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TOTAL SIGNED DOMINATION NUMBERS OF GRAPHS

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Abstract: Let G be a finite connected simple graph with vertex set $V(G)$ and edge set $E(G)$. A total signed domination function of G is a function $f : V(G) \cup E(G) \to \{-1, 1\}$. The weight of f is $w(f) = \sum_{x \in V(G) \cup E(G)} f(x)$. For an element $x \in V(G) \cup E(G)$, we define $f[x] = \sum_{y \in N_T[x]} f(y)$. A total signed domination function of G is a function $f : V(G) \cup E(G) \to \{-1, 1\}$ such that $f[x] \geq 1$ for all $x \in V(G) \cup E(G)$. The total signed domination number $\gamma_s^*(G)$ of G is the minimum weight of a total signed domination function on G.

In this paper, we obtained some lower bounds for the total signed domination number of a graph G and computed exact values of $\gamma_s^*(G)$ when G are C_n and P_n.

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1. Introduction

Let G be a finite connected simple graph with vertex set $V(G)$ and edge set $E(G)$. The neighborhood of v, denoted $N(v)$, is set $\{u | uv \in E(G)\}$ and the closed neighborhood of v, denoted $N[v]$, is set $N(v) \cup \{v\}$. A function $f : V(G) \to \{-1, 1\}$ is a signed domination function if for every vertex $v \in V(G), f[v] = f(N[v]) = \sum_{u \in N[v]} f(u) \geq 1$. The weight $w(f)$ of f is the sum of the function value of all vertices in G. The signed domination number $\gamma_s(G)$ of

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G is the minimum weight of a signed domination function on G. In [2] Dunbar et al introduced this concept and it has been studied several authors [2, 3, 6, 7].

In this paper, for an element $x \in V(G) \cup E(G)$, the closed neighborhood of x, denoted $N_T[x]$, $N_T[x] = \{y|\ y$ is adjacent to x or y is incident with x, $y \in V(G) \cup E(G)\} \cup \{x\}$. Let $\overline{G}$ be the complement of the graph G. First we give the definition of the total signed domination number of a graph as follows.

Definition 1. Let G be a graph and let $f : V(G) \cup E(G) \to \{-1, 1\}$ be a function. The weight of f is $w(f) = \sum_{x \in V(G) \cup E(G)} f(x)$. For an element $x \in V(G) \cup E(G)$, we define $f[x] = \sum_{y \in N_T[x]} f(y)$. A total signed domination function of G is a function $f : V(G) \cup E(G) \to \{-1, 1\}$ such that $f[x] \geq 1$ for all $x \in V(G) \cup E(G)$. The total signed domination number $\gamma_s^*(G)$ of G is the minimum weight of a total signed domination function on G. A total signed domination function of weight $\gamma_s^*(G)$, we call a γ_s^*-function of G.

2. Lower Bounds of Total Signed Domination Numbers

Theorem 2. *For any graph G, we have*

$$\gamma_s^*(G) \geq \left\lceil \frac{\delta(G) - \Delta(G) + 1}{\delta(G) + \Delta(G) + 1}(|E(G)| + |V(G)|) \right\rceil_{\mathcal{P}(|E(G)|+|V(G)|)}$$

and this bound is sharp.

Proof. Let f be a γ_s^*-function of G and let

$$V_1 = \{v \in V(G) | f(v) = 1\}, \quad V_2 = V(G) \backslash V_1,$$

$$E_1 = \{e \in E(G) | f(e) = 1\}, \quad E_2 = E(G) \backslash E_1.$$

Let $|V_1| = t$ and $|E_1| = s$. Then $|V_2| = |V(G)| - t$ and $|E_2| = |E(G)| - s$. Hence, we have

$$\gamma_s^*(G) = |V_1| + |E_1| - |V_2| - |E_2| = 2(t + s) - (|E(G)| + |V(G)|).$$

We observe that

$$\begin{aligned} \sum_{x \in V \cup E} f[x] &= \textstyle\sum_{v \in V} f[v] + \sum_{e = uv \in E} f[e] \\ &= \sum_{v \in V1} (2d(v) + 1) - \sum_{v \in V2} (2d(v) + 1) \\ &\quad + \sum_{e = uv \in E_1} (d(u) + d(v) + 1) - \sum_{e = uv \in E_2} (d(u) + d(v) + 1). \end{aligned}$$

For all $x \in V \cup E$, by Definition 1, $f[x] = \sum_{y \in N_T[x]} f(y) \geq 1$ and hence

$$\begin{aligned} |E(G)| + |V(G)| &\leq \sum_{x \in V \cup E} f[x] \\ &\leq t(2\Delta(G)+1) - (|V(G)| - t)(2\delta(G)+1) \\ &\quad + s(2\Delta(G)+1) - (|E(G)| - s)(2\delta(G)+1). \end{aligned}$$

From this inequality, we deduce that $t + s \geq \frac{\delta(G)+1}{\delta(G)+\Delta(G)+1}(|E(G)| + |V(G)|)$. Since $\gamma_s^*(G) = 2(t+s) - (|E(G)| + |V(G)|)$, we have the first statement of the theorem. In order to show that this bound is sharp, let us consider the cycle C_n. In Theorem 4, we will compute that $\gamma_s^*(C_n) = 2\lceil \frac{n}{5} \rceil$, which is coincide with the lower bound. This completes the proof. □

Theorem 3. *For any graph G, we have*

$$\gamma_s^*(G) \geq 2\left\lceil \frac{-1 + \sqrt{1 + 8(|V(G)| + |E(G)|)}}{2} \right\rceil - (|V(G)| + |E(G)|).$$

Proof. Let f be a γ_s^*-function of G and let

$$V_1 = \{v \in V(G) | f(v) = 1\}, \quad V_2 = V(G) \backslash V_1,$$

$$E_1 = \{e \in E(G) | f(e) = 1\}, \quad E_2 = E(G) \backslash E_1,$$

$$S_1 = V_1 \cup E_1, \quad S_2 = V_2 \cup E_2.$$

Let $|S_1| = t$. Then $|S_2| = |E(G)| + |V(G)| - t$ and

$$\gamma_s^*(G) = |S_1| - |S_2| = 2t - (|E(G)| + |V(G)|).$$

For convenience, let G^* be a graph whose vertex set is $S_1 \cup S_2$ and two vertices in G^* are adjacent if they are adjacent or incident in G. Let

$$E^*(S_1, S_2) = \{e = xy \in E(G^*) \,|\, x \in S_1,\ y \in S_2\}.$$

For each $y \in S_2$, by Definition 1, $|N_T[y] \cap S_1| = |N_{G^*}[y] \cap S_1| \geq 2$ and hence $|E^*(S_1, S_2)| \geq 2|S_2| = 2(|E(G)| + |V(G)| - t)$. So there exists at least one element $x \in S_1$ such that x is adjacent to at least $\lceil \frac{2(|E(G)|+|V(G)|-t)}{t} \rceil$ elements of S_2 in G^*. Hence

$$\begin{aligned} &|N_T[x] \cap S_1| \\ &\quad = |N_{G^*}[x] \cap S_1| \geq 1 + |N_{G^*}[x] \cap S_2| \geq 1 + \left\lceil \frac{2(|E(G)| + |V(G)| - t)}{t} \right\rceil, \end{aligned}$$

where the first inequality comes from the fact that f is a total domination function. Since $t = |S_1| \geq |N_T[x] \cap S_1|$, we have

$$t \geq \left\lceil \frac{2(|E(G)| + |V(G)| - t)}{t} \right\rceil + 1 \geq \frac{2(|E(G)| + |V(G)| - t)}{t} + 1.$$

From this inequality, we deduce that $t \geq \frac{-1+\sqrt{1+8(|E(G)|+|V(G)|)}}{2}$. Now, the theorem comes from the fact that $\gamma_s(G) = 2t - (|E(G)| + |V(G)|)$. □

3. Total Signed Domination Numbers of Some Classes of Graphs

Theorem 4. *For any cycle graph $C_n (n \geq 3)$, we have*

$$\gamma_s^*(C_n) = 2\left\lceil \frac{n}{5} \right\rceil.$$

Proof. By Theorem 2, we have $\gamma_s^*(C_n) \geq \lceil \frac{2n}{5} \rceil_e = 2\left\lceil \frac{n}{5} \right\rceil$. In order to prove that $\gamma_s^*(C_n) \leq 2\left\lceil \frac{n}{5} \right\rceil$, let $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(C_n) = \{e_1 = v_1v_2, e_2 = v_2v_3, \ldots, e_n = v_nv_1\}$. For all $x \in V \cup E$, then $|N_T[x]| = 5$. We construct a total signed domination function f of C_n as follows: if $n = 3, 4$, the proof is easy. Assume $n \geq 5$, we define $f : C_n \to \{-1, 1\}$ as follows:

If $n \equiv 0 \pmod 5$ or $n \equiv 3 \pmod 5$, then

$$\begin{cases} f(v_i) = -1 & \text{if } i \equiv 1 \text{ or } 2 \pmod{5}, \\ f(v_i) = 1 & \text{if } i \equiv 0, 3 \text{ or } 4 \pmod{5}, \\ f(e_i) = -1 & \text{if } i \equiv 3 \text{ or } 4 \pmod{5}, \\ f(e_i) = 1 & \text{if } i \equiv 0, 1 \text{ or } 2 \pmod{5}. \end{cases}$$

If $n \equiv 1 \pmod 5$ or $n \equiv 2 \pmod 5$, then

$$\begin{cases} f(v_i) = -1 & \text{if } i \equiv 1 \text{ or } 2 \pmod{5} \text{ and } i \neq n, \\ f(v_i) = 1 & \text{if } i \equiv 0, 3 \text{ or } 4 \pmod{5} \text{ or } i = n, \\ f(e_i) = -1 & \text{if } i \equiv 3 \text{ or } 4 \pmod{5}, \\ f(e_i) = 1 & \text{if } i \equiv 0, 1 \text{ or } 2 \pmod{5}. \end{cases}$$

If $n \equiv 4 \pmod 5$, then

$$\begin{cases} f(v_i) = -1 & \text{if } i \equiv 1 \text{ or } 2 \pmod{5}, \\ f(v_i) = 1 & \text{if } i \equiv 0, 3,\ 4 \pmod{5} \\ f(e_i) = -1 & \text{if } i \equiv 3 \text{ or } 4 \pmod{5} \text{ and } i \neq n, \\ f(e_i) = 1 & \text{if } i \equiv 0, 1 \text{ or } 2 \pmod{5} \text{ or } i = n. \end{cases}$$

It is not hard to show that, for each n, the function f is a total signed domination function of C_n and $\sum_{x\in V(C_n)\cup E(C_n)} f(x) = 2\left\lceil \frac{n}{5}\right\rceil$. For example, if $n \equiv 1 \pmod 5$, then f is a total domination function of C_n and

$$|\{v \mid f(v) = -1\}| = 2\left(\left\lceil\frac{n}{5}\right\rceil - 1\right) \quad \text{and} \quad |\{e \mid f(e) = -1\}| = 2\left(\left\lceil\frac{n}{5}\right\rceil - 1\right).$$

Hence

$$\sum_{x\in V\cup E} f(x) = 2n - 2\left(4\left\lceil\frac{n}{5}\right\rceil - 4\right) = 2n - 8\left\lceil\frac{n}{5}\right\rceil + 8 = 2\left\lceil\frac{n}{5}\right\rceil.$$

Since $\gamma_s^*(C_n) \le \sum_{x\in V(C_n)\cup E(C_n)} f(x) = 2\left\lceil\frac{n}{5}\right\rceil$, we have the theorem. □

Theorem 5. *For a path graph $P_n (n \ge 3)$, we have*

$$\gamma_s^*(P_n) = \begin{cases} \left\lceil\frac{2n-1}{5}\right\rceil + 1 & \text{if } n \equiv 0 \text{ or } 4 \pmod 5, \\ \left\lceil\frac{2n-1}{5}\right\rceil & \text{if } n \equiv 1 \text{ or } 3 \pmod 5, \\ \left\lceil\frac{2n-1}{5}\right\rceil + 2 & \text{if } n \equiv 2 \pmod 5. \end{cases}$$

Proof. When $n = 3$ or 4, it is easy to compute and the number is equal to ours. Now, we assume that $n \ge 5$. We use the same notations in the proof of Theorem 2, let $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ and let $E(P_n) = \{e_1, e_2, \ldots, e_{n-1}\}$. Let f be a total signed domination function of P_n. We notice that $|N_T[v_1]| = |N_T[v_n]| = 3$, $|N_T[e_1]| = N_T[e_{n-1}]| = 4$, and $|N_T[x]| = 5$ for each $x \in \{V \cup E\}\backslash\{v_1, v_n, e_1, e_{n-1}\}$. Since $f[e_1]$ and $f[e_{n-1}]$ are positive even numbers, $f[e_1] \ge 2, f[e_{n-1}] \ge 2$. So,

$$\sum_{x\in V\cup E} f[x] = \sum_{v\in V} f[v] + \sum_{e\in E} f[e] \ge n + (n-1) + 2 = 2n + 1.$$

In order to find an upper bound of $\sum_{x\in V\cup E} f[x]$, we consider the following three cases.

If $f(v_1) = f(v_n) = 1$, then

$$\begin{aligned} \textstyle\sum_{x\in V\cup E} f[x] &\le 5(t-2) + 6 - 5(n-t) + 5s - 5(n-s-3) - 8 \\ &= 5(2t + 2s - 2n) + 3. \end{aligned}$$

If exactly one of $f(v_1)$ or $f(v_n)$ is 1, then

$$\begin{aligned} \textstyle\sum_{x\in V\cup E} f[x] &\le 5(t-1) - 5(n-t-1) + 5(s-1) - 5(n-s-2) + 8 \\ &= 5(2t + 2s - 2n) + 5. \end{aligned}$$

If $f(v_1) = f(v_n) = -1$, then

$$\begin{array}{rcl}\sum_{x\in V\cup E} f[x] & \leq & 5t - 5(n-t-2) - 6 + 5(s-2) - 5(n-1-s) + 8 \\ & = & 5(2t+2s-2n) + 7.\end{array}$$

By summarizing the three cases, we can see $\sum_{x\in V\cup E} f[x] \leq 5(2t+2s-2n)+7$. From these two inequalities, we can get $2n+1 \leq 5(2t+2s-2n)+7$ and hence

$$\gamma_s^*(P_n) = 2(t+s) - (n+n-1) \geq \left\lceil \frac{2n-1}{5} \right\rceil.$$

Now, we aim to computer the total domination number of the path P_n. In order to this, we consider the following three cases.

Case 1. $n \equiv 0$ or $4 \pmod 5$. In this case, $\lceil \frac{2n-1}{5} \rceil$ is even. Since $\gamma_s^*(P_n) = 2(t+s) - 2n + 1$ is an odd integer, we have

$$\gamma_s^*(P_n) \geq \left\lceil \frac{2n-1}{5} \right\rceil + 1.$$

When $n \equiv 0 \pmod 5$, we define a function f on $V(P_n) \cup E(P_n)$ by

$$f_n(x) = \begin{cases} -1 & \text{if } x = v_i \text{ and } i \equiv 1 \text{ or } 3 \pmod 5, \\ -1 & \text{if } x = v_iv_{i+1},\ i \equiv 0 \text{ or } 3 \pmod 5, \\ 1 & \text{otherwise.} \end{cases}$$

When $n \equiv 4 \pmod 5$, we define a function f on $V(P_n) \cup E(P_n)$ by

$$f_n(x) = \begin{cases} -1 & \text{if } x = v_i \text{ and } i \equiv 1 \text{ or } 3 \pmod 5, \\ -1 & \text{if } x = v_iv_{i+1},\ i \equiv 0 \text{ or } 3 \pmod 5 \text{ and } i \neq n-1, \\ 1 & \text{otherwise.} \end{cases}$$

Then, for each $n \equiv 0$ or $4 \pmod 5$, f_n is a total domination function of P_n. Since $|\{v \mid f_n(v) = -1\}| = \lceil \frac{2n-1}{5} \rceil$ and $|\{e \mid f_n(e) = -1\}| = (\lceil \frac{2n-1}{5} \rceil - 1)$ for each $n \equiv 0$ or $4 \pmod 5$, we have

$$\sum_{x\in V\cup E} f_n(x) = (2n-1) - 2\left(2\left\lceil \frac{2n-1}{5} \right\rceil - 1\right) = \left\lceil \frac{2n-1}{5} \right\rceil + 1.$$

Hence we have $\gamma_s^*(P_n) = \lceil \frac{2n-1}{5} \rceil + 1$ for each $n \equiv 0$ or $4 \pmod 5$.

Case 2. $n \equiv 1$ or $3 \pmod 5$. In this case, $\lceil \frac{2n-1}{5} \rceil$ is odd, but $\gamma_s^*(P_n) = 2(t+s) - 2n + 1$ is an odd integer. Hence

$$\gamma_s^*(P_n) \geq \left\lceil \frac{2n-1}{5} \right\rceil.$$

When $n \equiv 1 \pmod 5$, we define a function f on $V(P_n) \cup E(P_n)$ by

$$f_n(x) = \begin{cases} -1 & \text{if } x = v_i \text{ and } i \equiv 1 \text{ or } 3 \pmod 5 \text{ and } i \neq n, \\ -1 & \text{if } x = v_i v_{i+1},\ i \equiv 0 \text{ or } 3 \pmod 5, \\ 1 & \text{otherwise.} \end{cases}$$

When $n \equiv 3 \pmod 5$, we define a function f on $V(P_n) \cup E(P_n)$ by

$$f_n(x) = \begin{cases} -1 & \text{if } x = v_i \text{ and } i \equiv 1 \text{ or } 3 \pmod 5, \\ -1 & \text{if } x = v_i v_{i+1},\ i \equiv 0 \text{ or } 3 \pmod 5, \\ 1 & \text{otherwise.} \end{cases}$$

Then, for each $n \equiv 1$ or $3 \pmod 5$, f_n is a total domination function of P_n. Since $|\{v \,|\, f_n(v) = -1\}| = \lceil \frac{2n-1}{5} \rceil$ and $|\{e \,|\, f_n(e) = -1\}| = \lceil \frac{2n-1}{5} \rceil$ for each $n \equiv 1$ or $3 \pmod 5$, we have

$$\sum_{x \in V \cup E} f(x) = (2n-1) - 4 \left\lceil \frac{2n-1}{5} \right\rceil = \left\lceil \frac{2n-1}{5} \right\rceil.$$

Hence we have $\gamma_s^*(P_n) = \lceil \frac{2n-1}{5} \rceil$ for each $n \equiv 1$ or $3 \pmod 5$.

Case 3. $n \equiv 2 \pmod 5$. We observe that for any total signed domination function f of P_n

$$\begin{aligned} \sum_{x \in V \cup E} f(x) &= \sum_{i \equiv 1 \pmod 5} (f[e_i] + f[v_{i+3}]) \\ &= f[e_1] + f[e_{n-1}] + \sum_{i \equiv 1 \pmod 5, i \neq 1, n-1} f[e_i] + \sum_{i \equiv 4 \pmod 5} f[v_i] \\ &\geq 2 + 2 + \frac{2n-9}{5}. \end{aligned}$$

From this, we can deduce that

$$\gamma_s^*(P_n) \geq \frac{2n+11}{5} = \left\lceil \frac{2n-1}{5} \right\rceil + 2.$$

We define a function f on $V(P_n) \cup E(P_n)$ by

$$f_n(x) = \begin{cases} -1 & \text{if } x = v_i,\ i \neq n-1 \text{ and } i \equiv 1 \text{ or } 3 \pmod 5, \\ -1 & \text{if } x = v_i v_{i+1},\ i \equiv 0 \text{ or } 3 \pmod 5, \\ 1 & \text{otherwise.} \end{cases}$$

Then f is a γ_s^*-function of P_n and f has $2(\frac{n-2}{5})$ vertices signed -1 and $2(\frac{n-2}{5})$ edges signed -1. Hence

$$\sum_{x \in V \cup E} f(x) = (2n-1) - 4\left(\frac{n-2}{5}\right) = \frac{2n+11}{5} = \left\lceil \frac{2n-1}{5} \right\rceil + 2. \qquad \square$$

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