International Journal of Pure and Applied Mathematics
Volume 29 No. 3 2006, 391-402

ON THE SOLUTION OF A MODIFIED REYNOLDS ELASTOHYDRODYNAMIC LUBRICATION EQUATION

Khalid Ait Hadi
Department of Mathematics
Faculty of Sciences Semlalia
Cadi Ayyad University
B.P. 2390, Marrakech, MOROCCO
e-mail: k.aithadi@ucam.ac.ma

Abstract: In this work a modified Reynolds equation issued from a consistent derivation of the equations of motion for elastohydrodynamic lubrication is studied. The existence and the uniqueness of this equation are, under some hypotheses on the data, obtained by means of both fixed point argument and monotonicity techniques.

AMS Subject Classification: 35Q35, 37C25, 35J65
Key Words: modified Reynolds equation, elastohydrodynamic lubrication, fixed point, monotonicity

1. Introduction

Reynolds's lubrication equation [11] has become one of the most important equations in fluid mechanics and the basis for the design of many machine elements. It is constructed on the assumption of constant viscosity, in particular, that the viscosity is independent of pressure. This assumption is valid at low pressures, and holds in a large number of applications, with one notable exception, elastohydrodynamic lubrication (EHL). In the elastohydrodynamic lubrication context, when liquid lubricant is subjected to extremely high pressures, the fact that the viscosity for lubricants could depend on the pressure and could change significantly with large variations of the pressure has been recognized [1, 2, 8]. Nevertheless, in the current literature the pressure dependence of

Received: May 19, 2006

viscosity in the derivation of the governing equations for elastohydrodynamic lubrication is only considered a posteriori, that is, after the Reynolds equation has been stated under the assumption of constant viscosity. This process neglects individual contributions form pressure dependence of viscosity. Individually these effects may be small, but they non-linearly reinforce one another, leading to pressure corrections that can be considerable. A consistent application of a Reynolds type thin-film approximation for variable viscosity lubricant, where the viscosity is function of pressure, yield terms in addition to those contained in the classical Reynolds analysis. In fact, due to the presence of such terms, it is no longer possible to derive a Reynolds type approximation unless added, and severe, assumptions are made.

The Reynolds equation for lubricants flowing between almost parallel surfaces is derived under the assumptions that (see [11]):

(i) the viscosity is a constant;

(ii) the film is thin;

(iii) the Reynolds number is negligible;

(iv) the body forces can be neglected.

The previous assumptions lead to the pressure being considered as constant across the film and allow one to extract the classical Reynolds equation (see [3])

$$\frac{\partial}{\partial x}\left(\frac{h^3}{\mu}\frac{\partial p}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{h^3}{\mu}\frac{\partial p}{\partial y}\right)=6U^*\frac{\partial h}{\partial x}, \tag{1.1}$$

where h is the film thickness, p is the pressure of the lubricant, μ is the viscosity and U^* is the characteristic velocity.

In the elastohydrodynamic lubrication problems, equation (1.1) is employed as a starting point, into which one substitutes the pressure-dependent viscosity. If the viscosity is not a constant, however, one cannot integrate the equations of motion to establish (1.1) in the first place. Thus, there is a need for a consistent development of the equations governing elastohydrodynamics, which systematically takes into account the fact that the viscosity is not constant and depends upon the pressure field.

We follow the procedure used in [10], which uses since the beginning, an implicit relation between the stress in the fluid and the velocity gradient and takes into consideration the fact that the viscosity relies on the pressure using the Barus formula ($\mu = \mu_0 \exp(\alpha p)$) which has much validity in the context of high pressures. It follows while making additional assumption $\frac{\partial p}{\partial y} = 0$ that [10],

$$\frac{d}{dx}\left[\left(\frac{h^3}{\mu_0}\exp(-\alpha p)+12\alpha\int_0^h y(y-h)\frac{\partial u}{\partial x}dy\right)\frac{dp}{dx}\right]=6U^*\frac{dh}{dx}. \tag{1.2}$$

Equation (1.2) is the new modified Reynolds equation [10], it will reduce to the equation that is currently employed in elastohydrodynamic only if $\alpha = 0$.
We further simplify equation (1.2), by substituting $\frac{du_m}{dx}$ for $\frac{\partial u}{\partial x}$ and observing that the flow rate, Q, is related to the average velocity, u_m, by

$$Q = h(x)u_m(x). \tag{1.3}$$

The modified Reynolds equation reduces to

$$\frac{d}{dx}\left[\left(\frac{h^3}{\mu_0}\exp(-\alpha p) + \alpha Q\frac{dh^2}{dx}\right)\frac{dp}{dx}\right] = 6U^*\frac{dh}{dx}. \tag{1.4}$$

This paper is organized as follows: Section 2 is concerned with the proof of the existence of weak solutions for (1.4) using some changes of variables techniques and fixed point argument. The last section is devoted to prove the uniqueness of solution using certain monotonicity techniques.

2. An Existence Result

In this section we will prove the existence of solutions for the more general Dirichlet problem:

$$(\mathcal{P}_1)\quad \begin{cases} \text{Find } p \in H^1([0,1]) \text{ such that} \\ \displaystyle\int_0^1 \left(\frac{h^3}{\mu_0}\exp(-\alpha p) + \alpha Q\frac{dh^2}{dx}\right)\frac{dp}{dx}\frac{d\varphi}{dx}\,dx = 6U^*\int_0^1 h\frac{d\varphi}{dx}\,dx \\ \forall \varphi \in H_0^1([0,1]) \ \text{ with } p(0) = 0 \text{ and } p(1) = p_1. \end{cases}$$

where p_1 is a positive given constant.

We assume that the functions $h : [0,1] \to \mathbb{R}$ and $Q : [0,1] \to \mathbb{R}$ satisfy the following hypotheses:

$$\begin{cases} h \in W^{1,\infty}([0,1]), \\ h \text{ is bounded in } W^{1,\infty}([0,1]) \text{ and } 0 < a \leq h(x) \leq b \text{ in } [0,1], \\ \text{with } a, b \text{ are two positives constants.} \end{cases}$$

$$\begin{cases} Q \in W^{1,\infty}([0,1]), \\ Q \text{ is bounded in } W^{1,\infty}([0,1]) \text{ and } Q(x) > 0 \text{ in } [0,1]. \end{cases}$$

It is very useful to avoid the nonlinearity due to the term $\exp(-\alpha p)$ by means of a classical Kirchoff change of variable, that in the lubrication context is known as Grubin's transformation (see, [3, 7], for example)

$$q = f(p) = \int_0^p \exp(-\alpha u)\,du = \frac{1-\exp(-\alpha p)}{\alpha}, \text{ for } \alpha > 0.$$

Now q is solution of

$$(\mathcal{P}_2)\quad \begin{cases} q \in H^1([0,1]) \text{ such that } q < \dfrac{1}{\alpha} \text{ and} \\ \displaystyle\int_0^1 \left(\frac{h^3}{\mu_0} + \alpha Q \frac{dh^2}{dx}\frac{1}{1-\alpha q}\right)\frac{dq}{dx}\frac{d\varphi}{dx}\,dx = 6U^* \int_0^1 h\frac{d\varphi}{dx}\,dx \\ \forall \varphi \in H_0^1([0,1]), \quad \text{where } q(0) = 0 \text{ and } q(1) = q_1 = f(p_1). \end{cases}$$

In order to prove that $(\mathcal{P}_1)$ admits a solution we will show that $(\mathcal{P}_2)$ has a solution. It is clear that the reciprocal function of f is not always defined for a given q. We will prove that for a specific data it is possible to get a solution of $(\mathcal{P}_2)$ such that $q < \frac{1}{\alpha}$, so $p = f^{-1}(q)$ is a well-defined solution of $(\mathcal{P}_1)$.

Let us write for $q \in H^1([0,1])$ (with $q < \frac{1}{\alpha}$)

$$\left(\frac{h^3}{\mu_0} + \alpha Q \frac{dh^2}{dx}\frac{1}{1-\alpha q}\right)\frac{dq}{dx} = \frac{h^3}{\mu_0}\frac{d}{dx}\left[q + \frac{\mu_0}{h^3}\log\left(\frac{1}{1-\alpha q}\right)Q\frac{dh^2}{dx}\right] - \frac{h^3}{\mu_0}\log\left(\frac{1}{1-\alpha q}\right)\frac{d}{dx}\left(\frac{\mu_0}{h^3}Q\frac{dh^2}{dx}\right). \quad (2.1)$$

The new unknown will be

$$r = q + \frac{\mu_0}{h^3}\log\left(\frac{1}{1-\alpha q}\right)Q\frac{dh^2}{dx}. \quad (2.2)$$

We consider the function $g : \left]-\infty, \frac{1}{\alpha}\right[\to \left]-\infty, +\infty\right[$

$$g(s) = s + \frac{\mu_0}{h^3}\log\left(\frac{1}{1-\alpha s}\right)Q\frac{dh^2}{dx}.$$

If $\frac{dh}{dx} \geq 0$ it is easy to prove that g is an increasing and bijective function.

For the following we consider that $\frac{dh}{dx}$ is positive.

Our problem $(\mathcal{P}_2)$ becomes in r:

$$(\mathcal{P}_3)\quad \begin{cases} \text{Find } r \in H^1([0,1]) \text{ such that} \\ \displaystyle\int_0^1 \frac{h^3}{\mu_0}\frac{dr}{dx}\frac{d\varphi}{dx}\,dx \\ \displaystyle = -\int_0^1 \frac{h^3}{\mu_0}\log\left(1 - \alpha g^{-1}(r)\right)\frac{d}{dx}\left(\frac{\mu_0}{h^3}Q\frac{dh^2}{dx}\right)\frac{d\varphi}{dx}\,dx \\ +6U^* \int_0^1 h\frac{d\varphi}{dx}\,dx \;\forall \varphi \in H_0^1([0,1]) \\ \text{where } r(0) = 0 \text{ and } r(1) = r_1 = g(q_1). \end{cases}$$

We remark that for all $z \in \mathbb{R}$

$$\left|\frac{d\left(g^{-1}\right)}{dz}(z)\right| = \left|\frac{h^3\left(1-\alpha g^{-1}(z)\right)}{h^3\left(1-\alpha g^{-1}(z)\right)+\mu_0 Q \frac{dh^2}{dx}}\right| \le 1, \tag{2.3}$$

$$\left|\frac{d\left(\log\left(1-\alpha g^{-1}\right)\right)}{dz}(z)\right| = \left|\frac{\alpha h^3}{h^3\left(1-\alpha g^{-1}(z)\right)+\mu_0 Q \frac{dh^2}{dx}}\right| \le \left|\frac{\alpha h^3}{\mu_0 Q \frac{dh^2}{dx}}\right| \le K. \tag{2.4}$$

With K a positive constant. The equivalence between $(\mathcal{P}_2)$ and $(\mathcal{P}_3)$ is given by the following lemma.

Lemma 2.1. *r is a weak solution of $(\mathcal{P}_3)$ if and only if $q = g^{-1}(r)$ is a weak solution of $(\mathcal{P}_2)$.*

Proof. It is clear from (2.1) that the two variational formulas are equivalent. And from (2.2) it is obvious that if $q \in H^1([0,1])$ with $1-\alpha q > 0$ then $r \in H^1([0,1])$. It remains to show that if r is a solution of $(\mathcal{P}_r)$ then $q \in H^1([0,1])$ such that $q < \frac{1}{\alpha}$. From $q = g^{-1}(r)$ we have that $q \in H^1([0,1])$ since $(g^{-1})'$ is bounded. On the other hand, we have that $r \in L^\infty([0,1])$, we deduce from $q = g^{-1}(r)$ that q belongs to $L^\infty([0,1])$ with q bounded away from $\frac{1}{\alpha}$, and the proof is ended. □

Let now prove existence of solutions for the problem $(\mathcal{P}_3)$.

Theorem 2.2. *Under the assumption $\frac{dh}{dx} > 0$ and if the problem parameters satisfy the condition*

$$a^3 > C_s b^3 \left\|\frac{d}{dx}\left(\frac{\mu_0}{h^3} Q \frac{dh^2}{dx}\right)\right\|_{L^\infty([0,1])}, \tag{2.5}$$

with C_s the Sobolev constant of the inclusion of $H_0^1([0,1])$ into $L^p([0,1])$ $(p > 2)$. Then the problem $(\mathcal{P}_3)$ admits at least one solution.

Before proving the theorem we state the following auxiliary lemma.

Let be $B_R = \left\{w \in L^2([0,1]) \;/\; w \le R\right\}$ and let A be the operator defined from B_R into $L^2([0,1])$ by $A(r) = u$, where u is the solution of the problem

$$(\widetilde{\mathcal{P}}) \quad \begin{cases} u \in H^1([0,1]) \text{ such that} \\ \displaystyle\int_0^1 \frac{h^3}{\mu_0}\frac{du}{dx}\frac{d\varphi}{dx}\,dx \\ \displaystyle = -\int_0^1 \frac{h^3}{\mu_0}\log\left(1-\alpha g^{-1}(r)\right)\frac{d}{dx}\left(\frac{\mu_0}{h^3}Q\frac{dh^2}{dx}\right)\frac{d\varphi}{dx}\,dx \\ \displaystyle +6U^*\int_0^1 h\frac{d\varphi}{dx}\,dx \quad \forall\varphi \in H_0^1([0,1]) \\ \text{where } u(0)=0 \text{ and } u(1)=r_1. \end{cases}$$

Lemma 2.3. *For $r \in B_R$ there exists at most one solution of problem $(\widetilde{\mathcal{P}})$. In addition we have the following estimates:*

$$\left\|\frac{du}{dx}\right\|_{L^2([0,1])} \le C_1(R), \tag{2.6}$$

$$\|u\|_{L^\infty([0,1])} \le C_2(R), \tag{2.7}$$

where $C_1(R)$ and $C_2(R)$ are two constants which depend on R.

Proof. The results of existence and uniqueness are classical (see, for exemple, [4, 9]). In order to prove (2.6) we take $\varphi = u - \psi_1$, where $\psi_1 \in H^1([0,1])$ such that $\psi_1(0) = 0$ and $\psi_1(1) = r_1$, as a test function in $(\widetilde{\mathcal{P}})$, then

$$\int_0^1 \frac{h^3}{\mu_0}\left(\frac{du}{dx}\right)^2 dx$$
$$= -\int_0^1 \frac{h^3}{\mu_0}\log\left(1-\alpha g^{-1}(r)\right)\frac{d}{dx}\left(\frac{\mu_0}{h^3}Q\frac{dh^2}{dx}\right)\frac{d(u-\psi_1)}{dx}\,dx$$
$$+ 6U^*\int_0^1 h\frac{d(u-\psi_1)}{dx}\,dx + \int_0^1 \frac{h^3}{\mu_0}\frac{du}{dx}\frac{d\psi_1}{dx}\,dx,$$

therefore

$$\frac{a^3}{\mu_0}\left\|\frac{du}{dx}\right\|^2_{L^2([0,1])}$$
$$\le \frac{Rb^3}{\mu_0}\left\|\frac{d}{dx}\left(\frac{\mu_0}{h^3}Q\frac{dh^2}{dx}\right)\right\|_{L^\infty([0,1])}\left(\left\|\frac{du}{dx}\right\|_{L^2([0,1])} + \left\|\frac{d\psi_1}{dx}\right\|_{L^2([0,1])}\right)$$
$$+ 6U^*b\left(\left\|\frac{du}{dx}\right\|_{L^2([0,1])} + \left\|\frac{d\psi_1}{dx}\right\|_{L^2([0,1])}\right) + \frac{b^3}{\mu_0}\left\|\frac{du}{dx}\right\|_{L^2([0,1])}\left\|\frac{d\psi_1}{dx}\right\|_{L^2([0,1])},$$

it follows that

$$\frac{a^3}{\mu_0}\left\|\frac{du}{dx}\right\|^2_{L^2([0,1])} \leq \left(\frac{Rb^3}{\mu_0}\left\|\frac{d}{dx}\left(\frac{\mu_0}{h^3}Q\frac{dh^2}{dx}\right)\right\|_{L^\infty([0,1])} + 6U^*b + \frac{b^3}{\mu_0}\left\|\frac{d\psi_1}{dx}\right\|_{L^2([0,1])}\right)\left\|\frac{du}{dx}\right\|_{L^2([0,1])} + \left(\frac{Rb^3}{\mu_0}\left\|\frac{d}{dx}\left(\frac{\mu_0}{h^3}Q\frac{dh^2}{dx}\right)\right\|_{L^\infty([0,1])} + 6U^*b\right)\left\|\frac{d\psi_1}{dx}\right\|_{L^2([0,1])},$$

so, there exists $C_1(R)$ such that

$$\left\|\frac{du}{dx}\right\|_{L^2([0,1])} \leq C_1(R).$$

In order to prove (2.7) we follow classical L^∞ estimates for elliptic Dirichlet problems as in [4, 9].

Let us consider for $m > \max(0, r_1)$, γ defined by

$$\gamma = (u-m)^+ - (u+m)^-.$$

It is clear that γ is a function in $H^1_0([0,1])$ which vanishes on $|u| \leq m$ and is equal to $\pm(|u|-m)$ on $S(m) = [|u| > m]$.

Taking $\varphi = \gamma$ as test function in $(\widetilde{\mathcal{P}})$, leads to

$$\int_0^1 \frac{h^3}{\mu_0}\left(\frac{d\gamma}{dx}\right)^2 dx = -\int_0^1 \frac{h^3}{\mu_0}\log\left(1-\alpha g^{-1}(r)\right)\frac{d}{dx}\left(\frac{\mu_0}{h^3}Q\frac{dh^2}{dx}\right)\frac{d\gamma}{dx}\,dx + 6U^*\int_0^1 h\frac{d\gamma}{dx}\,dx,$$

then

$$\frac{a^3}{\mu_0}\left\|\frac{d\gamma}{dx}\right\|^2_{L^2([0,1])} \leq \left[\frac{Rb^3}{\mu_0}\left\|\frac{d}{dx}\left(\frac{\mu_0}{h^3}Q\frac{dh^2}{dx}\right)\right\|_{L^\infty([0,1])}\left\|\frac{d\gamma}{dx}\right\|_{L^2([0,1])}\right]|S(m)|^{\frac{1}{2}} + 6U^*\|h\|_{L^2([0,1])}\left\|\frac{d\gamma}{dx}\right\|_{L^2([0,1])}|S(m)|^{\frac{1}{2}},$$

where $|S(m)|$ denotes the measure of $S(m)$.

Therefore,

$$\left\|\frac{d\gamma}{dx}\right\|_{L^2([0,1])} \leq |S(m)|^{\frac{1}{2}} \left[\frac{Rb^3}{a^3}\left\|\frac{d}{dx}\left(\frac{\mu_0}{h^3}Q\frac{dh^2}{dx}\right)\right\|_{L^\infty([0,1])} + \frac{\mu_0}{a^3}6U^*\|h\|_{L^2([0,1])}\right],$$

$$\left\|\frac{d\gamma}{dx}\right\|^2_{L^2([0,1])} \leq |S(m)| \left[\frac{Rb^3}{a^3}\left\|\frac{d}{dx}\left(\frac{\mu_0}{h^3}Q\frac{dh^2}{dx}\right)\right\|_{L^\infty([0,1])} + \frac{\mu_0}{a^3}6U^*\|h\|_{L^2([0,1])}\right]^2.$$

Now, let us note that for $n \geq m$, we have $S(n) \subset S(m)$ and so for such a n, we have

$$(n-m)^p\,|S(n)| = \int_{S(n)} (h-m)^p\ dx \leq \int_{S(n)} (|u|-m)^p\ dx = \int_{S(n)} |\gamma|^p\ dx.$$

By choosing $p > 2$ and using the continuous inclusion of $H_0^1([0,1])$ into $L^p([0,1])$ we obtain

$$\begin{aligned}(n-m)^p\,|S(n)| &\leq (C_s)^p\left(\int_0^1\left|\frac{d\gamma}{dx}\right|^2\ dx\right)^{\frac{p}{2}}\\ &\leq (C_s)^p\left[\frac{Rb^3}{a^3}\left\|\frac{d}{dx}\left(\frac{\mu_0}{h^3}Q\frac{dh^2}{dx}\right)\right\|_{L^\infty([0,1])} + \frac{\mu_0}{a^3}6U^*\|h\|_{L^2([0,1])}\right]^p |S(m)|^{\frac{p}{2}},\end{aligned}$$

where C_s is the Sobolev constant of the inclusion [5]. It follows that

$$|S(n)| \leq \left[\frac{C_s}{(n-m)}\left(\frac{Rb^3}{a^3}\left\|\frac{d}{dx}\left(\frac{\mu_0}{h^3}Q\frac{dh^2}{dx}\right)\right\|_{L^\infty([0,1])} + \frac{\mu_0}{a^3}6U^*\|h\|_{L^2([0,1])}\right)\right]^p |S(m)|^{\frac{p}{2}}.$$

Using Lemma B.1 from [9], we can get

$$\|u\|_{L^\infty([0,1])} \leq \max(0,r_1) + C_s\left(\frac{Rb^3}{a^3}\left\|\frac{d}{dx}\left(\frac{\mu_0}{h^3}Q\frac{dh^2}{dx}\right)\right\|_{L^\infty([0,1])} + \frac{\mu_0}{a^3}6U^*\|h\|_{L^2([0,1])}\right). \quad \square$$

Proof. (of Theorem 2.2) The operator A from B_R to $L^2([0,1])$ is compact and continuous. Indeed, the continuity is elementary and the compactness is a direct consequence of estimate (2.7) and the compact inclusion of $H^1([0,1])$ into $L^2([0,1])$.

Now, we choose a real number R such that $A(B_R) \subset B_R$. For this it is sufficient to take R such that

$$\max(0, r_1) + C_s \left(\frac{Rb^3}{a^3} \left\| \frac{d}{dx} \left(\frac{\mu_0}{h^3} Q \frac{dh^2}{dx} \right) \right\|_{L^\infty([0,1])} + \frac{\mu_0}{a^3} 6U^* \|h\|_{L^2([0,1])} \right) \leq R.$$

In fact, we need

$$R \left(1 - \frac{C_s b^3}{a^3} \left\| \frac{d}{dx} \left(\frac{\mu_0}{h^3} Q \frac{dh^2}{dx} \right) \right\|_{L^\infty([0,1])} \right) \geq \max(0, r_1) + \frac{C_s \mu_0}{a^3} 6U^* \|h\|_{L^2([0,1])} .$$

However if (2.5) is verified, we can take a big enough value of R such that

$$R \geq \frac{\max(0, r_1) + \frac{C_s \mu_0}{a^3} 6U^* \|h\|_{L^2([0,1])}}{\left(1 - \frac{C_s b^3}{a^3} \left\| \frac{d}{dx} \left(\frac{\mu_0}{h^3} Q \frac{dh^2}{dx} \right) \right\|_{L^\infty([0,1])} \right)}.$$

Finally, Theorem (2.2) follows from the Schauder Fixed Point Theorem. □

So, according to Lemma 2.1 there exists a solution q, defined by $q = g^{-1}(r)$, for the problem $(\mathcal{P}_2)$ such that $q < \frac{1}{\alpha}$ in $[0,1]$. Moreover, $p = f^{-1}(q)$ is a well-defined solution of $(\mathcal{P}_1)$.

3. Monotonicity and Uniqueness of Solutions

In the next we give an uniqueness result for the problem $(\mathcal{P}_1)$ using a general monotonicity for a class of semi-linear elliptic problems already developed by Gilbarg-Trudinger [6].

First, we prove a uniqueness and monotonicity result for weak solutions to the problem $(\mathcal{P}_3)$.

Theorem 3.1. *We suppose that r_i is a weak solution to $(\mathcal{P}_3)$ corresponding to the boundary data $r_a^i = (0, r_1^i)$, $i = 1, 2$. If $r_a^1 \geq r_a^2$, then $r_1 \geq r_2$ in $[0,1]$. Further, we have uniqueness among all weak solutions to problem $(\mathcal{P}_3)$.*

Proof. We take $r_3 = r_2 - r_1$ which satisfies the problem

$$\begin{cases} r_3 \in H^1([0,1]) \text{ such that} \\ \displaystyle\int_0^1 \frac{h^3}{\mu_0}\frac{dr_3}{dx}\frac{d\varphi}{dx}\,dx = -\int_0^1 (G(x,r_2) - G(x,r_1))\frac{d\varphi}{dx}\,dx \\ \forall\varphi \in H_0^1([0,1]) \text{ where } r_3(0) = 0 \text{ and } r_3(1) = r_1^2 - r_1^1. \end{cases} \tag{3.1}$$

where $G(x,r) = \frac{h^3}{\mu_0}\log\left(1 - \alpha g^{-1}(r)\right)\frac{d}{dx}\left(\frac{\mu_0}{h^3}Q\frac{dh^2}{dx}\right)$, we remark, according to (2.4), that $G : [0,1]\times\mathbb{R} \to \mathbb{R}$ is an uniform Lipschitz function:

$$\exists D_1 > 0,,\ |G(x,r_1) - G(x,r_2)| \le D_1\,|r_1 - r_2|\,,\ \forall x \in [0,1] \text{ and } r_1, r_2 \in \mathbb{R}. \tag{3.2}$$

We have that $r_3^+ \in H_0^1([0,1])$, so we can take $\varphi = \frac{r_3^+}{r_3^+ + \delta}$ as a test function in (3.1) with $\delta > 0$, which gives

$$\int_0^1 \frac{h^3}{\mu_0}\frac{dr_3^+}{dx}\frac{d}{dx}\left(\frac{r_3^+}{r_3^+ + \delta}\right)dx = -\int_0^1 (G(x,r_2) - G(x,r_1))\frac{d}{dx}\left(\frac{r_3^+}{r_3^+ + \delta}\right)dx. \tag{3.3}$$

However

$$\frac{d}{dx}\left(\frac{r_3^+}{r_3^+ + \delta}\right) = \delta\frac{\frac{d}{dx}r_3^+}{\left(r_3^+ + \delta\right)^2}, \qquad \frac{d}{dx}\log\left(1 + \frac{r_3^+}{\delta}\right) = \frac{\frac{d}{dx}r_3^+}{\left(r_3^+ + \delta\right)},$$

which implies

$$\int_0^1 \frac{h^3}{\mu_0}\frac{dr_3^+}{dx}\frac{d}{dx}\left(\frac{r_3^+}{r_3^+ + \delta}\right)dx = \delta\int_0^1 \frac{h^3}{\mu_0}\left|\frac{d}{dx}\log\left(1 + \frac{r_3^+}{\delta}\right)\right|^2 dx. \tag{3.4}$$

The right-hand side of (3.3) can be estimated with the help of (3.2) as

$$\begin{aligned} &\left|\int_0^1 (G(x,r_2) - G(x,r_1))\frac{d}{dx}\left(\frac{r_3^+}{r_3^+ + \delta}\right)dx.\right| \\ &\qquad\le \int_0^1 |G(x,r_2) - G(x,r_1)|\left|\frac{d}{dx}\left(\frac{r_3^+}{r_3^+ + \delta}\right)\right|dx \\ &\le D_1\int_0^1 |r_3|\left|\frac{d}{dx}\left(\frac{r_3^+}{r_3^+ + \delta}\right)\right|dx \ = D_1\int_0^1 \left|r_3^+\frac{d}{dx}\left(\frac{r_3^+}{r_3^+ + \delta}\right)\right|dx. \end{aligned} \tag{3.5}$$

However

$$\left|r_3^+ \frac{d}{dx}\left(\frac{r_3^+}{r_3^+ + \delta}\right)\right| = \delta \left|\frac{dr_3^+}{dx}\frac{r_3^+}{\left(r_3^+ + \delta\right)^2}\right| \leq \delta \left|\frac{dr_3^+}{dx}\left(\frac{1}{r_3^+ + \delta}\right)\right|$$
$$= \delta \left|\frac{d}{dx}\log\left(1 + \frac{r_3^+}{\delta}\right)\right|. \quad (3.6)$$

So, from (3.3) we obtain using also (3.4)-(3.6),

$$\frac{a^3}{\mu_0}\int_0^1 \frac{d}{dx}\left|\log\left(1 + \frac{r_3^+}{\delta}\right)\right|^2 dx \leq D_1 \int_0^1 \left|\frac{d}{dx}\log\left(1 + \frac{r_3^+}{\delta}\right)\right| dx. \quad (3.7)$$

Since $\log\left(1 + \frac{r_3^+}{\delta}\right) \in H_0^1([0,1])$, from Poincaré inequality we deduce

$$\int_0^1 \left|\log\left(1 + \frac{r_3^+}{\delta}\right)\right|^2 dx \leq C, \quad (3.8)$$

with C independent on δ. Then letting δ tends to zero, we will necessarily have $r_3^+ = 0$ in $[0,1]$. While substituting r_3 for $-r_3$ we get $r_3^- = 0$ in $[0,1]$, and the proof is ended. □

Proposition 3.2. *We have uniqueness among all weak solutions to the problem* $(\mathcal{P})$.

Proof. The proof is an obvious consequence of Theorem 3.1, Lemma 2.1 and the fact that $f^{-1}(q)$ is well-defined for $q < \frac{1}{\alpha}$. □

References

[1] E.J. Bissett, D. W. Glander, A highly accurate approach that resolves the pressure spike of elastohydrodynamic lubrication, *J. Tribol.*, **110** (1988), 241-246.

[2] P.W. Bridgman, *The Physics of High Pressure*, New York, MacMillan (1931).

[3] A. Cameron, *Basic Lubrication Theory*, Ellis Horwood Series (1981).

[4] M. Chipot, *Variational Inequalities and Flow on Porous Media*, Applied Math, Sciences Series 52, Springer-Verlag, New York (1984).

[5] L.C. Evans, R.F. Gariepy, *Measure Theory and Fine Properties of Functions*, Sudies in Advanced Mathematics, CRC Press (1992).

[6] D. Gilbarg, N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 2nd ed., Springer-Verlag, Berlin (1983).

[7] J.A. Greenwood, An extension of the Grubin theory of elastohydrodynamic lubrication, *J. Phys. D: Appl. Phys.*, **5** (1972), 2195-2211.

[8] K.L. Johnson, J.L. Tevaarwerk, Shear behaviour of elastohydrodynamic oil films, *Proc. R. Soc. Lond. A*, **356** (1977), 215-236.

[9] D. Kinderlehrer, G. Stampacchia, *An Introduction to Variational Inequalities and Applications*, Academic Press (1980).

[10] K.R. Rajagopal, A.Z. Szeri, On a inconsistency in the derivation of the equations of elastohydrodynamic lubrication, *Proc. R. Soc. Lond. A*, **459** (2003), 2771-2786.

[11] O. Reynolds, On the theory of lubrication and its application to Mr Tower's experiments, *Phil. Trans. R. Soc. Lond.*, **177** (1886), 159-209.