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THE LAWSON PARTIAL ORDER ON RPP SEMIGROUPS

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Abstract: In this paper,we consider the natural partial order "$\leq_\ell$" on a semigroup. In particular, we will characterize the *rpp* semigroup by using the Lawson partial order "$\leq_\ell$" which is (left; right) compatible with the semigroup multiplication. Our main result extends and generalizes a number of known results in the literature on the natural partial order on abundant semigroups and regular semigroups obtained by Nambooripad, Lawson, Blyth-Gomes and Guo-Luo, respectively.

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1. Introduction

We will consider a special kind of natural partial order, namely the Lawson partial order "$\leq_\ell$", on a semigroup. We find that this kind of natural partial order is quite useful in studying the properties of abundant semigroups and rpp semigroups (see [4], [5] and [6]). In particular, for regular semigroup which is a special abundant semigroup, Nambooripad proved the following well known theorem in 1980.

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Theorem 1.1. (see [13]) *Let S be a regular semigroup. Then the natural partial order "$\leq$" on S is compatible with the semigroup multiplication if and only if S is a locally inverse semigroup.*

The above theorem of Nambooripad was later generalized by Lawson from regular semigroups to concordant semigroups [12] in 1987. Since a concordant semigroup is an idempotent-connected abundant semigroup, in short, IC abundant semigroup satisfying some condition of regularity, it is natural to ask whether we can further generalize the result of Lawson to general abundant semigroups or to rpp semigroups? The answer to this question is "yes". In fact, Guo and Luo [8] have proved in 2005 that the natural partial order "$\leq$" on an abundant semigroup S is compatible with the semigroup multiplication if and only if S is a locally adequate semigroup which is also idempotent-connected.

As a generalization of F-regular semigroups and F-orthodox semigroups (see [2]), the authors together with Li [9] have considered the F-rpp semigroups which are the generalization of F-abundant semigroups (see [7]). In this paper, we will further generalize the result of Lawson from the IC abundant semigroups to rpp semigroups by considering the Lawson partial order "$\leq_\ell$" instead of the general natural order "$\leq$". We will also give some conditions for the Lawson partial order "$\leq_\ell$" to be (left; right) compatible with the semigroup multiplication on an rpp semigroup. In addition, a new version of the Nambooripad Theorem for IC abundant semigroups will also be given.

For terminologies and notations not given in this paper, the reader is referred to [4], [5] and [12].

2. Preliminaries

We first state some definitions and some basic results on rpp semigroups.

Definition 2.1. (see [12]) Let x, y be elements of an rpp semigroup S. Define the following Lawson partial order "$\leq_\ell$" on S by:

> $x \leq_\ell y$ if and only if $x \leq_{\mathcal{L}^*} y$ (that is, $L^*(x) \subseteq L^*(y)$) and there exists $f \in E(S) \bigcap L^*_x$ such that $x = yf$.

For the sake of convenience, we denote by a^* [resp. $a^\dagger$] the typical idempotents which is $\mathcal{L}^*$- [resp. $\mathcal{R}^*$-] related to a. The Lawson partial order $\leq_\ell$ on a rpp semigroup S is now described by the following lemma.

Lemma 2.2. (see [9]) *Let S be an rpp semigroup. Then the following statements hold:*

(i) "$\leq_\ell$" coincides with the usual idempotent order ω on $E(S)$, that is, $e\omega f$ if and only if $e = ef = fe$, where $e, f \in E(S)$.

(ii) if $x \leq_\ell e$, then $x^2 = x \in S$.

(iii) if $x \leq_\ell y$ and y is a regular element of S, then x is also a regular element of S.

(iv) $x \leq_\ell y$ if and only if for all y^*, there exists $f \in \omega(y^*)$ such that $x = yf$.

(v) if $x \leq_\ell y$ and $x\mathcal{L}^*y$, then $x = y$.

Let B be a band. Then, we call B a *left regular band* if B satisfies the identity $xy = xyx$. The band B is called *left quasi-normal* if it satisfies the identity $xyz = xyxz$ (see [11]). Obviously, the left regular bands are special left quasi-normal bands.

We now consider the left regular band of a semigroup. We first give the following definition.

Definition 2.3. (1) A semigroup S is called $\mathcal{R}$-unipotent if $E(S)$ of S forms a left regular band, where $E(S)$ is the set of all idempotents of the semigroup S.

(2) A semigroup S is called a locally $\mathcal{P}$ semigroup if the local submonoid eSe is itself a semigroup with the property $\mathcal{P}$, for all $e \in E(S)$.

We also cite below some useful results given by Hall in [10].

Lemma 2.4. (see [10]) *Let S be a semigroup with $e \in E(S)$. Then the following statements hold:*

(1) If $E(eSe)$ is a band, then $E(eS)$ and $E(Se)$ are both bands.

(2) If each $\mathcal{R}$-class of eSe contains at most one idempotent, then Se also has this property.

3. Main Results

We first establish a characterization theorem for an rpp semigroup S in which the Lawson partial order "$\leq_\ell$" is left compatible with the semigroup multiplication.

Theorem 3.1. *Let S be an rpp semigroup. Then the following statements are equivalent:*

(1) S is a locally $\mathcal{R}$-unipotent semigroup;

(2) "$\leq_\ell$" is left compatible with the multiplication of S;

(3) for all $e \in E(S)$, $E(Se)$ is a left regular band;

(4) for all $e \in E(S)$, $E(eS)$ is a left quasinormal band;

(5) for all $e, f \in E(S)$, $E(eSf)$ is a left regular band.

Proof. We will show that (1) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1), (1) $\Rightarrow$ (4) $\Rightarrow$ (1) and (3) $\Rightarrow$ (5) $\Rightarrow$ (1). Since (3) $\Rightarrow$ (5) $\Rightarrow$ (1) are trivial, we need only prove the first two implications.

(1) $\Rightarrow$ (3) By Lemma 2.4(1), $E(Se)$ is clearly a band. Now, by our hypothesis, the local submonoid eSe is a $\mathcal{R}$-unipotent semigroup, and so $E(eSe)$ is a left regular band. Consequently, each $\mathcal{R}$-class of eSe contains at most one idempotent. Thus, by Lemma 2.4(2), $E(Se)$ forms a semilattice of left zero semigroups under semigroup multiplication and hence $E(Se)$ is a left regular band.

(3) $\Rightarrow$ (2) Suppose that (3) holds. Let $a, b \in S$ with $a \leq_\ell b$. Since S is an rpp semigroup, for all b^* and some $a^* \in \omega(b^*)$, we have $a = ba^*$ so that $ca = cba^*$ for any $c \in S$. Since $cb = cbb^*$, we also have $(cb)^* = (cb)^* b^*$. This implies that

$$(b^*(cb)^*)^2 = b^*(cb)^* b^*(cb)^* = b^*(cb)^*$$

and since $(b^*(cb)^*)(cb)^* = b^*(cb)^*$ and

$$((cb)^*)(b^*(cb)^*) = ((cb)^* b^*)(cb)^* = (cb)^*(cb)^* = (cb)^*,$$

we can now easily verify that $b^*(cb)^* \mathcal{L} (cb)^*$ and hence $cb \mathcal{L}^* b^*(cb)^*$. Clearly, $a^*, b^*(cb)^* \in E(b^* S b^*)$ and whence $b^*(cb)^* a^* \in E(b^* S b^*)$ because $E(Se)$ is a band. Again, since $E(Se)$ is a left regular band, we can deduce that

$$b^*(cb)^* a^* = (b^*(cb)^*) a^* = (b^*(cb)^*) a^* (b^*(cb)^*).$$

Consequently, we have $b^*(cb)^* a^* \in \omega(b^*(cb)^*)$. Now, by

$$ca = cba^* \mathcal{L}^* b^*(cb)^* a^* \text{ and } ca = cb(b^*(cb)^* a^*),$$

We have $ca \leq_\ell cb$. This proves (2).

(2) $\Rightarrow$ (1) Assume that condition (2) holds. Let $e \in E(S)$ and $x, y \in E(eSe)$. Then $y \leq_\ell e$ and so $xy \leq_\ell xe = x$. Hence, by Lemma 2.2, $xy \in E(eSe)$. Again since $xy \leq_\ell x$, by Lemma 2.2 again, we have $xy \omega x$ and so $xyx = xy$. This shows that $E(eSe)$ is a left regular band,proving (1).

(1) $\Rightarrow$ (4) Suppose that S is a locally $\mathcal{R}$-unipotent semigroup. Then $E(eSe)$ is clearly a left regular band, for all $e \in E(S)$, and by Lemma 2.4, $E(eS)$ is a band. Now, for all $x, y, z \in E(eS)$, we have $exe, eye \in E(eSe)$. Hence, we can derive that

$$\begin{aligned} xyz &= (ex)(ey)(ez) = (exe)(eye)z \\ &= (exe)(eye)(exe)z = xyxz. \end{aligned}$$

This shows that $E(eS)$ is indeed a left quasi-normal band.

(4) $\Rightarrow$ (1) Let $x, y \in E(eSe)$. Then, it is clear that $x, y \in E(eS)$ and $xy \in E(eS)$. It follows that $xy \in E(eSe)$. Since $E(eS)$ is a left quasi-normal band, we have

$$xy = xye = xyxe = xyx.$$

This shows that $E(eSe)$ is a left regular band, and consequently, S is a locally $\mathcal{R}$-unipotent semigroup. □

We are now going to find the conditions for an rpp semigroup S on which the Lawson partial order "$\leq_\ell$" is right compatible with the semigroup multiplication. We first prove the following lemma.

Lemma 3.2. *Let S be an rpp semigroup. Then the Lawson partial order "$\leq_\ell$" is right compatible with the semigroup multiplication of S if and only if there exist $(ea)^*$ and $g \in \omega((ea)^*)$ such that $fa = eag$ for all $(ea)^*$, where $a \in S, e \in E(S)$ and $f \in \omega(e)$.*

Proof. (Necessity) Suppose that $a \in S, e \in E(S)$ and $f \in \omega(e)$. Then, by (1), we have $f \leq_\ell e$ and so, $fa \leq_\ell ea$. Now, by the definition of $\leq_\ell$, we can find an element $g \in \omega((ea)^*)$ such that $fa = eag$, for all $(ea)^*$.

(Sufficiency) Assume that there exist $(ea)^*$ and $g \in \omega((ea)^*)$ such that $fa = eag$ for all $(ea)^*$, where $a \in S, e \in E(S)$ and $f \in \omega(e)$. Let $a, b, c \in S$ and $a \leq_\ell b$ on S. Then for all b^* and some $a^* \in \omega(b^*)$, we have $a = ba^*$. Consider $a^* \leq_\ell b^*$. Then, by our hypothesis, we have $e \in \omega((b^*c)^*)$ such that $a^*c = b^*ce$. This leads to $ac = ba^*c = bb^*ce = bc \circ e$. On the other hand, because $\mathcal{L}^*$ is a right congruence on S, we have $bc\mathcal{L}^*b^*c\mathcal{L}^*(b^*c)^*$. This shows that $ac \leq_\ell bc$. □

We now ask when will an abundant semigroup whose Lawson partial order "$\leq_\ell$" be compatible with the semigroup multiplication? We first state the definition of idempotent-connected semigroups.

Definition 3.3. An abundant semigroup S is called a right IC abundant semigroup if for each $a \in S$ and some a^+, a^*, there exists a mapping $\theta : \langle a^+ \rangle \to \langle a^* \rangle$ such that $xa = a(x\theta)$, for all $x \in \langle a^+ \rangle$.

We now characterize the right IC abundant semigroups.

Theorem 3.4. *Let S be an abundant semigroup. Then the following statements are equivalent:*

(1) S is a right IC semigroup;

(2) for each $a \in S$ and for some a^+, there exists $b \in S$ such that $ea = ab$, for all $e \in \omega(a^+)$;

(3) for each $a \in S$ and for some a^+, there exists $f \in \omega(a^*)$ such that $ea = af$, for all $e \in \omega(a^+)$.

Proof. The proof of this Theorem is similar to the proof of (Proposition 2.1 in [12]). We hence omit the details. □

Lemma 3.5. *Let S be an abundant semigroup. If the Lawson partial order "$\leq_\ell$" is right compatible with the semigroup multiplication, then S is an abundant right IC semigroup.*

Proof. The proof of this lemma is an immediate consequence of Lemma 2.2. □

We now formulate the following theorem for IC abundant semigroups. In fact, this is a new version of Nambooripad Theorem of IC abundant semigroups.

Theorem 3.6. *Let S be an abundant right IC semigroup. Then the Lawson partial order "$\leq_\ell$" on S is compatible with the semigroup multiplication if and only if S is a locally adequate semigroup, see [3].*

Proof. (Necessity) We first suppose that the Lawson partial order "$\leq_\ell$" on the abundant right IC semigroup S is compatible with the semigroup multiplication. Then, by Theorem 2.1, eSe itself is a locally $\mathcal{R}$-unipotent semigroup so that $E(eSe)$ is a left regular band. Now, by our hypothesis, we can easily see that in the regular semigroup $E(eSe)$, the Lawson partial order "$\leq_\ell$" is compatible with the semigroup multiplication. Now, we observe that $\omega =$ " $\leq_\ell$" (by Lemma 2.2) and together with the well known theorem of Nambooripad (see Theorem 1.1), we obtain immediately that $E(eSe)$ is a locally inverse semigroup and so $E(eSe)$ is a semilattice. Therefore eSe must be an adequate semigroup and whence S is a locally adequate semigroup.

(Sufficiency) Assume that S is a locally adequate semigroup. Then, we can let $a, b \in S$ with $a \leq_\ell b$. Clearly, for all b^* and some $a^* \in \omega(b^*)$, we have $a = ba^*$ and so $ac = ba^*c$, for any $c \in S$. Since $b^*c = b^*(b^*c)$, we also have $b^*(b^*c)^\dagger = (b^*c)^\dagger$ and hence $(b^*c)^\dagger b^* \in E(S)$. This leads to $(b^*c)^\dagger b^* \omega b^*$, and $(b^*c)^\dagger b^* \mathcal{R} (b^*c)^\dagger$. Since b^*Sb^* is an adequate semigroup, $E(b^*Sb^*)$ is a semilattice, see [3]. Observe that $a^*, (b^*c)^\dagger b^* \in E(b^*Sb^*)$, we hence deduce that there exists $g \in \omega((b^*c)^*)$ such that

$$a^*c = a^* \circ (b^*c)^\dagger b^*(b^*c)c = (b^*c)^\dagger b^* a^* (b^*c)^\dagger b^* \circ (b^*c) = (b^*c)g.$$

Thus, we have $ac = ba^*c = b(b^*c)g = bcg$. On the other hand, since $\mathcal{L}^*$ is a right congruence on S, we have $bc\mathcal{L}^*b^*c\mathcal{L}^*(b^*c)^*$. Hence, we have proved that

$ac \leq_\ell bc$ and thus "$\leq_\ell$" is compatible with the semigroup multiplication of S. The proof is now completed. □

4. Applications

Since the regular semigroups are special rpp semigroups, by using a result of Lawson in [13], we know that "$\leq_\ell$"="$\leq$" on a semigroup S if the semigroup S itself is a regular semigroup, where the natural partial order "$\leq$" is defined by

$$a \leq b \Leftrightarrow (\exists e, f \in E(S))a = eb = bf.$$

In view of Theorem 2.1, we immediately obtain the following Corollary which is essentially the result proved by Blyth and Gomes [1] in 1983.

Corollary 4.1. (see [1]) *Let S be a regular semigroup. Then the following statements are equivalent:*

(1) S is a locally $\mathcal{R}$-unipotent semigroup;

(2) the natural partial order "$\leq$" is left compatible with the semigroup multiplication of S;

(3) for all $e \in E(S)$, $E(Se)$ forms a left regular band under semigroup multiplication;

(4) for all $e \in E(S)$, $E(eS)$ forms a left quasi-normal band under semigroup multiplication;

(5) for all $e, f \in E(S)$, $E(eSf)$ forms a left regular band under semigroup multiplication.

Observe that a semilattice is both a left regular band and a right regular band. Hence, by using the above Corollary and its dual, we can easily re-obtain the well known theorem of Nambooripad (see [13], Theorem 0.1).

We recall here that it was proved proved by Lawson in [12] that an abundant semigroup is idempotent-connected if its natural partial order "$\leq$" is compatible with the semigroup multiplication. At the same time, Lawson also pointed out that an abundant semigroup S is idempotent-connected if and only if "$\leq_\ell$" = "$\leq=\leq_r$" on S. Observe that the intersection of a right regular band and a left quasi-normal band is a right normal band and the intersection of a right normal (regular) band and a left normal (regular) band is a semilattice (see [11]). Hence, by Theorem 2.1 and its dual, we can re-obtain the following result of Guo and Luo [8] given in 2005.

Corollary 4.2. (see [8]) *Let S be an idempotent-connected abundant semigroup. Then the following statements are equivalent:*

(1) S is a locally adequate semigroup;

(2) the Lawson partial order "$\leq$" is left compatible with the semigroup multiplication of S;

(3) for all $e \in E(S)$, $E(Se)$ forms a left normal band under semigroup multiplication;

(4) for all $e \in E(S)$, $E(eS)$ forms a right normal band under semigroup multiplication.

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