International Journal of Pure and Applied Mathematics
Volume 29 No. 3 2006, 349-382

GENERALIZED SOLUTION OF MIXED PROBLEMS FOR FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS WITH STATE DEPENDENT DELAYS

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Abstract: A theorem on the existence and continuous dependence upon initial-boundary conditions is given. The method of bicharacteristics is used to transform the mixed problem into a system of integral functional equations with operators of the Volterra type as a functional variable. A method of succesive approximations is used. The uniqueness of solutions of differential equation with the initial-boundary condition is proved by using a comparison technique. Classical solutions of integral functional equations lead to generalized solutions of the original problem. Differential equations with deviated variables and differential integral problems can be obtained from a general model by specializing given operators.

AMS Subject Classification: 35F20, 39B22
Key Words: functional differential equations, generalized solutions, comparison technique, method of succesive approximations, method of bicharacteristics

1. Introduction

Let $R_+ = [0, +\infty)$. We use vectorial inequalities with the understanding that the same inequalities hold between their corresponding components. For $h, a \in R_+$, and $b = (b_1, \dots, b_n), d = (d_1, \dots, d_n) \in R_+^n$, such that $b \le d$, and for $t \in [0, c]$, and $c \in (0, a]$ we define the sets as follows

Received: May 16, 2006

$$E_t = [0,t] \times [-b,b], \qquad \partial_0 E_t = \partial_0 E_t^+ \cup \partial_0 E_t^-,$$

where

$$\partial_0 E_t^+ = [0,t] \times [b,d], \qquad \partial_0 E_t^- = [0,t] \times [-d,-b].$$

Let $E_0 = [-h,0] \times [-d,d]$, and $D = [-h,0] \times [-r,r]$, where $r = (r_1,\ldots,r_n)$, $r = d - b$, and $\Omega_t = \partial_0 E_t \cup E_t$ for $t \in [0,c]$, $c \in (0,a]$. Let $C(X,Y)$ be the class of all continuous functions from X to Y. For any $x = (x_1,\ldots,x_n) \in R^n$ we define the norm $\sum_{i=1}^{n} |x_i|$. The symbol $\|\cdot\|_0$ means the maximum norm in the space $C(D,R)$. For a function $z : [-h,c] \times R^n \to R$, where $0 < c \le a$, and a point $(t,x) \in [0,c] \times R^n$ we define a function $z_{(t,x)} : D \to R$ by $z_{(t,x)}(\tau,y) = z(t+\tau, x+y)$ for $(\tau,y) \in D$. Suppose that

$$f : \Omega_a \times C(D,R) \times R^n \to R, \quad \varphi : [-h,0] \times [-d,d] \to R,$$

and

$$\Psi = (\Psi_0, \Psi_\star), \quad \Psi_\star = (\Psi_1,\ldots,\Psi_n) : [0,a] \times R^n \times C(D,R) \to R^n$$

are given functions. We also write that

$$\Psi(t,x,w) = (\Psi_0(t), \Psi_\star(t,x,w)) \quad \text{for} \quad (t,x,w) \in [0,a] \times R^n \times C(D,R).$$

We put

$$V[z](t,x) = z_{\Psi(t,x,z_{(t,x)})} = z_{(\Psi_0(t),\Psi_\star(t,x,z_{(t,x)}))}.$$

Note that the symbol $z_{\Psi(t,x,z_{(t,x)})}$ means the restriction of z to the set

$$[\Psi_0(t) - h, \Psi_0(t)] \times [\Psi_\star(t,x,z_{(t,x)}) - b, \Psi_\star(t,x,z_{(t,x)}) + b],$$

and this restriction is shifted to the set D.

Consider the following mixed problem

$$\partial_t z(t,x) = f(t,x,V[z](t,x), \partial_x z(t,x)) \quad \text{on} \quad E_c, \tag{1}$$

$$z(t,x) = \varphi(t,x) \quad \text{on} \quad E_0 \cup \partial_0 E_c, \tag{2}$$

where (1) is the nonlinear differential equation with state dependent delays, and (2) is the initial-boundary condition.

A function $\hat{z} : [-h,c] \times [-d,d] \to R$, $c \in (0,a]$, is a generalized solution of (1), (2), if $\hat{z}$ is a continuous function, and the following conditions for $\hat{z}$ hold:

(i) the derivatives $(\partial_{x_1}\hat{z},\ldots,\partial_{x_n}\hat{z}) = \partial_x \hat{z}$ exist on $[0,c] \times [-d,d]$,

(ii) $\hat{z}(\cdot,x) : [-h,c] \to R$ is absolutely continuous on $[0,c]$, and for $x \in [-d,d]$,

(iii) equation (1) is satisfied for $x \in [-d, d]$, and for almost all $t \in [0, c]$, and condition (2) holds.

Recent developments of hyperbolic functional defferential systems of equations are included in the monograph [18]. Different classes of weak solutions of initial, and mixed problems to partial differential equations are considered in many papers. General results on the uniqueness of semiclassical solutions for nonlinear partial differential equations with initial conditions are contained in [28]. Generalized solutions in Cinquini-Cibrario sense for equation without a functional variable were first consider in [7], [10], and [11]. Then it was also adopted for functional differential equation in [4], and [26]. This class of solutions is placed between classical solutions and solutions in Carathéodory sense. The monograph [33] treats about generalized solutions in the Cinquini-Cibrario sense for nonlinear equations, and systems without the functional dependence. Many papers addresses the existence, and uniqueness of quasilinear systems with initial-boundary conditions in the class of almost everywhere solutions (see [32]). The theory of Carathéodory solutions for such equations with the initial conditions is contained in [3], [25] and [27]. Also Carathéodory solutions are considered in [17], [18], [21]-[25], and [29]-[32]. The existence of Carathéodory solutions of functional differential problems with the unknown function of two variables is received by a difference method in [14].

In many papers different authors consider the equations

$$\partial_t z(t, x) = F(t, x, V[z](t, x), \partial_x z(t, x)),$$

where V is an operator of the Volterra type and F is defined on finite-demensional Euclidean space. Papers [3], and [19] concern the existence theory of these problems.

Equations in the second canonic form have been studied in a large number of papers, for example [1], [2], [5], [6], and [9]. Papers [25] and [27] initiated the study of the existence theory for first order functional partial differential equation with state dependent delays.

Differential functional equations with state dependent delays are used in technical science. Delay systems with state dependent delays occur as models for the dynamics of diseases when the mechanism of infection is such that the infections dosage received by an individual has to reach a threshold value before the resistance of the individual becomes infections. We can read about such mechanism in [12].

The bicharacteristics method and Banach Fixed Point Theorem were used for getting results of the existence of solutions of quasilinear hyperbolic functional differential systems in [13], [15], [16], [20], [25], [27], and also in this paper.

In our paper we use the quasilinearisation (which was also introduced in [8]) for getting results of the existence of solutions of mixed problem (1) (2). We will consider continuous dependence of solutions on initial-boundary conditions. We also use the method of succesive approximations based on the idea, which was first introduced by T. Ważewski in [34]. He used this method for systems without the functional dependence. We will define a sequence of functions $\{z^{(m)}\}$, where $z^{(m)} : [-h, c] \times [-d, d] \to R$, $m \in N \cup \{0\}$, such that $z^{(0)}$ is an arbitrary function satisfying adequate regularity conditions, and if $z^{(m)}$ is known, then $z^{(m+1)}$ is a solution of the Cauchy problem

$$\partial_t z(t,x) = F(t, x, z^{(m)}_{(t,x)}, \partial_x z(t,x)).$$

We will prove that under suitable assumptions on given functions the sequence $\{z^{(m)}\}$, $m \in N \cup \{0\}$, exists and it is convergent to a generalized solution of the original problem. We also give a theorem on the estimate of the difference between solutions of two initial problems of the type (1), (2).

2. Bicharacteristics

We define now some special function spaces needed to our considerations. Put $s = (s_0, s_1) \in R^2_+$, $k = (k_0, k_1) \in R^2_+$, and $p = (p_0, p_1) \in R^2_+$. Denote by $C^{1.L}[s]$ the set of all functions $\varphi \in C([-h, 0] \times R^n, R)$, such that:

(i) the partial derivatives $\partial_{x_i}\varphi$, $i = 1, \ldots, n$, exist on $[-h, 0] \times R^n$;

(ii) $\partial_x \varphi = (\partial_{x_1}\varphi, \ldots, \partial_{x_n}\varphi) \in C([-h, 0] \times R^n, R^n)$, and

$$|\varphi(t,x)| \le s_0, \qquad \|\partial_x \varphi(t,x)\| \le s_0,$$
$$\|\partial_x \varphi(t,x) - \partial_x \varphi(\bar{t}, \bar{x})\| \le s_1 \Big[|t - \bar{t}| + \|x - \bar{x}\|\Big]$$

on $[-h, 0] \times R^n$.

Let $\varphi \in C^{1.L}[s]$ be a given function, and let $0 < c \le a$. For $k_0 \ge s_0$, and $k_1 \ge s_1$ we write that $C^{1.L}_{\varphi.c}[k]$ is the class of all functions $z \in C([-h, 0] \times R^n, R)$, such that:

(i) $z(t,x) = \varphi(t,x)$ on $E_0 \cup \partial_0 E_c$;

(ii) there exists $\partial_x z = (\partial_{x_1}, \ldots, \partial_{x_n})$ on E, and we have estimations

$$|z(t,x)| \le k_0, \quad \text{and} \quad \|\partial_x z(t,x)\| \le k_0 \quad \text{on} \quad E_c,$$

and

$$|z(t,x) - z(\bar{t}, \bar{x})| \le k_1 \Big[|t - \bar{t}| + \|x - \bar{x}\|\Big],$$

$$\|\partial_x z(t,x) - \partial_x z(\bar{t},\bar{x})\| \le k_1\Big[|t-\bar{t}| + \|x-\bar{x}\|\Big],$$

for $(t,x),(\bar{t},\bar{x}) \in [-h,c]\times[-d,d]$, $c \in (0,a]$.

Let $\varphi \in C^{1.L}[s]$ be a given function. For $p_0 \ge s_0$, and $p_1 \ge s_1$ we write that $C^{0.L}_{\partial\varphi.c}[p]$ is the class of all functions $u \in C([-h,c]\times R^n, R^n)$, such that:

(i) $u(t,x) = \partial_x\varphi(t,x)$ on $[-h,0]\times R^n$;

(ii) $\|u(t,x)\| \le p_0$ on $[-h,c]\times R^n$;

(iii) $\|u(t,x)-u(\bar{t},\bar{x})\| \le p_1\Big[|t-\bar{t}| + \|x-\bar{x}\|\Big]$ for $(t,x),(\bar{t},\bar{x}) \in [0,c]\times R^n$.

For functions $z \in C([-h,c]\times R^n, R)$, and $u \in C([-h,c]\times R^n, R^n)$ we put

$$\|z\|_t = \sup\{|z(\tau,y)| : (\tau,y) \in [-h,t]\times R^n\},$$
$$\|u\|_t = \sup\{\|u(\tau,y)\| : (\tau,y) \in [-h,t]\times R^n\},$$

where $t \in [0,c]$.

We will prove that under suitable assumptions on given functions, and for sufficiently small $c \in (0,a]$, there exists a solution v of problem (1), (2), such that

$$v \in C^{1.L}_{\partial\varphi.c}[k], \quad \text{and} \quad \partial_x v \in C^{0.L}_{\partial\varphi.c}[p].$$

Put

$$\Delta^+ = \bigcup_{i=1}^n \Delta_i^+, \quad \Delta^- = \bigcup_{i=1}^n \Delta_i^-,$$

and

$$\Delta = \Delta^+ \cup \Delta^-,$$

where

$$\Delta_i^+ = \{x \in [-b,b] : x_i = b_i\}, \qquad \Delta_i^- = \{x \in [-b,b] : x_i = -b_i\},$$

and $i = 1,\ldots,n$.

Assumption $\boldsymbol{H[\Psi]}$. Suppose that:

(i) Ψ_0, and $\Psi_\star$ are continuous functions, and $0 \le \Psi_0(t) \le t$ for $t \in [0,a]$;

(ii) there exists $\bar{\beta} \in R_+$, such that

$$\|\Psi\star(t,x,w) - \Psi\star(t,\bar{x},\bar{w})\| \le \bar{\beta}\Big[\|x-\bar{x}\| + \|w-\bar{w}\|_0\Big]$$

on $[0,a]\times R^n \times C(D,R)$.

Assumption $\boldsymbol{H[\partial_q f]}$. Suppose that f is the function of the variables (t,x,w,q), and:

(i) $f(\cdot,x,w,q) : [0,a] \to R$ is measurable for $(x,w,q) \in R^n \times C(D,R)\times R^n$, and the partial derivatives $\partial_{q_i} f$, $i = 1,\ldots,n$, exist for $(x,w,q) \in \Omega_a$, and for almost all $t \in [0,a]$;

(ii) $\partial_q f(\cdot,x,w,q) = (\partial_{q_1} f(\cdot,x,w,q),\ldots,\partial_{q_n} f(\cdot,x,w,q)) : [0,a] \to R^n$ is measurable, and there exist $\tilde\alpha$, and $\bar\alpha \in R_+$, such that

$$\|\partial_q f(t,x,w,q)\| \le \tilde\alpha,$$

and

$$\|\partial_q f(t,x,w,q) - \partial_q f(t,\bar x,\bar w,\bar q)\| \le \bar\alpha\Big[\|x-\bar x\| + \|w-\bar w\|_0 + \|q-\bar q\|\Big]$$

for (x,w,q), $(\bar x,\bar w,\bar q) \in R^n \times C(D,R)\times R^n$, and for almost all $t\in[0,a]$;

(iii) there is $\kappa>0$, such that

$$\partial_{q_i} f(t,x,w,q) \ge \kappa \quad \text{for} \quad \Delta_i^+ \times C(D,R)\times R^n, \quad i=1,\ldots,n,$$

and

$$\partial_{q_i} f(t,x,w,q) \le -\kappa \quad \text{for} \quad \Delta_i^- \times C(D,R)\times R^n, \quad i=1,\ldots,n,$$

and for almost all $t\in[0,a]$.

Suppose that $\varphi \in C^{1.L}[s]$, $z \in C^{1.L}_{\varphi.c}[k]$, $u \in C^{0.L}_{\partial\varphi.c}[p]$. Consider the Cauchy problem

$$\eta'(\tau) = -\partial_q f(\tau,\eta(\tau),V[z](\tau,\eta(\tau)),u(\tau,\eta(\tau))), \tag{3}$$

$$\eta(t) = x, \tag{4}$$

where $(t,x)\in[0,c]\times R^n$. Let us denote by $g[z,u](\cdot,t,x)$ the solution of (3), (4). The function $g[z,u]$ is the bicharacteristic of problem (1), (2) corresponding to functions z, and u. Let $I_{(t,x)}$ be the domain of $g[z,u](\cdot,t,x)$, and $\tau[z,u](t,x)$ is the left end of the maximal interval, on which the bicharacteristic $g[z,u](\cdot,t,x)$ is defined.

We prove below lemmas on the existence, and uniqueness, and on the regularity of bicharacteristics.

Lemma 1. *Suppose that Assumptions $H[\Psi]$, $H[\partial_q f]$ are satisfied, $c\in(0,a]$, φ, $\bar\varphi \in C^{1.L}[s]$, $z\in C^{1.L}_{\varphi.c}[k]$, $\bar z \in C^{1.L}_{\bar\varphi.c}[k]$, $u\in C^{0.L}_{\partial\varphi.c}[p]$, and $\bar u \in C^{0.L}_{\partial\bar\varphi.c}[p]$. Then solutions $g[z,u](\cdot,t,x)$, and $g[\bar z,\bar u](\cdot,t,x)$ exist on the intervals $I_{(t,x)}$, and $\bar I_{(t,x)}$ respectively, and $g[z,u](\xi,t,x), g[\bar z,\bar u](\xi,t,x)\in\Delta$, where $\xi=\tau[z,u](t,x)$, and $\bar\xi = \tau[\bar z,\bar u](t,x)$. The bicharacteristics are unique on domains $I_{(t,x)}, \bar I_{(t,x)}$. Moreover we have estimates*

$$\|g[z,u](\tau,t,x) - g[z,u](\tau,\bar t,\bar x)\| \le \Theta_c\Big[\tilde\alpha|t-\bar t| + \|x-\bar x\|\Big] \tag{5}$$

for $\tau \in I_{(t,x)} \cap I_{(\bar{t},\bar{x})}$, $(t,x), (\bar{t},\bar{x}) \in E_c$, *and*

$$\|g[z,u](\tau,t,x) - g[\bar{z},\bar{u}](\tau,t,x)\| \leq \Theta_c \Big| \int_\tau^t [\|z-\bar{z}\|_\xi + \|u-\bar{u}\|_\xi] d\xi \Big| \tag{6}$$

for $\tau \in I_{(t,x)} \cap \bar{I}_{(t,x)}$, $(t,x) \in E_c$, *where*

$$\Theta_c = \max\left\{(1+k_1\bar{\beta})\bar{\alpha}, 1\right\} \exp(N_1\bar{\alpha}c), \tag{7}$$

and

$$N_1 = 1 + p_1 + k_1\bar{\beta}(1+k_1). \tag{8}$$

Proof. Note that

$$\|z_{(\tau,x)} - \bar{z}_{(\tau,\bar{x})}\|_0 \leq k_1\|x-\bar{x}\| + \|z-\bar{z}\|_\tau,$$

and

$$\|V[z](\tau,x) - V[\bar{z}](\tau,\bar{x})\|_0 \leq (1+k_1)k_1\bar{\beta}\|x-\bar{x}\| + (1+k_1\bar{\beta})\|z-\bar{z}\|_\tau$$

for $(\tau,x), (\tau,\bar{x}) \in [0,c] \times R^n$. The existence and uniqueness of the solution of the Cauchy problem (3), (4) follows from classical theorems. From Assumptions $H[\Psi]$ and $H[\partial_q f]$ it is easy to obtain the integral inequalities

$$\|g[z,u](\tau,t,x) - g[z,u](\tau,\bar{t},\bar{x})\| \leq \tilde{\alpha}|t-\bar{t}| + \|x-\bar{x}\|$$

$$+N_1\bar{\alpha}\Big|\int_\tau^t \|g[z,u](\xi,t,x) - g[z,u](\xi,\bar{t},\bar{x})\| d\xi\Big|,$$

for $\tau \in I_{(t,x)} \cap I_{(\bar{t},\bar{x})}$, $(t,x), (\bar{t},\bar{x}) \in E_c$, and

$$\|g[z,u](\tau,t,x,) - g[\bar{z},\bar{u}](\tau,t,x)\|$$

$$\leq \bar{\alpha}\left\{(1+p_1)\Big|\int_\tau^t \|g[z,u](\xi,t,x) - g[\bar{z},\bar{u}](\xi,t,x)\| d\xi\Big|\right.$$

$$\left. +(1+k_1\bar{\beta})\Big|\int_\tau^t [\|z-\bar{z}\|_\xi + \|u-\bar{u}\|_\xi] d\xi\Big|\right\}$$

for $\tau \in I_{(t,x)} \cap \bar{I}_{(t,x)}$ and $(t,x) \in E_c$. Hence by the Gronwall inequality we deduce (5), (6). □

Now we prove a lemma on a regularity of the function $\tau[z,u]$.

Lemma 2. *Suppose that Assumptions $H[\psi]$, $H[\partial_q f]$ are satisfied, $c \in (0,a]$, and φ, $\bar\varphi \in C^{1.L}[s]$, $z \in C^{1.L}_{\varphi.c}[k]$, $\bar z \in C^{1.L}_{\bar\varphi.c}[k]$, $u \in C^{0.L}_{\partial\varphi.c}[p]$ $\bar u \in C^{0.L}_{\partial\bar\varphi.c}[p]$. Then functions $\tau[z,u]$, $\tau[\bar z,\bar u]$ are continuous on E_c. Moreover the estimates*

$$|\tau[z,u](t,x) - \tau[z,u](\bar t,\bar x)| \le 2\Theta_c\kappa^{-1}\Big[\tilde\alpha|t-\bar t| + \|x-\bar x\|\Big], \tag{9}$$

$$|\tau[z,u](t,x) - \tau[\bar z,\bar u](t,x)| \le 2\Theta_c\kappa^{-1}|\int_0^t [\|z-\bar z\|_s + \|u-\bar u\|_s]ds| \tag{10}$$

hold for (t,x), $(\bar t,\bar x) \in E_c$, where Θ_c, and N_1 are given by formulas (7), and (8), respectively.

Proof. The continuity of functions $\tau[z,u]$, $\tau[\bar z,\bar u]$ on E_c follows from classical theorems of continuous dependence of initial problems for ordinary differential equations on given initial conditions. Now we prove (9). This estimate is obvious, if $\tau[z,u](t,x) = \tau[z,u](\bar t,\bar x) = 0$. We assume that $0 \le \tau[z,u](t,x) < \tau[z,u](\bar t,\bar x)$. Then for $\bar\xi = \tau[\bar z,\bar u](t,x)$ we have $g[z,u](\bar\xi,\bar t,\bar x) \in \Delta$. It is easily seen, that there are two possibilities:

$$g_i[z,u](\bar\xi,\bar t,\bar x) = b_i \quad \text{and} \quad g_i[z,u](\bar\xi,\bar t,\bar x) = -b_i.$$

Suppose that $g_i[z,u](\bar\xi,\bar t,\bar x) = b_i$, and let

$$\bar x = (x_1,\dots,x_{i-1},b_i,x_{i+1},\dots,x_n), \quad x = (x_1,\dots,x_{i-1},x_i,x_{i+1},\dots,x_n),$$

where $x_i \in (-b_i,b_i)$. Then

$$\begin{aligned} &|\partial_{q_i} f(t,x,V[z](t,x),u(t,x)) - \partial_{q_i} f(t,\bar x,V[z](t,\bar x,u(t,\bar x)))| \\ &\le \bar\alpha\Big[\|x-\bar x\| + \|V[z](t,x) - V[z](t,\bar x)\|_0 + \|u(t,x)-u(t,\bar x)\|\Big] \\ &\le \bar\alpha(1+(1+k_1)\bar\beta k_1 + p_1)\|x-\bar x\| \le \bar\alpha N_1(b_i - x_i). \end{aligned}$$

Let $b_i - x_1 \le \kappa(2N_1\bar\alpha)^{-1}$. Then

$$|\partial_{q_i} f(t,x,V[z](t,x),u(t,x)) - \partial_{q_i} f(t,\bar x,V[z](t,\bar x,u(t,\bar x)))| \le \frac{\kappa}{2}.$$

Because $\bar x \in \Delta_i^+$, so

$$\partial_{q_i} f(t,x,V[z](t,x),u(t,x)) \ge \frac{\kappa}{2},$$

if $x_i \ge b_i - \kappa(2N_1\bar\alpha)^{-1}$. Note that

$$b_i - g_i[z,u](\bar\xi,t,x) \le \|g[z,u](\xi,t,x) - g[z,u](\bar\xi,\bar t,\bar x)\|.$$

Using Lemma 1 we get the estimate

$$b_i - g_i[z,u](\bar{\xi},t,x) \le \Theta_c\Big[\tilde{\alpha}|t-\bar{t}| + \|x-\bar{x}\|\Big].$$

Provided that

$$\tilde{\alpha}|t-\bar{t}| + \|x-\bar{x}\| \le \kappa(2N_1\bar{\alpha}\Theta_c)^{-1}, \tag{11}$$

we get

$$b_i - g_i[z,u](\bar{\xi},t,x) \le \kappa(2N_1\bar{\alpha})^{-1},$$

and

$$\partial_{q_i} f(\bar{\xi}, g[z,u](\bar{\xi},t,x), V[z](\bar{\xi}, g[z,u](\bar{\xi},t,x)), u(\bar{\xi}, g[z,u](\bar{\xi},t,x))) \ge \frac{\kappa}{2} > 0.$$

Therefore

$$\partial_t g_i[z,u](\tau[z,u](\bar{t},\bar{x}),t,x) < 0,$$

provided that (11) holds. Of course

$$b_i - g_i[z,u](\tau,t,x) \le \kappa(2N_1\bar{\alpha})^{-1},$$

and

$$|\partial_{q_i} f(s,x,V[z](s,x),u(s,x)) - \partial_{q_i} f(s,\bar{x},V[z](s,\bar{x},u(s,\bar{x})))| \ge \frac{\kappa}{2}$$

for $s \in (\tau[z,u](t,x), \tau[z,u](\bar{t},\bar{x}))$, and if condition (11) holds. It is easy to get

$$-\frac{\kappa}{2}(\tau[z,u](\bar{t},\bar{x}), \tau[z,u](t,x))$$

$$\ge -\int\limits_{\tau[z,u](t,x)}^{\tau[z,u](\bar{t},\bar{x})} \Big[\partial_{q_i} f(s,x,V[z](s,x),u(s,x)) - \partial_{q_i} f(s,\bar{x},V[z](s,\bar{x},u(s,\bar{x})))\Big]ds$$

$$\ge g_i[z,u](\tau[z,u](\bar{t},\bar{x}),t,x) - g_i[z,u](\tau[z,u](\bar{t},\bar{x}),\bar{t},\bar{x})$$

$$\ge -\Theta_c\Big[\tilde{\alpha}|t-\bar{t}| + \|x-\bar{x}\|\Big],$$

and

$$\tau[z,u](\bar{t},\bar{x}) - \tau[z,u](t,x) \le 2\Theta_c\kappa^{-1}\Big[\tilde{\alpha}|t-\bar{t}| + \|x-\bar{x}\|\Big],$$

where (t,x), $(\bar{t},\bar{x}) \in E_c$, provided that (11) holds. For $g[z,u](t,x) = -b_i$, in a similar way, we get

$$\tau[z,u](t,x) - \tau[z,u](\bar{t},\bar{x}) \ge -2\Theta_c\kappa^{-1}\Big[\tilde{\alpha}|t-\bar{t}| + \|x-\bar{x}\|\Big],$$

and

$$|\tau[z,u](t,x)-\tau[z,u](\bar t,\bar x)| \geq 2\Theta_c\kappa^{-1}\Big[\tilde\alpha|t-\bar t| + \|x-\bar x\|\Big],$$

where (t,x), $(\bar t,\bar x) \in E_c$, provided that (11) holds. Let $c \in (0,a]$, and (t,x), $(\bar t,\bar x) \in E_c$ be chosen arbitrary. We define

$$M(t,\bar t,x,\bar x) = \tilde\alpha|t-\bar t| + \|x-\bar x\|.$$

Let $N(t,\bar t,x,\bar x) \in N$, such that

$$\kappa\Big[N(t,\bar t,x,\bar x)-1\Big](2N_1\bar\alpha\Theta_c)^{-1} \leq \kappa(2N_1\bar\alpha\Theta_c)^{-1}N(t,\bar t,x,\bar x),$$

and let

$$\varepsilon(t,\bar t,x,\bar x) = N^{-1}(t,\bar t,x,\bar x).$$

For $i=0,\dots,N(t,\bar t,x,\bar x)$ we write

$$x^{(i)} = i\varepsilon(t,\bar t,x,\bar x)x + (1-i\varepsilon(t,\bar t,x,\bar x))\bar x,$$

and

$$t^{(i)} = i\varepsilon(t,\bar t,x,\bar x)t + (1-i\varepsilon(t,\bar t,x,\bar x))\bar t.$$

Note that $(t^{(0)},x^{(0)}) = (\bar t,\bar x)$, $(t^{(N_0)},x^{(N_0)}) = (t,x)$, and

$$|t^{(i)}-t^{(i+1)}| + \|x^{(i)}-x^{(i+1)}\| \leq \kappa(2N_1\bar\alpha\Theta_c)^{-1}.$$

Of course

$$|t-\bar t| = \sum_{i=0}^{N(t,\bar t,x,\bar x)-1}|t^{(i)}-t^{(i+1)}|, \quad \text{and}$$
$$\|x-\bar x\| = \sum_{i=0}^{N(t,\bar t,x,\bar x)-1}\|x^{(i)}-x^{(i+1)}\|.$$

We get

$$\begin{aligned}
&|\tau[z,u](t,x)-\tau[z,u](\bar t,\bar x)|\\
&\qquad\leq \sum_{i=0}^{N(t,\bar t,x,\bar x)-1}|\tau[z,u](t^{(i)}),x^{(i)})-\tau[z,u](t^{(i+1)},x^{(i+1)})|\\
&\qquad\leq \sum_{i=0}^{N(t,\bar t,x,\bar x)-1}2\Theta_c\kappa^{-1}\Big[\tilde\alpha|t^{(i)}-t^{(i+1)}| + \|x^{(i)}-x^{(i+1)}\|\Big]
\end{aligned}$$

$$\leq 2\Theta_c \kappa^{-1}\Big[\tilde{\alpha}|t-\bar{t}| + \|x-\bar{x}\|\Big],$$

so estimate (9) is true for all (t,x), $(\bar{t},\bar{x}) \in E_c$.

Now we prove (10). We assume that

$$0 < \tau[z,u](t,x) \leq \tau[\bar{z},\bar{u}](t,x)$$

for $(t,x) \in E_c$, where $c \in (0,a]$. Of course $\bar{\xi} = \tau[\bar{z},\bar{u}](t,x)$, if $g[\bar{z},\bar{u}](\bar{\xi},t,x) \in \Delta$, and there exists $i = 1,\ldots,n$, such that $|g_i[\bar{z},\bar{u}](\bar{\xi},t,x)| = b_i$. Assume that $g_i[\bar{z},\bar{u}](\bar{\xi},t,x) = b_i$. Because

$$b_i - g_i[z,u](\bar{\xi},t,x) \leq \|g[z,u](\bar{\xi},t,x) - g[\bar{z},\bar{u}](\bar{\xi},t,x)\|,$$

so using Lemma 1 we get

$$b_i - g_i[z,u](\bar{\xi},t,x) \leq \bar{\alpha}\Theta_c \int_{\bar{\xi}}^{t} \Big[|z-\bar{z}|_s + \|u-\bar{u}\|_s\Big] ds.$$

Because of Assumption $H[\partial_q f]$ we have

$$|\partial_{q_i} f(t,x,V[z](t,x),u(t,x)) - \partial_{q_i} f(t,x,V[\bar{z}](t,x),\bar{u}(t,x))| \leq N_1\bar{\alpha}\Big[|z-\bar{z}|_t + \|u-\bar{u}\|_t\Big]. \quad (12)$$

Provided, that

$$|z-\bar{z}|_t + \|u-\bar{u}\|_t \leq \kappa(2\bar{\alpha}N_1 c\Theta_c)^{-1},$$

we have

$$b_i - g_i[z,u](\bar{\xi},t,x) \leq \kappa(2N_1)^{-1},$$

and

$$\partial_{q_i} f(\bar{\xi}, g[z,u](\bar{\xi},t,x), V[z](\bar{\xi}, g[z,u](\bar{\xi},t,x)), u(\bar{\xi}, g[z,u](\bar{\xi},t,x))) \geq \frac{\kappa}{2} > 0.$$

Therefore

$$\partial_t g_i[z,u](\tau[\bar{z},\bar{u}](t,x),t,x) < 0$$

for $(t,x) \in E_c$, provided that condition (2) holds. The function $g_i[z,u](\cdot, t,x)$ is decreasing on the interval $(\tau[z,u](t,x), \tau[\bar{z},\bar{u}](t,x))$. The estimates

$$b_i - g_i[z,u](\tau,t,x) \leq c\kappa(2N_1)^{-1},$$

and

$$\partial_{q_i} f(s, g[z,u](s,t,x), V[z](s, g[z,u](s,t,x)), u(s, g[z,u](s,t,x))) \geq \frac{\kappa}{2}$$

hold for $s \in (\tau[z,u](t,x), \tau[\bar{z},\bar{z}](t,x))$, $(t,x) \in E_c$, provided the condition (2) holds. We get

$$\begin{aligned} &-\frac{\kappa}{2}\Big[\tau[z,u](t,x) - \tau[\bar{z},\bar{u}](t,x)\Big] \geq \\ &-\int\limits_{\tau[z,u](t,x)}^{\tau[\bar{z},\bar{u}](t,x)} \partial_{q_i} f(s, g[z,u](s,t,x), V[z](s, g[z,u](s,t,x)), u(s, g[z,u](s,t,x)))ds \\ &\qquad \geq g_i[z,u](\tau[\bar{z},\bar{u}](t,x),t,x) - g_i[\bar{z},\bar{u}](\tau[\bar{z},\bar{u}](t,x),t,x) \\ &\qquad\qquad \geq -\Theta_c\bar{\alpha}\int\limits_0^t \Big[|z-\bar{z}|_s + \|u-\bar{u}\|_s\Big]ds, \end{aligned}$$

if condition (2) holds. We define

$$M_t(z,\bar{z},u,\bar{u}) = |z-\bar{z}|_t + \|u-\bar{u}\|_t, \quad t \in [0,c], \quad \text{and} \quad c \in (0,a].$$

There exists $K(t,\bar{t},x,\bar{x}) \in N$, such that

$$\begin{aligned} \kappa\Big[K(z,\bar{z},u,\bar{u}) - 1\Big](2\bar{\alpha}N_1\Theta_c)^{-1} &\leq M_t(z,\bar{z},u,\bar{u}) \\ &\leq \kappa K(z,\bar{z},u,\bar{u})(2\bar{\alpha}N_1\Theta_c)^{-1}. \end{aligned}$$

Let $\varepsilon(z,\bar{z},u,\bar{u}) = (K(z,\bar{z},u,\bar{u}))^{-1}$, and

$$z^{(j)} = j\varepsilon(z,\bar{z},u,\bar{u})\bar{z} + (1 - j\varepsilon(z,\bar{z},u,\bar{u}))z,$$

and

$$u^{(j)} = j\varepsilon(z,\bar{z},u,\bar{u})\bar{u} + (1 - j\varepsilon(z,\bar{z},u,\bar{u}))u,$$

where $j = 0,1,\ldots,K(z,\bar{z},u,\bar{u})$. Of course $z^{(0)} = z$, $u^{(0)} = u$, $z^{(K(z,\bar{z},u,}$ $\bar{u})) = \bar{z}$, $u^{(K(z,\bar{z},u,\bar{u}))} = \bar{u}$, and

$$|z^{(j)} - z^{(j+1)}| + \|u^{(j)} - u^{(j+1)}\| \leq \kappa(2\bar{\alpha}N_1 c\Theta_c)^{-1}$$

for $j = 0,\ldots,K(z,\bar{z},u,\bar{u}) - 1$, and also

$$|z-\bar{z}|_t = \sum_{j=0}^{K(z,\bar{z},u,\bar{u})-1} |z^{(j)} - z^{(j+1)}|_t, \quad \|u-\bar{u}\|_t = \sum_{j=0}^{K(z,\bar{z},u,\bar{u})-1} \|u^{(j)} - u^{(j+1)}\|_t.$$

We get

$$
\begin{aligned}
&|\tau[z,u](t,x)] - \tau[\bar z,\bar u](t,x)| \\
&\qquad \le \sum_{j=0}^{K(z,\bar z,u,\bar u)-1} |\tau[z^{(j)},u^{(j)}](t,x) - \tau[z^{(j+1)},u^{(j+1)}](t,x)| \\
&\qquad \le \sum_{j=0}^{K(z,\bar z,u,\bar u)-1} 2\Theta_c\kappa^{-1}\int_0^t \Big[|z^{(j)}-z^{(j+1)}|_s + \|u^{(j)}-u^{(j+1)}\|_s\Big]ds \\
&\qquad\qquad \le 2\Theta_c\kappa^{-1}\int_0^t \Big[|z-\bar z|_s + \|u-\bar u\|_s\Big]ds.
\end{aligned}
$$

Therefore estimate (10) holds for all $u \in C^{1.L}_{\varphi.c}[k]$, $\bar z \in C^{1.L}_{\bar\varphi.c}[k]$, $u \in C^{0.L}_{\partial\varphi.c}[p]$, $\bar u \in C^{0.L}_{\partial\bar\varphi.c}$, where φ, $\bar\varphi \in C^{1.L}[s]$, and $c \in (0,a]$. The proof is finished. $\square$

3. Uniqueness of Solutions

Consider problems (1), (2), and

$$\partial_t z(t,x) = \bar f(t,x,\bar V[z](t,x),\partial_x z(t,x)) \quad \text{on } E_a, \tag{13}$$

$$z(t,x) = \bar\varphi(t,x) \quad \text{on } E_0\cup\partial_0 E_a, \tag{14}$$

where

$$\bar V[z](t,x) = z_{\bar\Psi(t,x,z_{(t,x)})},$$

$$\bar f : \Omega_a \to R,\quad \bar\varphi : [-h,0]\times[-d,d] \to R,$$
$$\bar\Psi = (\bar\Psi_0,\bar\Psi_\star),\quad \text{and}$$
$$\bar\Psi_0 : [0,a]\to R,\quad \bar\Psi_\star : [0,a]\times[-d,d]\times C(B,R)\to R^n$$

are given functions.

Now we write some conditions for functions $\Gamma : [0,a]\times R_+ \to R_+$, and $\gamma : [0,a]\times R_+\to R_+$.

Assumption $\boldsymbol{H[\Gamma,\gamma]}$. Suppose that:

(i) Γ satisfies Carathéodory conditions;

(ii) γ is continuous, and $\Gamma(t,\cdot) : R_+\to R_+$, $\gamma(t,\cdot) : R_+\to R_+$ are nondecreasing for almost all $t\in[0,a]$;

(iii) for $\iota_0,\iota_1 \in R_+$, and $\delta \in C([0,a],R_+)$ the maximal solution of the problem

$$\zeta'(t) = \Gamma(t,\delta(t)+\iota_1\gamma(t,\zeta(t))),\quad \zeta(0)=\iota_0,$$

is defined on $[0,a]$.

We need some conditions for functions $\bar{f}$, $\bar{\Psi}$ to formulate a theorem on the uniqueness of solutions of problem (1), (2).

Assumption $H[\bar{f},\bar{\Psi}]$. Suppose that:

(i) $\bar{f}(\cdot,x,w,q)$ is measurable for $(x,w,q)\in[-d,d]\times C(B,R)\times R^n$, and $\bar{f}(t,\cdot):[-d,d]\times C(B,R)\times R^n$ is continuous for almost all $t\in[0,a]$;

(ii) functions $\bar{\Psi}_0$, and $\bar{\Psi}_\star$ are continuous, and $0\le\bar{\Psi}_0(t)\le t$ for $t\in[0,a]$, and there exist functions Γ, γ satisfying Assumption $H[\Gamma,\gamma]$, such that

$$|f(t,x,w,q)-\bar{f}(t,x,\bar{w},q)|\le\Gamma(t,\|w-\bar{w}\|_0),$$

and

$$\|\Psi_\star(t,x,w)-\bar{\Psi}_\star(t,x,\bar{w})\|\le\gamma(t,\|w-\bar{w}\|_0),$$

where (t,x,w), $(t,x,\bar{w})\in[0,a]\times[-d,d]\times C(B,R)$, and $q\in R^n$.

Theorem 1. *Suppose that φ, $\bar{\varphi}\in C^{1.L}[s]$, and Assumptions $H[\partial_q f]$, $H[\Psi]$, $H[\bar{f},\bar{\Psi}]$ are satisfied. Let $c\in(0,a]$, and $v,\bar{v}:[-h,c]\times[-d,d]\to R$ be solutions of problems (1), (2), and (13), (14), respectively. Suppose that $v\in C^{1.L}_{\varphi.c}[k]$, $\bar{\varphi}\in C^{1.L}_{\bar{\varphi}.c}[k]$, $\partial_x v\in C^{0.L}_{\partial\varphi.c}[p]$, and $\partial_x\bar{v}\in C^{0.L}_{\partial\bar{\varphi}.c}[p]$. Then*

$$\|v-\bar{v}\|_t\le\omega(t)\quad\text{on } [0,c], \tag{15}$$

where ω is the maximal solution of the problem

$$\zeta'(t)=\Gamma(t,\delta(t)+\iota_1\gamma(t,\zeta(t)))\quad\text{on}\quad E_c, \tag{16}$$

$$\zeta(0)=\|\varphi-\bar{\varphi}\|_0, \tag{17}$$

and $\delta(t)=\tilde{\alpha}t$.

Proof. We prove, that for the function $\tilde{\eta}=\|v-\bar{v}\|_t$, $t\in[0,c]$, we have the integral inequality

$$\tilde{\eta}(t)\le\tilde{\eta}(0)+\int_0^t\Gamma(s,\delta(s)+\iota_1\gamma(s,\tilde{\eta}(s)))ds,$$

$t\in[0,c]$. From (1), (13), and Hadamard Theorem it follows, that

$$\partial_t(v-\bar{v})=\sum_{j=1}^n\int_0^t\partial_{q_j}f(\tilde{Q}(t,x,\xi))d\xi\partial_{x_j}(v-\bar{v})(t,x)$$

$$+f(t,x,V[v](t,x),\partial_x\bar{v}(t,x))-\bar{f}(t,x,\bar{V}[\bar{v}](t,x),\partial_x\bar{v}(t,x)),$$

where
$$\tilde{Q}(t,x,\xi) = (t,x,V[v](t,x), \partial_x\bar{v}(t,x) + \xi\partial_x(v-\bar{v})(t,x)).$$
Let $g(\cdot,t,x)$ means a solution of the problem
$$\eta'(t) = -\int_0^1 \partial_q f(\tilde{Q}(s,\eta(s),\xi))d\xi, \quad \eta(t) = x.$$
We get
$$(v-\bar{v})(t,x) = (v-\bar{v})(0,\tilde{g}(0,t,x))$$
$$+\int_0^t f(s,\tilde{g}(s,t,x),V[v](s,\tilde{g}(s,t,x)),\partial_x\bar{v}(s,\tilde{g}(s,t,x)))ds$$
$$-\int_0^t f(s,\tilde{g}(s,t,x),\bar{V}[\bar{v}](s,\tilde{g}(s,t,x)),\partial_x\bar{\bar{v}}(s,\tilde{g}(s,t,x)))ds.$$
Because of Assumptions $H[\bar{f},\bar{\Psi}]$, $H[\Psi]$ it is easy to obtain the estimate
$$\|V[v](s,y) - \bar{V}[\bar{v}](s,y)\|_0 \le \tilde{\alpha}s + \iota_1\gamma(s,\|v-\bar{v}\|_s).$$
So we have
$$\|v-\bar{v}\|_t \le \|v-\bar{v}\|_0 + \int_0^t \Gamma(s,\tilde{\alpha}s + \iota_1\gamma(s,\|v-\bar{s}\|_s))ds,$$
and therefore
$$\tilde{\eta}(t) \le \tilde{\eta}(0) + \int_0^t \Gamma(s,\delta(s) + \iota_1\gamma(s,\|v-\bar{v}\|_s))ds.$$
Because ω is the maximal solution of the problem (13), (14), so now using the classical theorem on the ordinary differential inequalities we get (15). The proof is finished. □

Theorem 2. *Suppose that* φ, $\bar{\varphi} \in C^{1.L}[s]$, *and Assumptions* $H[\partial_q f]$, $H[\Psi]$, $H[\bar{f},\bar{\Psi}]$, $H[\Gamma,\gamma]$ *are satisfied, and* $c \in (0,a]$. *Then problem (1), (2) has at most one solution* v, *such that* $v \in C^{1.L}_{\varphi.c}[k]$, *and* $\partial_x v \in C^{0.L}_{\partial\varphi.c}[p]$.

Theorem 2 is the consequence of Theorem 1 for $f = \bar{f}$, and $\Psi = \bar{\Psi}$.

4. Functional Integral Equations

Now we need some more assumptions to write the next lemma.

Assumption *H*[*D*Ψ]. Suppose that Assumption $H[\Psi]$ is satisfied, and:

(i) the partial derivatives

$$\Big[\partial_{x_j}\Psi_i(t,x,w)\Big]_{i,j=1,\dots,n}=\partial_x\Psi_\star(t,x,w),$$

and Frechet's derivatives

$$\partial_w\Psi_\star(t,x,w)=(\partial_w\Psi_1(t,x,w),\dots,\partial_w\Psi_n(t,x,w))$$

exist on $[0,a]\times[-d,d]\times C(D,R)$;

(ii) there exists $\tilde\beta\in R_+$, such that

$$\|\partial_x\Psi_\star(t,x,w)\|\le\tilde\beta,\quad \|\partial_w\Psi_\star(t,x,w)\|\le\tilde\beta$$

for $(t,x,w)\in[0,a]\times[-d,d]\times C(D,R)$;

(iii) the following estimates hold:

$$\|\partial_x\Psi_\star(t,x,w)-\partial_x\Psi_\star(t,\bar x,\bar w)\|\le\bar\beta\Big[\|x-\bar x\|+\|w-\bar w\|_0\Big],$$
$$\|\partial_w\Psi_\star(t,x,w)-\partial_w\Psi_\star(t,\bar x,\bar w)\|\le\bar\beta\Big[\|x-\bar x\|+\|w-\bar w\|_0\Big]$$

for (t,x,w), $(t,\bar x,\bar w)\in[0,a]\times[-d,d]\times C(D,R)$.

Assumption *H*[*f*]. Suppose that Assumption $H[\partial_q f]$ is satisfied, and:

(i) the partial derivatives

$$\partial_x f(\tilde P)=(\partial_{x_1}f(\tilde P),\dots,\partial_{x_n}f(\tilde P)),$$

where $\tilde P=(t,x,w,q)$, and the Frechet's derivative $\partial_w f(\tilde P)$ exists for $(x,w,q)\in[-d,d]\times C(D,R)\times R^n$, and for almost all $t\in[0,a]$;

(ii) $|f(t,x,p,q)|\le\tilde\alpha$, $\|\partial_x f(t,x,p,q)\|\le\tilde\alpha$, $\|\partial_w f(t,x,p,q)\|\le\tilde\alpha$ for (x,p,q)
$\in[-d,d]\times C(D,R)\times R^n$, and for almost all $t\in[0,a]$;

(iii) the following estimates hold:

$$\|\partial_x f(t,x,w,q)-\partial_x f(t,\bar x,\bar w,\bar q)\|\le\bar\alpha\Big[\|x-\bar x\|+\|w-\bar w\|_0+\|q-\bar q\|\Big],$$
$$\|\partial_w f(t,x,w,q)-\partial_w f(t,\bar x,\bar w,\bar q)\|\le\bar\alpha\Big[\|x-\bar x\|+\|w-\bar w\|_0+\|q-\bar q\|\Big],$$

for (x,w,q), $(\bar x,\bar w,\bar q)\in[-d,d]\times C(D,R)\times R^n$, and for almost all $t\in[0,a]$.

Now we formulate a system of integral functional equations, generated by problem (1), (2). Put

$$P[z,u](\tau,t,x) = (\tau, g[z,u](\tau,t,x), V[z](\tau, g[z,u](\tau,t,x)), u(\tau, g[z,u](\tau,t,x))),$$

and

$$W[z,u](t,x) = \sum_{j=1}^{n} (u_j)_{\Psi(t,x,z_{(t,x)})} \Big[\partial_x \Psi_j(t,x,z_{(t,x)}) + \partial_w \Psi_j(t,x,z_{(t,x)})(u_j)_{(t,x)}\Big],$$

and $W = (W_1, \dots, W_n)$. Let

$$\begin{aligned} Q[z,u](t,x) &= (\tau[z,u](t,x), g[z,u](\tau[z,u](t,x),t,x)), \\ \Phi[z,u](t,x) &= \varphi(Q[z,u](t,x)), \\ \Theta[z,u](t,x) &= \partial_x \varphi(Q[z,u](t,x)), \end{aligned}$$

where $\Theta = (\Theta_1, \dots, \Theta_n)$.

Suppose that $\varphi \in C^{1.L}[s]$, $z \in C^{1.L}_{\varphi.c}[k]$, and $u \in C^{0.L}_{\partial\varphi.c}[p]$, where $c \in (0,a]$, are given functions. We define

$$\begin{aligned} F[z,u](t,x) = \Phi[z,u](t,x) &+ \int_{\tau[z,u](t,x)}^{t} f(P[z,u](s,t,x))ds \\ &- \sum_{j=1}^{n} \int_{\tau[z,u](t,x)}^{t} f(P[z,u](s,t,x)) u_j(s, g[z,u](s,t,x))ds, \end{aligned} \tag{18}$$

and

$$\begin{aligned} G[z,u](t,x) = \Theta[z,u](t,x) &+ \int_{\tau[z,u](t,x)}^{t} \Big[\partial_x f(P[z,u](s,t,x)) \\ &+ \partial_w f(P[z,u](s,t,x)) W[z,u](s, g[z,u](s,t,x))\Big] ds. \end{aligned} \tag{19}$$

Consider the following system of equations:

$$z(t,x) = F[z,u](t,x), \tag{20}$$

$$u(t,x) = G[z,u](t,x), \tag{21}$$

$$g[z,u](s,t,x) = x + \int_s^t \partial_q f(P[z,u](\xi,t,x))d\xi \tag{22}$$

with the initial-boundary condition

$$z(t,x) = \varphi(t,x), \tag{23}$$

$$u(t,x) = \partial_x \varphi(t,x) \tag{24}$$

on $E_0 \cup \partial_0 E_c$, $c \in (0,a]$. The existence of solutions of system (20), (23), and (23), (24) will be proved by method of succesive approximations. Suppose that $\varphi \in C^{1.L}[s]$ and Assumptions $H[f]$, $H[D\Psi]$ are satisfied. We define sequences $\{z^{(m)}\}$, $\{u^{(m)}\}$, $m \in N \cup \{0\}$, in the following way: let $z^{(0)} \in C^{1.L}_{\varphi.c}[k]$, $u^{(0)} \in C^{0.L}_{\partial\varphi.c}[p]$ be given functions, such that

$$\partial_x z^{(0)}(t,x) = u^{(0)}(t,x) \quad \text{on } E_c. \tag{25}$$

If $z^{(m)} \in C^{1.L}_{\varphi.c}[k]$, $u^{(m)} \in C^{0.L}_{\partial\varphi.c}[p]$ are known functions, then the functions $z^{(m+1)}$, $u^{(m+1)}$ we can find by the system

$$z^{(m+1)}(t,x) = F[z^{(m)}, u^{(m+1)}](t,x) \quad \text{on } E_c,$$
$$z^{(m+1)}(t,x) = \varphi(t,x) \quad \text{on } E_0 \cup \partial_0 E_c,$$

where $c \in (0,a]$, and $u^{(m+1)}$ is a solution of equation $u = G^{(m)}[u]$, and

$$G^{(m)}[u](t,x) = \Theta[z^{(m)},u](t,x) + \int_{\tau[z^{(m)},u](t,x)}^{t} \Big[\partial_x f(P[z^{(m)},u](s,t,x)) \tag{26}$$

$$+\partial_w f(P[z^{(m)},u](s,t,x)) W[z^{(m)},u^{(m)}](s, g[z^{(m)},u](s,t,x))\Big] ds.$$

Further we prove, that sequences $\{z^{(m)}\}$, $\{u^{(m)}\}$, $m \in N \cup \{0\}$, exist and are uniformly convergent on $[-h,c] \times [-d,d]$ for sufficienly small $c \in (0,a]$. To prove this fact we need some more assumptions.

Assumption $H[k,p]$. Suppose that:

(i) $p_0 > s_0$;

(ii) $p_1 > s_2 \Theta_c \{2(1+\tilde{\alpha})\kappa^{-1} + 1\} \max\{1, \tilde{\alpha}\}$;

Let T_φ, where $\varphi \in C^{1.L}[s]$, be a set of all functions $\omega : E_a \to R$, satisfying following conditions:

(i) ω is continuous;

(ii) $\omega\Big|_{E_0\cup\partial_0 E_a} = \varphi$.

Put $\mathcal{I} = \{i : d_i > 0\}$. We write next assumption needed to formulate lemma 3.

Assumption $\boldsymbol{H[f,\varphi]}$. Suppose that:

(i) if ω, $\tilde\omega \in T_\varphi$, then $f(t,x,V[\omega],q) = f(t,x,V[\tilde\omega],q)$ for $x \in \Delta$, $q \in R^n$, and for almost all $t \in [0,a]$;

(ii) there exists a function $\varrho : E_0 \cup \partial_0 E_a \to R^n$, $\varrho = (\varrho_1,\dots,\varrho_n)$, such that

$$\partial_t\varphi(t,x) = f(t,x,V[\tilde\varphi](t,x),\varrho(t,x))$$

for $x \in \Delta$, and for almost all $t \in [0,a]$, where φ, $\tilde\varphi \in T_\varphi$, and

$$\varrho_i(t,x) = \partial_{x_i}\varphi(t,x)$$

for $(t,x) \in [0,a] \times (\Delta_i^+ \cup \Delta_i^-)$, $i \in \mathcal{I}$.

Lemma 3. *Suppose that $\varphi \in C^{1.L}[s]$, and Assumptions $H[f]$, $H[D\Psi]$, $H[f,\varphi]$, $H[k,p]$ are satisfied. Then there exists $c \in (0,a]$, such that for any $m \in N \cup \{0\}$ the following conditions hold:*

(I_m) functions $z^{(m)}$, $u^{(m)}$ are defined on $[-h,c]\times[-d,d]$, and $z^{(m)} \in C^{1.L}_{\varphi.c}[k]$, $u^{(m)} \in C^{0.L}_{\partial\varphi.c}[p]$,

(II_m) $\partial_x z^{(m)}(t,x) = u^{(m)}(t,x)$ on $[0,c]\times[-d,d]$.

Proof. We will prove conditions (I_m), and (II_m) by induction. From (25) conditions (I_0), (II_0) hold. Suppose that (I_i), (II_i) are satisfied for $i = 1,\dots,m$, $m \in N$. Using the Assumption $H[k,p]$ it is easy to obtain that there exist such small $c \in (0,a]$, that

$$G^{(m)} : C^{0.L}_{\partial\varphi.c}[p] \to C^{0.L}_{\partial\varphi.c}[p].$$

We prove that $G^{(m)}$ is a contraction on $C^{0.L}_{\partial\varphi.c}[p]$. We define the norm

$$\Big[|u-\bar u|\Big]_\lambda = \sup\Big\{\|u-\bar u\|_t \exp(-\lambda t) : (t,x) \in E_c\Big\}.$$

From Assumptions $H[f]$ and $H[D\Psi]$ we get

$$\|G^{(m)}[u](t,x) - G^{(m)}[\tilde u](t,x)\| \le K_c\Big|\int_0^t \|u-\tilde u\|_s ds\Big|,$$

where

$$K_c = M_1 + cM_2,$$

and
$$M_1 = 2s_1\Theta_c\kappa^{-1}\left\{1+\tilde{\alpha}[1+p_0\tilde{\beta}(1+p_0)]\right\},$$

$$M_2 = \Theta_c\Big\{\bar{\alpha}(1+p_1)[1+p_0\tilde{\beta}(1+p_0)] + \tilde{\alpha}p_0\bar{\beta}(1+k_1)(1+p_1) \\ + \tilde{\alpha}\tilde{\beta}p_1[p_0+\bar{\beta}(1+p_0)(1+k_1)]\Big\},$$

and $u,\ \tilde{u} \in C^{0.L}_{\partial\varphi.c}[p]$, $(t,x) \in [0,c]\times[-d,d]$. For $\lambda > K_c$ we get

$$\|G^{(m)}[u](t,x) - G^{(m)}[\tilde{u}](t,x)\| \\ \leq K_c\Big[|u-\tilde{u}|\Big]_\lambda \Big|\int_0^t \exp(\lambda s)ds\Big| \leq K_c\lambda^{-1}\Big[|u-\tilde{u}|\Big]_\lambda \exp(\lambda t),$$

and
$$\Big[|G^{(m)}[u] - G^{(m)}[\tilde{u}]|\Big]_\lambda \leq K_c\lambda^{-1}\Big[|u-\tilde{u}|\Big]_\lambda.$$
Therefore there exists $R_1 \in (0,1)$, such that
$$\Big[|G^{(m)}[u] - G^{(m)}[\tilde{u}]|\Big]_\lambda \leq R_1\Big[|u-\tilde{u}|\Big]_\lambda,$$

so indeed $G^{(m)}$ is a contraction. Now using the Banach Fixed Point Theorem there exists exactly one $\bar{u} \in C^{0.L}_{\partial\varphi.c}[p]$, such that $\bar{u} = G^{(m)}[\bar{u}]$. Put

$$\Delta^{(m)}(t,x,\bar{x}) = z^{(m+1)}(t,\bar{x}) - z^{(m+1)}(t,x) - u^{(m+1)}(t,x)(\bar{x}-x) \quad \text{on } E_c.$$

We prove now that there exists $R_2 \in R_+$, such that
$$|\Delta^{(m)}(t,x,\bar{x})| \leq R_2\|x-\bar{x}\|^2 \quad \text{on } E_c, \tag{27}$$

what is equivalent to condition (II_{m+1}). Of course

$$\Delta^{(m)}(t,x,\bar{x}) = F[z^{(m)},u^{(m+1)}](t,\bar{x}) \\ - F[z^{(m)},u^{(m+1)}](t,x) - G^{(m)}[u^{(m+1)}](t,x)(\bar{x}-x).$$

We put some new symbols to make our considerations a little easier. Let

$$g^{(m)}(s,t,x) = g[z^{(m)},u^{(m+1)}](s,t,x),$$
$$\tau^{(m)}(t,x) = \tau[z^{(m)},u^{(m+1)}](t,x),$$
$$S^{(m)}(t,x) = (\tau^{(m)}(t,x), g^{(m)}(\tau^{(m)}(t,x),t,x)),$$

and

$$Q^{(m)}(s,\xi,t,x,\bar{x}) = sP[z^{(m)},u^{(m+1)}](\xi,t,\bar{x}) + (1-s)P[z^{(m)},u^{(m+1)}](\xi,t,x).$$

Suppose that $\tau^{(m)}(t,\bar{x}) \leq \tau^{(m)}(t,x)$ (the case $\tau^{(m)}(t,\bar{x}) > \tau^{(m)}(t,x)$ is analogous). Let also

$$A^{(m)}(t,x,\bar{x}) = \varphi(S^{(m)}(t,\bar{x})) - \varphi(S^{(m)}(t,x))$$
$$-\partial_x\varphi(S^{(m)}(t,x))\Big[g^{(m)}(\tau^{(m)}(t,\bar{x}),t,\bar{x}) - g^{(m)}(\tau^{(m)}(t,x),t,x)\Big]$$
$$-\partial_t\varphi(S^{(m)}(t,x))\Big[\tau^{(m)}(t,\bar{x}) - \tau^{(m)}(t,x)\Big],$$

$$B^{(m)}(t,x,\bar{x}) = \partial_t\varphi(S^{(m)}(t,x))\Big[\tau^{(m)}(t,\bar{x}) - \tau^{(m)}(t,x)\Big]$$
$$+\partial_x\varphi(S^{(m)}(t,x))\Big[g^{(m)}(\tau^{(m)}(t,\bar{x}),t,\bar{x}) - g^{(m)}(\tau^{(m)}(t,x),t,x) - (\bar{x}-x)\Big],$$

$$C^{(m)}(\tau,t,x,\bar{x})$$
$$= \int_\tau^t \Big[\partial_q f(P[z^{(m)},u^{(m+1)}](s,t,\bar{x})) - \partial_q f(P[z^{(m)},u^{(m+1)}](s,t,x))\Big]ds,$$

and

$$\Lambda^{(m)}(t,x,\bar{x}) = \int_{\tau^{(m)}(t,\bar{x})}^{\tau^{(m)}(t,x)} \Big[f(P[z^{(m)},u^{(m+1)}](s,t,\bar{x}))$$
$$-\partial_q f(P[z^{(m)},u^{(m+1)}](s,t,\bar{x}))u^{(m+1)}(s,g^{(m)}(s,t,\bar{x}))\Big]ds.$$

Then we get

$$\Delta^{(m)} = \varphi(S^{(m)}(t,\bar{x})) - \varphi(S^{(m)}(t,x))$$
$$+\int_{\tau^{(m)}(t,x)}^t \Big[f(P[z^{(m)},u^{(m+1)}](s,t,\bar{x})) - f(P[z^{(m)},u^{(m+1)}](s,t,x))\Big]ds$$
$$-\int_{\tau^{(m)}(t,x)}^t \partial_q f(P[z^{(m)},u^{(m+1)}](s,t,\bar{x}))u^{(m+1)}(s,g^{(m)}(s,t,\bar{x}))ds$$
$$+\int_{\tau^{(m)}(t,x)}^t \partial_q f(P[z^{(m)},u^{(m+1)}](s,t,x))u^{(m+1)}(s,g^{(m)}(s,t,x))ds$$

$$+\Lambda^{(m)}(t,x,\bar{x}) - G^{(m)}[u^{(m+1)}](t,x)(\bar{x}-x).$$

Now let

$$\Delta^{(m)}(t,x,\bar{x}) = \tilde{\Delta}^{(m)}(t,x,\bar{x}) + \bar{\Delta}^{(m)}(t,x,\bar{x}),$$

where

$$\begin{aligned}
\tilde{\Delta}^{(m)}(t,x,\bar{x}) &= B^{(m)}(t,x,\bar{x}) \\
&+ \int_{\tau^{(m)}(t,x)}^{t} \partial_x f(P[z^{(m)},u^{(m+1)}](s,t,x))\Big[g^{(m)}(s,t,\bar{x}) - g^{(m)}(s,t,x) - (\bar{x}-x)\Big]ds \\
&+ \int_{\tau^{(m)}(t,x)}^{t} \partial_w f(P[z^{(m)},u^{(m+1)}](s,t,x))W[z^{(m)},u^{(m+1)}](s,t,x)\Big[g^{(m)}(s,t,\bar{x}) \\
&\qquad - g^{(m)}(s,t,x)\Big]ds \\
&- \int_{\tau^{(m)}(t,x)}^{t} \partial_w f(P[z^{(m)},u^{(m+1)}](s,t,x))W[z^{(m)},u^{(m+1)}](s,t,x)(\bar{x}-x)ds \\
&- \int_{\tau^{(m)}(t,x)}^{t} \Big[\partial_q f(P[z^{(m)},u^{(m+1)}](s,t,\bar{x})) \\
&- \partial_q f(P[z^{(m)},u^{(m+1)}](s,t,x))\Big]u^{(m+1)}(s,g^{(m)}(s,t,x))ds + \Lambda^{(m)}(t,x,\bar{x}),
\end{aligned}$$

and

$$\begin{aligned}
\bar{\Delta}^{(m)}(t,x,\bar{x}) &= A^{(m)}(t,x,\bar{x}) \\
&+ \int_{\tau^{(m)}(t,x)}^{t} \int_0^t \Big[\partial_x f(Q^{(m)}(\xi,s,t,x,\bar{x})) - \partial_x f(P[z^{(m)},u^{(m+1)}](s,t,x))\Big]d\xi \\
&\times g^{(m)}(s,t,\bar{x})ds - \int_{\tau^{(m)}(t,x)}^{t} \int_0^t \Big[\partial_x f(Q^{(m)}(\xi,s,t,x,\bar{x})) \\
&- \partial_x f(P[z^{(m)},u^{(m+1)}](s,t,x))\Big]d\xi g^{(m)}(s,t,x)ds \\
&+ \int_{\tau^{(m)}(t,x)}^{t} \int_0^t \Big[\partial_w f(Q^{(m)}(\xi,s,t,x,\bar{x}))
\end{aligned}$$

$$- \partial_w f(P[z^{(m)}, u^{(m+1)}](s,t,x))\Big] d\xi V[z^{(m)}](s, g^{(m)}(s,t,\bar{x}))ds$$

$$- \int_{\tau^{(m)}(t,x)}^{t} \int_0^t \Big[\partial_w f(Q^{(m)}(\xi,s,t,x,\bar{x}))$$

$$- \partial_w f(P[z^{(m)}, u^{(m+1)}](s,t,x))\Big] d\xi V[z^{(m)}](s, g^{(m)}(s,t,x))ds$$

$$+ \int_{\tau^{(m)}(t,x)}^{t} \int_0^t \Big[\partial_q f(Q^{(m)}(\xi,s,t,x,\bar{x}))$$

$$- \partial_q f(P[z^{(m)}, u^{(m+1)}](s,t,\bar{x}))\Big] d\xi u^{(m+1)}(s, g^{(m)}(s,t,\bar{x}))ds$$

$$- \int_{\tau^{(m)}(t,x)}^{t} \int_0^t \Big[\partial_q f(Q^{(m)}(\xi,s,t,x,\bar{x}))$$

$$- \partial_q f(P[z^{(m)}, u^{(m+1)}](s,t,\bar{x}))\Big] d\xi u^{(m+1)}(s, g^{(m)}(s,t,x))ds$$

$$+ \int_{\tau^{(m)}(t,x)}^{t} \partial_w f(P[z^{(m)}, u^{(m+1)}](s,t,x)) V[z^{(m)}](s, g^{(m)}(s,t,\bar{x}))$$

$$- V[z^{(m)}](s, g^{(m)}(s,t,x))ds$$

$$- \int_{\tau^{(m)}(t,x)}^{t} \partial_w f(P[z^{(m)}, u^{(m+1)}](s,t,x)) W[z^{(m)}, u^{(m+1)}](s,t,x)$$

$$\times \Big[g^{(m)}(s,t,\bar{x}) - g^{(m)}(s,t,x)\Big] ds.$$

Note that

$$g^{(m)}(s,t,\bar{x}) - g^{(m)}(s,t,x) - (\bar{x} - x) = C^{(m)}(s,t,x,\bar{x}).$$

Therefore by changing the order of integrating it is easy to obtain, that

$$\tilde{\Delta}^{(m)}(t,x,\bar{x}) = D^{(m)}(t,x,\bar{x})$$

$$+ \int_{\tau^{(m)}(t,x)}^{t} \Big[\partial_q f(P[z^{(m)}, u^{(m+1)}](\xi,s,t,\bar{x})) - \partial_q f(P[z^{(m)}, u^{(m+1)}](s,t,x))\Big]$$

$$\times E^{(m)}(s,t,x)ds,$$

where

$$D^{(m)}(t,x,\bar{x}) = \int\limits_{\tau^{(m)}(t,\bar{x})}^{\tau^{(m)}(t,x)} \Big[f(P[z^{(m)},u^{(m+1)}](s,t,\bar{x})) - \partial_t\varphi(S^{(m)}(t,x))\Big]ds$$
$$+ \int\limits_{\tau^{(m)}(t,\bar{x})}^{\tau^{(m)}(t,x)} \Big[\partial_x\varphi(S^{(m)}(t,x)) - u^{(m+1)}(s,g^{(m)}(s,t,\bar{x}))\Big]$$
$$\times\, \partial_q f(P[z^{(m)},u^{(m+1)}](s,t,\bar{x}))ds,$$

and

$$E^{(m)}(s,t,x) = -u^{(m+1)}(s,g^{(m)}(s,t,x)) + \partial_x\varphi(S^{(m)}(t,x))$$
$$+ \int\limits_{\tau^{(m)}(t,x)}^{t} \Big[\partial_x f(P[z^{(m)},u^{(m+1)}](s,t,\bar{x}))$$
$$+\partial_w f(P[z^{(m)},u^{(m+1)}](s,t,\bar{x}))W[z^{(m)},u^{(m+1)}](s,t,x)\Big]ds.$$

Of course $E^{(m)}(s,t,x) = 0$ for $(t,x) \in E_c$, and $s \in I_{(t,x)}$. So $\tilde{\Delta}(t,x,\bar{x}) = D^{(m)}(t,x,\bar{x})$. It is easy to note, that there exists $\tilde{R} \in R_+$, such that

$$|D^{(m)}(t,x,\bar{x})| \leq \tilde{R}\|x-\bar{x}\|^2,$$

and consequently

$$|\tilde{\Delta}^{(m)}(t,x,\bar{x})| \leq \tilde{R}\|x-\bar{x}\|^2 \quad \text{on } E_c.$$

Now we write the estimate for $|\bar{\Delta}^{(m)}(t,x,\bar{x})|$. Note, that there exists $R_3 \in R_+$, such that

$$\|\partial_x f(P[z^{(m)},u^{(m+1)}](s,t,\bar{x})) - \partial_x f(P[z^{(m)},u^{(m+1)}](s,t,x))\|$$
$$\leq R_3\|g^{(m)}(s,t,\bar{x}) - g^{(m)}(s,t,x)\|,$$
$$\|\partial_w f(P[z^{(m)},u^{(m+1)}](s,t,\bar{x})) - \partial_w f(P[z^{(m)},u^{(m+1)}](s,t,x))\|$$
$$\leq R_3\|g^{(m)}(s,t,\bar{x}) - g^{(m)}(s,t,x)\|,$$
$$\|\partial_q f(P[z^{(m)},u^{(m+1)}](s,t,\bar{x})) - \partial_q f(P[z^{(m)},u^{(m+1)}](s,t,x))\|$$
$$\leq R_3\|g^{(m)}(s,t,\bar{x}) - g^{(m)}(s,t,x)\|.$$

Using the Taylor's formula for the function φ we get

$$\|A^{(m)}(s,t,x)\| \leq s_1\|g^{(m)}(s,t,\bar{x}) - g^{(m)}(s,t,x)\|^2$$

and we conclude, that there exists $R_4 \in R_+$, such that

$$\|V[z^{(m)}](s, g^{(m)}(s,t,\bar{x})) - V[z^{(m)}](s, g^{(m)}(s,t,x))$$

$$-W[z^{(m)}, u^{(m+1)}](s,t,x)\Big[g^{(m)}(s,t,\bar{x}) - g^{(m)}(s,t,x)\Big]\|$$

$$\leq R_4 \|g^{(m)}(s,t,\bar{x}) - g^{(m)}(s,t,x)\|.$$

Because

$$\|g^{(m)}(s,t,\bar{x}) - g^{(m)}(s,t,x)\| \leq \Theta_c \|x - \bar{x}\|,$$

so it is easily seen that there indeed exists $\bar{R} \in R_+$, such that

$$|\bar{\Delta}(t,x,\bar{x})| \leq \bar{R}\|x - \bar{x}\|^2.$$

Now we see, that (27) holds for $R_2 = \tilde{R} + \bar{R}$. So condition (II_{m+1}), where $m \in N \cup \{0\}$, is proved.

We prove now, that $z^{(m+1)} \in C^{1.L}_{\varphi.c}[d]$. Of course $z^{(m+1)}$ is continuous function on E_c, and

$$z^{(m+1)}(t,x) = \varphi(t,x) \quad \text{on } E_0 \cup \partial_0 E.$$

Of course the following estimates hold:

$$\|\partial_x z^{(m+1)}(t,x)\| \leq k_0,$$

$$\|\partial_x z^{(m+1)}(t,x) - \partial_x z^{(m+1)}(t,x)\| \leq k_1 \Big[|t - \bar{t}| + \|x - \bar{x}\|\Big],$$

$$|z^{(m+1)}(t,x)| \leq k_0,$$

and

$$\|z^{(m+1)}(t,x) - z^{(m+1)}(t,x)\| \leq k_1 \Big[|t - \bar{t}| + \|x - \bar{x}\|\Big]$$

on E_c. Now the proof is finished. □

We need to write a lemma about uniformly convergency of the sequence $\{z^{(m)}\}$, $\{u^{(m)}\}$, where $m \in N \cup \{0\}$.

Lemma 4. *Suppose that Assumptions $H[f]$, $H[D\Psi]$, $H[k,p]$, and $H[f,\varphi]$ are satisfied. Then there exists $c \in (0,a]$, such that sequences $\{z^{(m)}\}$, $\{u^{(m)}\}$, $m \in N \cup \{0\}$, are uniformly convergence on E_c.*

Proof. Let $c \in (0,a]$ be chosen arbitrary. For $t \in [0,c]$ we define

$$Z^{(m)}(t) = \|z^{(m)} - z^{(m-1)}\|_t, \quad \text{and} \quad U^{(m)}(t) = \|u^{(m)} - u^{(m-1)}\|_t.$$

Using Lemma 1 and Lemma 2 we conclude, that there exist $\bar{\eta}$, $\tilde{\eta} \in R_+$, such that for $t \in (0, c]$ the following estimate hold

$$U^{(m)}(t) \leq \bar{\eta}\Big|\int_0^t U^{(m)}(s)ds\Big| + \tilde{\eta}\Big|\int_0^t \Big[Z^{(m-1)}(s) + U^{(m-1)}(s)\Big]ds\Big|,$$

and

$$Z^{(m)}(t) \leq \eta\Big|\int_0^t \Big[Z^{(m-1)}(s) + U^{(m)}(s)\Big]ds\Big|.$$

Using the Gronwall enequality we get

$$U^{(m)}(t) \leq \tilde{\eta}\Big|\int_0^t \Big[Z^{(m-1)}(s) + U^{(m-1)}(s)\Big]\exp(c\bar{\eta})ds\Big|,$$

where $c \in (0, a]$. Further we have consequently

$$Z^{(m)}(t) \leq (\eta + c\tilde{\eta}\exp(c\bar{\eta}))\Big|\int_0^t \Big[Z^{(m-1)}(s) + U^{(m-1)}(s)\Big]ds\Big|.$$

Of course

$$Z^{(m)}(t) + U^{(m)}(t) \leq \Big[\eta + (1+c)\tilde{\eta}\exp(c\bar{\eta})\Big]\Big|\int_0^t \Big[Z^{(m-1)}(s) + U^{(m-1)}(s)\Big]ds\Big|,$$

and thus we have

$$Z^{(m)}(t) + U^{(m)}(t) \leq c\Big[\eta + (1+c)\tilde{\eta}\exp(c\bar{\eta})\Big]\Big(Z^{(m-1)}(t) + U^{(m-1)}(t)\Big).$$

Moreover there exists $\eta_0 \in R_+$, such that

$$Z^{(1)} + U^{(1)} \leq \eta_0 \quad \text{for } t \in [0, c].$$

For $c \in (0, a]$, such that

$$c\Big[\eta + (1+c)\tilde{\eta}\exp(c\bar{\eta})\Big] < 1, \tag{28}$$

sequence $\{U^{(m)} + Z^{(m)}\}$, $m \in N \cup \{0\}$, is uniformly convergence for $t \in [0, c]$. So for sufficiently small $c \in (0, a]$ sequences $\{z^{(m)}\}$, $\{u^{(m)}\}$, $m \in N \cup \{0\}$, are uniformly convergence on $t \in (0, c]$, where $c \in (0, a]$, and condition (28) holds. The proof is finished. □

5. The Main Theorem

Theorem 3. *Suppose that Assumptions* $H[f]$, $H[D\Psi]$, $H[k,p]$, $H[f,\varphi]$, *and* φ, $\bar{\varphi} \in C^{1.L}[s]$. *Then:*

(I) there exists $c \in (0,a]$, *such that problem (1), (2) has a solution* $v \in [-h,c] \times [-d,d] \to R$, *and moreover* $v \in C^{1.L}_{\varphi.c}[k]$, *and* $\partial_x v \in C^{0.L}_{\partial\varphi.c}[p]$;

(II) if $\bar{v} \in C^{1.L}_{\bar{\varphi}.c}[k]$ *is another solution of (1) with the initial-boundary condition*

$$z(t,x) = \bar{\varphi}(t,x) \quad \text{for } (t,x) \in E_0 \cup \partial_0 E, \tag{29}$$

then there exist $c \in (0,a]$, *and* $\Lambda_c \in R_+$, *such that*

$$\|v - \bar{v}\|_t + \|\partial_x v - \partial_x \bar{v}\|_t \leq \Lambda_c \Big[\|\varphi - \bar{\varphi}\|_t + \|\partial_x \varphi - \partial_x \bar{\varphi}\|_t \Big], \tag{30}$$

where $t \in [0,c]$.

Proof. From Lemma 3 and Lemma 4 there exist functions $v \in C^{1.L}_{\varphi.c}[k]$, and $u \in C^{0.L}_{\partial\varphi.c}[p]$, such that sequences $\{z^{(m)}\}$, $\{u^{(m)}\}$, $m \in N \cup \{0\}$, are uniformly convergent on $[0,c] \times [-d,d]$ properly to functions v, and u. Moreover $\partial_x v$ exists on $[0,c] \times [-d,d]$, and $\partial_x v = u$. Therefore

$$v(t,x) = F[v, \partial_x v](t,x), \quad \partial_x v(t,x) = G[v, \partial_x v](t,x),$$

and

$$g[v, \partial_x v](s,t,x) = x + \int_s^t \partial_q f(P[v, \partial_x v](\xi,t,x)) d\xi.$$

The initial-boundary condition $v(t,x) = \varphi(t,x)$, $(t,x) \in E_0 \cup \partial_0 E$, holds obviously, and $\partial_x v(t,x) = \partial_x \varphi(t,x)$ on $E_0 \cup \partial_0 E$. So v is a solution of (1), (2). Part (I) is proved.

Now we prove part (II). From our assumptions we have

$$\bar{v}(t,x) = F[\bar{v}, \partial_x \bar{v}](t,x), \quad \partial_x \bar{v}(t,x) = G[\bar{v}, \partial_x \bar{v}](t,x),$$

and

$$g[\bar{v}, \partial_x \bar{v}](s,t,x) = x + \int_s^t \partial_q f(P[\bar{v}, \partial_x \bar{v}](\xi,t,x)) d\xi.$$

Of course the initial-boundary condition $\bar{v} = \bar{\varphi}(t,x)$, $(t,x) \in E_0 \cup \partial_0 E$, holds. Note, that there exist $\bar{\Lambda}$, $\tilde{\Lambda} \in R_+$, such that

$$\|v - \bar{v}\|_t + \|\partial_x v - \partial_x \bar{v}\|$$

$$\leq \tilde{\Lambda}\Big[\|\varphi-\bar{\varphi}\|_t+\|\partial_x\varphi-\partial_x\bar{\varphi}\|_t\Big]+\bar{\Lambda}\int\limits_0^t\Big[\|v-\bar{v}\|_s+\|\partial_x v-\partial_x\bar{v}\|_s\Big]ds,$$

where $t\in[0,c]$. Using the Gronwall inequality for $\Lambda_c=\tilde{\Lambda}\exp(\bar{\Lambda}c)$ we get (30). The theorem is proved. □

6. Some Special Cases

Now we list some new assumptions needed to formulate theorems for some special situations.

Assumption *H*[Φ]. Let $\Phi(t,x,w)=(\Phi_0(t),\Phi_\star(t,x,w))$, where $\Phi_0:[0,a]\to R$, $\Phi_\star:[0,a]\times[-d,d]\times C(D,R^n)\to R^n$, $\Phi_\star=(\Phi_1,\ldots,\Phi_n)$, and:

(i) $0\leq\Phi_0(t)\leq t$ for $t\in[0,a]$;

(ii) the partial derivative $\partial_x\Phi_\star$ exists on $[0,a]\times[-d,d]\times C(D,R^n)$;

(iii) there exist $\tilde{\beta}_\star$, $\bar{\beta}_0$, $\bar{\beta}_\star\in R_+$, such that

$$\|\partial_x\Phi_\star(t,x,w)\|\leq\tilde{\beta}_\star,$$
$$\|\partial_x\Phi_\star(t,x,w)\|-\partial_x\Phi_\star(\bar{t},\bar{x},\bar{w})\|\leq\bar{\beta}_\star\Big[|t-\bar{t}|+\|x-\bar{x}\|+\|w-\bar{w}\|_0\Big]$$

for (t,x,w), $(\bar{t},\bar{x},\bar{w})\in[0,a]\times[-d,d]\times C(D,R^n)$, and

$$|\Phi_0(t)-\Phi_0(\bar{t})|\leq\bar{\beta}_0|t-\bar{t}|$$

for t, $\bar{t}\in[0,a]$.

Assumption *H*[Φ, Ψ]. Suppose that Assumptions $H[\Psi]$, and $H[\Phi]$ are satisfied, and:

(i) $-h\leq\Phi_0(t)-\Psi_0(t)\leq 0$;

(ii) $-r\leq\Phi_\star(t,x,w)-\Psi_\star(t,x,w)\leq r$ for $t\in[0,a]$, $(x,w)\in[-d,d]\times C(D,R^n)$.

Assumption *H*[*F*]. Suppose that function $F:[0,a]\times[-d,d]\times R\times R^n\to R$, and:

(i) $F(\cdot,x,p,q):[0,a]\to R$ is measurable for all $(x,p,q)\in[-d,d]\times R\times R^n$, and there exists $\alpha_\star\in R_+$, such that

$$|F(t,x,p,q)|\leq\alpha_\star$$

on $[0,a]\times[-d,d]\times R\times R^n$;

(ii) partial derivatives ∂_xF, ∂_pF, ∂_qF exist for $(x,p,q)\in[-d,d]\times R\times R^n$, and for almost all $t\in[0,a]$, functions $\partial_pF(\cdot,x,p,q):[0,a]\to R$, $\partial_xF(\cdot,x,p,q)$, and $\partial_q F(\cdot, x, p, q) : [0, a] \to R^n$ are measurable, and there exists $\tilde{\alpha}_\star \in R_+$, such that

$$\|\partial_x F(t, x, p, q)\| \leq \tilde{\alpha}_\star, \quad |\partial_p F(t, x, p, q)| \leq \tilde{\alpha}_\star, \quad \text{and} \quad \|\partial_q F(t, x, p, q)\| \leq \tilde{\alpha}_\star$$

for $(x, p, q) \in [-d, d] \times R \times R^n$, and for almost $t \in [0, a]$;

(iii) there exists $\bar{\alpha}_\star \in R_+$, such that

$$\begin{aligned}\|\partial_x F(t, x, p, q) - \partial_x F(t, \bar{x}, \bar{p}, \bar{q})\| &\leq \bar{\alpha}_\star \Big[\|x - \bar{x}\| + \|p - \bar{p}\|_0 + \|q - \bar{q}\|\Big],\\ |\partial_p F(t, x, p, q) - \partial_p F(t, \bar{x}, \bar{p}, \bar{q})| &\leq \bar{\alpha}_\star \Big[\|x - \bar{x}\| + \|p - \bar{p}\|_0 + \|q - \bar{q}\|\Big],\\ \|\partial_q F(t, x, p, q) - \partial_q F(t, \bar{x}, \bar{p}, \bar{q})\| &\leq \bar{\alpha}_\star \Big[\|x - \bar{x}\| + \|p - \bar{p}\|_0 + \|q - \bar{q}\|\Big]\end{aligned}$$

for (x, p, q), $(\bar{x}, \bar{p}, \bar{q}) \in [-d, d] \times R \times R^n$, and for almost all $t \in [0, a]$.

Now we formulate the next theorem, and we prove, that it is a special case of Theorem 3.

Theorem 4. *Suppose that assumptions $H[F]$, $H[\Phi, \Psi]$ are satisfied, $\varphi \in C^{1.L}[s]$, and there exists a function $\tilde{\Phi} : \partial_0 E \to R^n$, such that*

$$\partial_t \varphi(t, x) = F(t, x, \varphi(\Phi(t, x, \varphi_{(t,x)}), \tilde{\Phi}(t, x)))$$

for $x \in \Delta$, and for almost all $t \in [0, a]$. Then for sufficiently small $c \in (0, a]$ there exists a solution $v : E_c \to R$ of equation

$$\partial_t z(t, x) = F(t, x, z(\Phi(t, x, z_{,(t,x)})), \partial_x z(t, x)) \tag{31}$$

with the initial-boundary condition (2), and $v \in C^{1.L}_{\varphi.c}[k]$, and $\partial_x v \in C^{0.L}_{\partial\varphi.c}[p]$. Furthermore if

$$\partial_t \bar{\varphi}(t, x) = F(t, x, \bar{\varphi}(\Phi(t, x, \bar{\varphi}_{(t,x)}), \tilde{\Phi}(t, x)))$$

for $\bar{\varphi} \in C^{1.L}[s]$, and $\bar{v} \in C^{1.L}_{\bar{\varphi}.c}[d]$ is a solution of (31), (29), then there exists $\bar{\Lambda}_c \in R_+$, such that

$$\|v - \bar{v}\|_t + \|\partial_x v - \partial_x \bar{v}\|_t \leq \bar{\Lambda}_c \Big[\|\varphi - \bar{\varphi}\|_t + \|\partial_x \varphi - \partial_x \bar{\varphi}\|_t\Big].$$

Proof. We define $\Xi(t, x, w) = (\Xi_0(t), \Xi_\star(t, x, w))$ as follows

$$\Xi(t) = \Phi_0(t) - \Psi_0(t), \quad \Xi(t, x, w) = \Phi_\star(t, x, w) - \Psi_\star(t, x, w),$$

where $(t, x, w) \in [0, a] \times [-d, d] \times C(D, R)$. Let

$$f(t, x, w, q) = F(t, x, w(\Xi(t, x, w)), q)$$

for $(t,x,w,q) \in [0,a] \times [-d,d] \times C(D,R) \times R^n$. Then

$$\begin{gathered}\partial_x f(t,x,w,q) = \partial_x F(t,x,w(\Xi(t,x,w)),q)\\ +\partial_p F(t,x,w(\Xi(t,x,w)),q)\partial_x w(\Xi(t,x,w))\partial_x \Xi_\star(t,x,w),\\ \partial_q f(t,x,w,q) = \partial_q F(t,x,w(\Xi(t,x,w)),q),\\ \partial_w f(t,x,w,q)\bar{w} = \partial_w F(t,x,w(\Xi(t,x,w)),q)\bar{w}(\Xi(t,x,w)),\end{gathered}$$

where $(t,x,q) \in [0,a] \times [-d,d] \times R^n$, and $w, \bar{w} \in C(D,R)$. Now it is easy to note, that assumptions of Theorem 3 are satisfied.

Now we formulate the integral equation. Suppose that $\bar{\Phi}(t,x) = (\bar{\Phi}_0(t), \bar{\Phi}_\star(t,x))$, and $\bar{\Psi}(t,x) = (\bar{\Psi}_0(t), \bar{\Psi}_\star(t,x))$, where $\bar{\Phi}_0$, $\bar{\Psi}_0 : [0,a] \to R$, and $\bar{\Phi}_\star$, $\bar{\Psi}_\star : [0,a] \times [-d,d] \to R$. Note that functions $\Phi_\star$, and $\Psi_\star$ are independent from functional variable. Let

$$f(t,x,w,q) = F(t,x, \int\limits_{\Phi_0(t)-\Psi_0(t)}^{\bar{\Phi}_0(t)-\Psi_0(t)} \int\limits_{\Phi_\star(t,x)-\Psi_\star(t,x)}^{\bar{\Phi}_\star(t,x)-\Psi_\star(t,x)} w(s,\eta)d\eta ds, q)$$

for $(t,x,w,q) \in [0,a] \times [-d,d] \times C(D,R) \times R^n$. Then

$$f(t,x,V[z](t,x),q) = F(t,x, \int\limits_{\Phi(t,x)}^{\bar{\Phi}(t,x)} z_{(\Psi_0(t),\Psi_\star(t,x))}(s,\eta)dsd\eta, \partial_x z(t,x))$$

and the equation (1) is reduced to

$$\partial_t z(t,x) = F(t,x, \int_{\Phi(t,x)}^{\bar{\Phi}(t,x)} z_{(\Psi_0(t),\Psi_\star(t,x))}(s,\eta)dsd\eta, \partial_x z(t,x)). \tag{32}$$

Assumption $\tilde{H}[\Psi]$. Suppose that the function $\Psi(t,x) = (\Psi_0(t), \Psi_\star(t,x))$, where $\Psi_0 : [0,a] \to R$, $\Psi_\star : [0,a] \times [-d,d] \to R^n$, is continuous, and the following conditions hold:

(i) $0 \le \Psi_0(t) \le t$ for $t \in [0,a]$;

(ii) the partial derivative $\partial_x \Psi_\star(t,x)$ exists on $[0,a] \times R^n$;

(iii) there exists $\beta_\star \in R_+$, such that

$$\begin{gathered}\|\partial_x \Psi_\star(t,x)\| \le \beta_\star,\\ \|\partial_x \Psi_\star(t,x) - \partial_x \Psi_\star(t,\bar{x})\| \le \beta_\star \|x - \bar{x}\|\end{gathered}$$

on $[0,a] \times [-d,d]$.

Theorem 5. *Suppose that Assumption $\tilde{H}[\Psi]$ is satisfied, and $\varphi \in C^{1.L}[s]$. Then for sufficiently small $c \in (0,a]$ there exists a solution $v : [-h,c] \times [-d,d] \to R$ of the equation*

$$\partial_z(t,x) = f(t,x, \int\limits_D z_{(\Psi_0(t),\Psi_\star(t,x))}(\xi,\eta)d\xi d\eta, \partial_x z(t,x)) \tag{33}$$

with the initial-boundary condition (2). Moreover $v \in C^{1.L}_{\varphi.c}[k]$, $\partial_x v \in C^{0.L}_{\partial\varphi.c}[p]$, and if $\bar{\varphi} \in C^{1.L}[s]$, and $\bar{v} \in C^{0.L}_{\varphi.c}[p]$ is a solution of (33), (29), then there exists $\hat{\Lambda}_c \in R_+$, such that

$$\|v - \bar{v}\|_t + \|\partial_x v - \partial_x \bar{v}\|_t \le \hat{\Lambda}_c \Big[\|\varphi - \bar{\varphi}\|_t + \|\partial_x \varphi - \partial_x \bar{\varphi}\|_t \Big], \quad t \in [0,c].$$

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