearly maximal subgroups forms a characteristic subgroup called the *upper near Frattini subgroup* of G, denoted by $\mu(G)$ (if there are no nearly maximal subgroups, then $\mu(G) = G$). The set of all non-near generators of G also forms a characteristic subgroup of G called the lower near Frattini subgroup of G, denoted by $\lambda(G)$. In general, $\lambda(G) \leq \mu(G)$. If $\lambda(G) = \mu(G)$, then their common value is called the near Frattini subgroup of G, denoted by $\psi(G)$. We know that $\psi(G)$ is a characteristic subgroup, but if G is Abelian, then $\psi(G)$ is fully invariant [18, Corollary 2.4, p. 398]. If G is any Abelian group, then $\psi(G)$ exists [18, Theorem 2.1, p. 396] and any Abelian group is the near Frattini subgroup of a suitably chosen Abelian group [18, Corollary 5.1, p. 396]. We say G is $\psi-$free ($\lambda-$free) if the (lower)near Frattini subgroup is the trivial subgroup. Definitions and terminologies involving the near Frattini subgroup are due to J.B. Riles [28].

Remark 2.1. As we indicated above, one must keep in mind that $\lambda(G)$ and $\mu(G)$ may not coincide and as a result $\psi(G)$ for an arbitrary group G may not exists. In some papers, some confusion may have risen where the authors were not careful in dealing with the near Frattini subgroup of a group. For example, J.C. Lennox and D.J.S. Robinson [25, p. 290], and S. Franciosi and F. de Giovanni [22, p. 19], defined the near Frattini subgroup as the intersection of all nearly maximal subgroups with no mention of the upper near Frattini subgroup. The author has not verified whether or not the confusion in the definition of the near Frattini subgroup affected any of the results in the above articles.

3. Results

In order to make this paper self contained we will restate known results from previous work in here. In [9, Theorem A, p. 524] we have shown that the lower near Frattini subgroup of a free product of groups is trivial. Also, Allenby [1, Theorem 1, p. 400] has shown that the upper near Frattini subgroup of a free product of groups is trivial as well. Combining these two theorems we obtain the following fundamental theorem regarding the near Frattini subgroups of ordinary free products of groups.

Theorem 3.1. *If G is any ordinary free product of groups, then G is $\psi-$free.*

The above result may not hold for generalized free product of groups. If A and B are arbitrary groups and $G = A *_H B$, then $\psi(G)$ may not exists. The existence of $\psi(G)$ for an arbitrary generalized free product of groups has been open for over a decade until R.B.J.T. Allenby [2, Example, p. 467] provided an example where $\lambda(G) \neq \mu(G)$. J.B. Riles [28, Example 1, p. 162] has shown that if $G = \mathbb{Z}(p^\infty)wrC(q)$ is the wreath product of a group of type p^∞ by a cyclic group of order q, then $\lambda(G) \neq \mu(G)$. Allenby used Riles' idea to construct a generalized free product of groups where $\lambda(G) \neq \mu(G)$. Allenby has shown that if $A = C(2) \times (\mathbb{Z}(2^\infty)wrC(2))$ and B is isomorphic to A, then $\lambda(A *_H B) \neq \mu(A *_H B)$, where $H = \mathbb{Z}(2^\infty)wr(C(2))$.

Also, we recall that another useful result in the study of the lower near Frattini subgroups of generalized free product of groups has been [9, Proposition

4, p. 526] where we have shown that if $G = A *_H B$ and if there exists an element c in G such that $H^c \cap H = 1$, then G is $\lambda-$free. Similarly, Allenby [3, Corollary (b) of Theorem 1] obtained the exact analogue of this theorem for $\mu(G)$. We combine these two theorems to state the following useful theorem for the near Frattini subgroup of generalized free product of groups.

Theorem 3.2. *Let $G = A *_H B$, where A and B are any arbitrary groups. If there exists an element c in $G \backslash H$ such that $H^c \cap H = 1$, then G is $\psi-$free.*

Theorem 3.3. *Let $G = A *_H B$, where A and B are any arbitrary groups. If $\mu(G) \cap H = 1$, then G is $\psi-$free.*

Proof follows from [3, Corollary (a) of Theorem 1], and the fact that if $\mu(G) = 1$, then $\lambda(G) = 1$, and therefore G is $\psi-$free.

Theorem 3.4. *Let $G = A *_H B$, where A and B are any $\lambda-$free groups. If H is infinite cyclic, then G is $\psi-$free.*

Proof. By [2, Theorem 2, p. 465], $\lambda(G) = \mu(G) = 1$ or $\lambda(G) = \mu(G) = K(G, H)$. However, since $\lambda(A) = \lambda(B) = 1$, by [2, Theorem 3, p. 465], $\lambda(G) = \mu(G) = 1$. Therefore, G is $\psi-$free. □

Theorem 3.5. *Let $G = A *_H B$, where A and B are arbitrary groups. If H is non-trivial and malnormal in at least one of A or B, then G is $\psi-$free.*

Proof. Without loss of generality we may assume that H is malnormal in A. If H is malnormal in A, then for every $a \in A \backslash H$, $H^a \cap H = 1$. Therefore, by Theorem 3.2, G is $\psi-$free. □

Corollary 3.6. *Let $G = A *_H B$, where A and B are free groups. If H is non-trivial and malnormal in at least one of A or B, then G is $\psi-$free.*

Theorem 3.7. *The group G is $\psi-$free in each of the following cases:*

(i) *G is a surface group;*

(ii) *G is the group of an unknotted circle in $\mathbb{R}^3$.*

Proof. (i) According to Fine and Rosenberger [20, p. 2175] if A and B are free groups and $H =< h >$ where h is an element which is not a proper power, then H is malnormal in both A and B. Thus, by Corollary 3.6, $\psi(G) = \psi(A *_H B) = 1$. As they have stated, this is exactly the structure of a surface group. For proof of (ii), we use the fact that if G is the group of an unknotted circle in $\mathbb{R}^3$, then G is an infinite cyclic group [26, Proposition 6.1, p. 139]. Therefore, again $\psi(G) = \psi(C(\infty)) = 1$, for the only non-near generator of $C(\infty)$ is the identity element and the only nearly maximal subgroup of $C(\infty)$ is the identity subgroup. □

Theorem 3.8. *The group G is $\psi-$free in each of the following cases:*

(i) *G is a group of $F-$type with only odd torsion where neither U nor V is a proper power;*

(ii) *G is a non-elementary planar discontinuous group with only odd torsion.*

Proof. (i) A group of $F-$type has the structure of a free product of two free products of cyclic groups amalgamated over an infinite cyclic subgroup. And if neither U nor V is a proper power then the amalgamated subgroup is malnormal in both factors (Fine and Rosenberger [20, p. 2175]). Therefore, by Theorem 3.5, $\psi(G) = 1$. Similarly, proof of (ii) follows from the fact that a non-elementary planar discontinuous group is a group of $F-$type with the same structure as the group G as in part (i) ([20, p. 2176]), and hence again $\psi(G) = 1$. □

Theorem 3.9. *Let $G = A *_H B$. If both A and B are nilpotent, then $\lambda(G) \le H$. Moreover, if both A and B are finitely generated and nilpotent, then $\psi(G) \le H$.*

Proof. The first part of the proof is parallel to the proof of the Frattini version of this theorem by R.B.J.T. Allenby and C.Y. Tang [7, Theorem 5.1, p. 467]. So, we may assume that $\lambda(G) \not\le H$, and thus there exists $a \in A\backslash H$ such that $a \in \lambda(G)$. Now, since A is nilpotent $H \lneqq H(A \cap \lambda(G))$ and hence $H \lneqq N_{H(A\cap\lambda(G))}(H)$. But, every element of $H(A \cap \lambda(G))$ is of the form ha_1 where $h \in H$ and $a_1 \in A\cap\lambda(G)$. Hence, there exist $h \in H$, $a_1 \in A\cap\lambda(G)$ such that ha_1 normalizes H and $ha_1 \notin H$. That is, $a_1 \notin H$, and a_1 normalizes H. Next, if $b \in N_B(H)\backslash H$, then $z = b^{-1}a_1^{-1}ba_1$ normalizes H and $z \in \lambda(G)$. Now, we show that this is not possible, by showing that z is a near generator of G and hence $z \notin \lambda(G)$. That is, we conclude that the assumption that $\lambda(G) \not\le H$ reaches a contradiction.

To prove that z is a near generator we need to show that there exists a subset S of G such that $|G :< S > | = \infty$, but $|G :< S, z > | < \infty$. If we set $< S >=< A^z, B >$, then clearly, $|G :< S, z > | < \infty$. To show that $|G :< S > | = \infty$, we show that

$$z < S >, z^2 < S >, z^3 < S >, ..., z^n < S >, ...$$

(n a natural number) are infinitely many distinct cosets of S in G. That is, we need to show that if m and k are distinct natural numbers, then $z^m(z^k)^{-1} = z^{m-k} \notin< S >$. Now, if $m - k = 1$, then by the argument given by Allenby and Tang [7, p. 467], $z \notin< S >$. Also, when $m - k > 1$ or $m - k < 1$, it can again be shown by a similar argument that $z^{m-k} \notin< S >$. Therefore, we have established that $\lambda(G) \le H$. Finally, if both A and B are finitely generated and nilpotent, then $\lambda(G) = \mu(G) = \psi(G)$, and hence $\psi(G) \le H$. This completes the proof of this theorem. □

From our experience in dealing with the lower near Frattini subgroups as well as the upper near Frattini subgroups of generalized free products of groups, we suspect that the result of Theorem 3.9 is valid for $\mu(G)$. Therefore, we state the following conjecture.

Conjecture 3.10. *Let $G = A *_H B$. If both A and B are nilpotent, then $\mu(G) \leq H$.*

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