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ON THE TOTAL COLORING OF $S_n \vee P_n \vee P_n$

Dong-Han Zhang[1], Xiao-Ping Wang[2], Peng-Xiang Qiu[3],
Yao-Dong Cheng[4], Zhong-Fu Zhang[5 §]
[1,2,3,4,5]Institute of Applied Mathematics
Lanzhou Jiaotong University
Lanzhou, 730070, P.R. CHINA
[1]e-mail: zhanghan227@sohu.com

Abstract: In this paper, the total chromatic number of $S_n \vee P_n \vee P_n$ was obtained.

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1. Introduction

The graph coloring is one of the chief topics in graph research. The total coloring of graphs is NP-hard problem in graph theory. In this paper, the total chromatic number of $S_n \vee P_n \vee P_n$ is obtained.

Definition 1. (see [2, 3, 6, 7]) A proper total k-coloring of a graph G (k-TC of G in brief) is a mapping f from $V(G) \cup E(G)$ to the set of colors $\{1, 2, \cdots, k\}$ such that $f(x) \neq f(y)$ for every pair of adjacent or incident elements $x, y \in V(G) \cup E(G)$. The graph G is total k-colorable if it has a total k-coloring. The total chromatic number χ_t of G is the smallest integer k such that G is total k-colorable.

Conjecture 1. (see [4], [1]) *For any simple graph G(V, E), then*

$$\chi_t(G) \leq \Delta(G) + 2,$$

where $\Delta(G)$ is maximum degree of G.

Definition 2. (see [1]) For graph G and H $(V(G) \cap V(H) = \phi,\ E(G) \cap E(H) = \phi)$ a new graph denoted by $G \vee H$ is called the join of G and H. If

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§Correspondence author

$V(G \vee H) = V(G) \vee V(H)$, $E(G \vee H) = E(G) \cup E(H) \cup \{uv | u \in V(G) v \in V(H)\}$.

2. Main Results

Lemma 1. (see [3], [5]) *Suppose G is a graph, then*

$$\chi_t(G) \geq \Delta(G) + 1,$$

where $\Delta(G)$ is maximum degree of G.

Lemma 2. *Suppose S_n is star with order n+1, P_n is path with order n, then*

$$\Delta(S_n \vee P_n \vee P_n) = 3n.$$

Lemma 3. (see [1]) *Suppose K_n is complete graph with order n, then*

$$\chi'(K_n) = \begin{cases} n, & n \equiv 1(\mod 2); \\ n-1, & n \equiv 0(\mod 2); \end{cases}$$

Lemma 4. *If $n \equiv 0(\mod 2)$, $\Delta(G) = n - 1$, then*

$$\chi'(G) = \Delta(G),$$

where n= $|V(G)|$.

Lemma 5. (see [7]) *If V_Δ is forest, then*

$$\chi'(G) = \Delta,$$

where $V_\Delta = \{v | d(v) = \Delta\}$.

Lemma 6. *If $\Delta(G) = n - 1$ and $n \equiv 1(\mod 2)$, then*

$$\chi_t(G) = n,$$

where $n = |V(G)|$.

Proof. Suppose $w \notin V(G)$, $G^* = G \vee \{w\}$, then $\chi'(G^*) = \Delta(G^*) = n$ by Lemma 4. let f^* be n-EC of G^*. For $v \in V(G)$, $uv \in E(G)$, f(v)= $f^*(wv)$, f(uv)= $f^*(uv)$. Obviously f is n-TC of G. So Lemma 6 is ture. □

Lemma 7. *Suppose K_n is complete graph with order n, then*

$$\chi_t(K_n) = \begin{cases} n, & n \equiv 1(\mod 2); \\ n+1, & n \equiv 0(\mod 2). \end{cases}$$

Theorem 1. *For n=1, then*

$$\chi_t(S_1 \vee P_1 \vee P_1) = 5.$$

Proof. Note $S_1 \vee P_1 \vee P_1 = K_4 = 5$, so Theorem 1 is ture by Lemma 7. □

Theorem 2. *If* $n \equiv 0(\ mod\ 2)$, *then*

$$\chi_t(S_n \vee P_n \vee P_n) = 3n + 1.$$

Proof. It is corollary of Lemma 6. □

Theorem 3. *For n=3, then*

$$\chi_t(S_3 \vee P_3 \vee P_3) = 10.$$

Proof. Only to prove exists 10-TC of $S_3 \vee P_3 \vee P_3$ by Lemma 1 and Lemma 2. Suppose $V(S_3) = \{u_i|i = 0, 1, 2, 3\}$, $E(S_3) = \{u_0u_i|i = 1, 2, 3\}$

$$V(P_3) = \{v_i|i = 1, 2, 3\}, \quad E(P_3) = \{v_1v_2, v_2v_3\}$$

another P_3, $V(P_3) = \{w_i|i = 1, 2, 3\}$, $E(P_3) = \{w_1w_2, w_2w_3\}$.

We define a mapping f as follows:

$$f(u_0u_i) = i,\ i = 1, 2, 3; \quad f(u_0v_i) = 3 + i,\ i = 1, 2, 3;$$

$$f(u_0w_i) = 6 + i,\ i = 1, 2, 3; \quad f(u_0) = 10;\ f(u_1) = 3;\ f(u_2) = 6;\ f(u_3) = 2;$$

$$f(v_2u_i) = 5 + i,\ i = 1, 2, 3; \quad f(w_2u_i) = i + 1,\ i = 1, 2, 3;$$

$$f(v_2w_1) = 9;\ f(v_2w_2) = 10;\ f(v_2w_3) = 1;\ f(v_2v_1) = 3;\ f(v_2v_3) = 2;$$

$$f(v_1) = 1;\ f(v_2) = 4;\ f(v_3) = 1; \quad f(w_2v_1) = 5;\ f(w_2v_3) = 7;$$

$$f(w_2w_1) = 1;\ f(w_2w_3) = 6;\ f(w_1) = 8;\ f(w_2) = 9;\ f(w_3) = 8;$$

So f is 10-TC of $S_3 \vee P_3 \vee P_3$. □

Theorem 4. *If* $n \geq 4$, *then* $\chi_t(S_n \vee P_n \vee P_n)$ *=3n+1.*

Proof. We seperate two situations:

Case 1. If $n \equiv 0(\ \mathrm{mod}\ 2)$, then it is ture by Theorem 2.

Case 2. If $n \equiv 1(\ \mathrm{mod}\ 2)$, then it is ture by Lemma 6.

So, Theorem 4 is ture. □

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