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FOLDINGS AND RETRACTIONS OF MANIFOLDS AND THEIR FUNDAMENTAL GROUPS

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Abstract: In this paper, we introduce the definition of the unfolding on the fundamental group. The folding on a wedge sum of some types of manifolds which are determined by their fundamental group are obtained. Some types of conditional foldings restricted on the elements of the fundamental groups are deduced. The effect of retraction of manifolds on the fundamental group is obtained. The folding of variation curvature of manifolds on their fundamental group are presented. Theorems governing these relations are achieved.

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1. Introduction

The folding of a manifold was introduced by S.A. Robertson 1977, see [14]. More studies of the folding of manifolds are [3], [4], [7]-[10]. Various folding problems arising in the physics of membrane and polymers are reviewed by Francesco [2]. The unfolding of a manifold introduced in [5]. The retraction of a manifold was defined and discussed in [1], [3], [6]. The fundamental groups of some types of a manifold are discussed in [11]-[13].

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2. Definitions and Background

In this section, we give the introduce some necessary definitions.

Definition 1. (see [12]) The set of homotopy classes of loops based at the point x_0 with the product operation $[f][g] = [f.g]$ is called the fundamental group and denoted by $\pi_1(X, x_0)$.

Definition 2. (see [14], [10]) Let M and N be two manifolds of dimensions m and n respectively. A map $f : M \to N$ is said to be an isometric folding of M into N if for every piecewise geodesic path $\gamma : I \to M$, the induced path $f \circ \gamma : I \to N$ is piecewise geodesic and of the same length as γ. If f does not preserve length, it is called a topological folding.

Definition 3. (see [5]) Let M and N be two manifolds of the same dimension. A map $g : M \to N$ is said to be unfolding of M into N if, for every piecewise geodesic path $\gamma : I \to M$, the induced path $g \circ \gamma : I \to N$ is piecewise geodesic with length greater than γ.

Definition 4. (see [13]) A subset A of a topological space X is called a retract of X if there exists a continuous map $r : X \to A$ (called a retraction) such that $r(a) = a \; \forall a \in A$.

Definition 5. (see [12]) Given spaces X and Y with chosen points $x_0 \in X$ and $y_0 \in Y$, then the wedge sum $X \vee Y$ is the quotient of the disjoint union $X \cup Y$ obtained identifying x_0 and y_0 to a single point.

3. The Main Results

Aiming to our study, we will introduce the folowing.

Definition 6. Let M be a manifold. A map $\overline{\text{unf}} : \pi_1(M) \to \pi_1(M^/)$ is said to be unfolding of $\pi_1(M)$ into $\pi_1(M^/)$ if $\overline{\text{unf}}([\alpha]) = \overline{[\alpha]}$, where

$$\max(\text{radius}\,\overline{[\alpha]}) > \max(\text{radius}\,[\alpha])$$

as in Figure 1.

Lemma 1. *Let C_1, C_2 be two disjoint circles in R^2. Then there is unfolding* $\text{unf} : C_1 \cup C_2 \to C_1^/ \cup C_2^/$ *such that* $\pi_1(\text{unf}(C_1 \cup C_2)) \approx Z * Z$.

Proof. Let $\text{unf} : C_1 \cup C_2 \to C_1^/ \cup C_2^/$ be unfolding such that $\text{unf}(C_1 \cup C_2) = \text{unf}(C_1) \vee \text{unf}(C_2)$ as in Figure 2 thus $\pi_1(\text{unf}(C_1 \cup C_2)) = \pi_1(\text{unf}(C_1) \vee \text{unf}(C_2))$

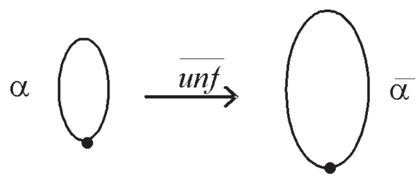

Figure 1:

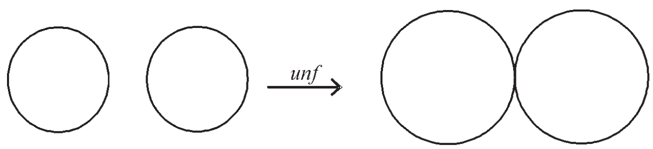

Figure 2:

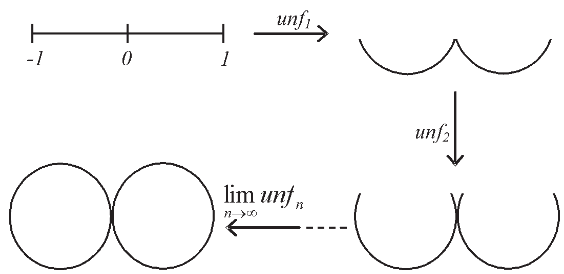

Figure 3:

and so $\pi_1(\mathrm{unf}(C_1 \cup C_2)) \approx \pi_1(\mathrm{unf}(C_1)) * \pi_1(\mathrm{unf}(C_2))$. Since $\pi_1(\mathrm{unf}(C_i)) \approx Z$, $i = 1, 2$, it follows that $\pi_1(\mathrm{unf}(C_1 \cup C_2)) \approx Z * Z$. $\square$

Lemma 2. *Let I be the closed interval $[0, 1]$ and let $\bar{I}$ be the closed interval $[-1, 0]$. Then there are unfoldings $\mathrm{unf} : I \cup \bar{I} \to J \cup \overline{J}$ with variation curvature such that $\pi_1(\lim_{n\to\infty}(\mathrm{unf}_n(I \cup \bar{I}))) \approx Z * Z$.*

Proof. Consider the sequence of unfoldings with variation curvature such that $\mathrm{unf}_1 : I \cup \bar{I} \to X_1, \mathrm{unf}_2 : X_1 \to X_2, ..., \mathrm{unf}_n : X_{n-1} \to X_n$ such that $\lim_{n\to\infty}(\mathrm{unf}_n(I \cup \bar{I})) = S_1^1 \vee S_2^1$ as in Figure 3, thus $\pi_1(\lim_{n\to\infty}(\mathrm{unf}_n(I \cup \bar{I})))$ $\approx Z * Z$. $\square$

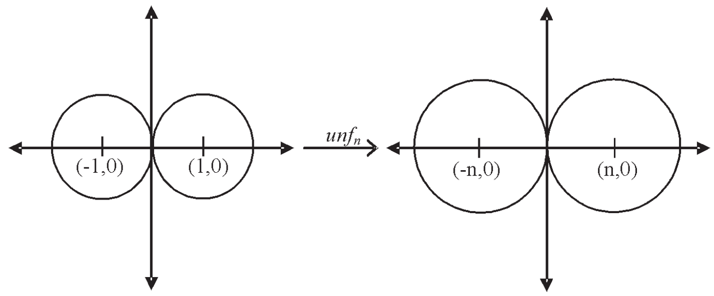

Figure 4:

Theorem 1. *Let C be the circle of radius 1 and centre $(-1,0)$. Then there are infinite number of unfoldings* $\text{unf}_n : C \vee \overline{C} \to C_1 \vee \overline{C}_1, n = 1, 2, ...$*which induces unfoldings* $\overline{\text{unf}}_n : \pi_1(C) * \pi_1(\overline{C}) \to \pi_1(C_1) * \pi_1(\overline{C}_1)$ *such that* $\overline{\text{unf}}_n(\pi_1(C) * \pi_1(\overline{C})) = \pi_1(\text{unf}_n(C)) * \pi_1(\text{unf}_n(\overline{C}))$ *or* $\overline{\text{unf}}_n(\pi_1(C) * \pi_1(\overline{C})) = \pi_1(\text{unf}_n(C)) * \pi_1(\overline{C})$ *or* $\overline{\text{unf}}_n(\pi_1(C) * \pi_1(\overline{C})) = \pi_1(C) * \pi_1(\text{unf}_n(\overline{C}))$.

Proof. Let $\text{unf}_n : C \vee \overline{C} \to C_1 \vee \overline{C}_1, n = 1, 2, ...$ are unfoldings which are preserving curvature or not preserving curvature such that $\text{unf}_n(C \vee \overline{C}) = \text{unf}_n(C) \vee \text{unf}_n(\overline{C})$ where $\text{unf}_n(C)$ is the circle of radius n and center $(n, 0)$ and $\text{unf}_n(\overline{C})$ is the circle of radius n and center $(-n, 0)$ as in Figure 4. So we have an induced unfolding $\overline{\text{unf}}_n : \pi_1(C) * \pi_1(\overline{C}) \to \pi_1(C_1) * \pi_1(\overline{C}_1)$ such that $\overline{\text{unf}}_n(\pi_1(C) * \pi_1(\overline{C})) = \pi_1(\text{unf}_n(C)) * \pi_1(\text{unf}_n(\overline{C}))$. Similarely, we can get the induced unfolding $\overline{\text{unf}}_n : \pi_1(C) * \pi_1(\overline{C}) \to \pi_1(C_1) * \pi_1(\overline{C}_1)$ such that $\overline{\text{unf}}_n(\pi_1(C) * \pi_1(\overline{C})) = \pi_1(\text{unf}_n(C)) * \pi_1(\overline{C})$ or $\overline{\text{unf}}_n(\pi_1(C) * \pi_1(\overline{C})) = \pi_1(C) * \pi_1(\text{unf}_n(\overline{C}))$. □

Theorem 2. *Let X be the subspace of R^2 that is the union of the circles C_n of radii $\frac{1}{n}$ and centres $(\frac{1}{n}, 0)$ for $n = 1, 2, ...$ then there is unfolding* $\text{unf} : X \to \overline{X}$ *which induced unfolding* $\overline{\text{unf}} : \pi_1(X) \to \pi_1(\overline{X})$ *such that* $\overline{\text{unf}}(\pi_1(X))$ *is a free group on a countable set of generators.*

Proof. Let $\text{unf} : X \to \overline{X}$ be unfolding such that $\text{unf}(X) = \text{unf}(\bigcup_{n=1}^{\infty} C_n) = \bigcup_{n=1}^{\infty} \text{unf}(C_n)$ and $\text{unf}(C_n)$ is the circles of radius n and centre $(n, 0)$ as in Figure 5. Then the induced unfolding $\overline{\text{unf}} : \pi_1(X) \to \pi_1(\overline{X})$ satisfies $\overline{\text{unf}}(\pi_1(X)) = \pi_1(\text{unf}(X))$. Now, we want to show that, $\pi_1(\text{unf}(X))$ is a free group on a countable set of generators. It follows from $\text{unf}(X) = \bigcup_{n=1}^{\infty} \text{unf}(C_n)$ that $\text{unf}(X)$ is

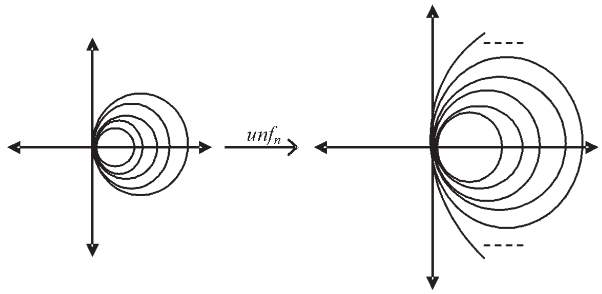

Figure 5:

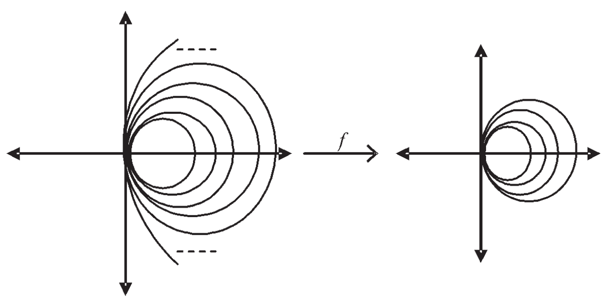

Figure 6:

the union of the circles of radius n and center $(n, 0)$, since $\mathrm{unf}(X)$ and $\underset{\infty}{\vee} S^1$ are homotopy equivalent, we have $\pi_1(\mathrm{unf}(X)) \approx \pi_1(\underset{\infty}{\vee} S^1)$ and so $\pi_1(\mathrm{unf}(X)) \approx \pi_1(Z * Z * ...)$, thus $\pi_1(\mathrm{unf}(X)) \approx \pi_1(Z) * \pi_1(Z) * ...\infty)$. Therefore, $\overline{\mathrm{unf}}_n(\pi_1(X))$ is a free group on a countable set of generators. $\square$

Theorem 3. *Let X be the subspace of R^2 that is the union of the circles C_n of radius n and center $(n, 0)$ for $n = 1, 2, ...$ Then there is a folding $F : X \to \overline{X}$ without singularity, which induces a folding $\overline{F} : \pi_1(X) \to \pi_1(\overline{X})$ such that $\overline{F}(\pi_1(X))$ is uncountable.*

Proof. Let $F : X \to \overline{X}$ be a folding without singularity such that $F(X) = F(\overset{\infty}{\underset{n=1}{\cup}} C_n) = \overset{\infty}{\underset{n=1}{\cup}} F(C_n)$ and $F(C_n)$ is the circle of radius $\frac{1}{n}$ and centre $(\frac{1}{n}, 0)$ as in Figure 6. Then the induced folding $\overline{F} : \pi_1(X) \to \pi_1(\overline{X})$ satisfies $\overline{F}(\pi_1(X)) = \pi_1(F(\overline{X}))$. Now we want to show that $\pi_1(F(X))$ is uncountable. Consider the retracion $r_n : F(X) \to F(C_n)$ which collapsing all $F(C_i)$ exept $F(C_n)$ to origin. Each r_n induces a surjection. $\overline{r}_n : \pi_1(F(X)) \to \pi_1(F(C_n)) \approx Z$, where

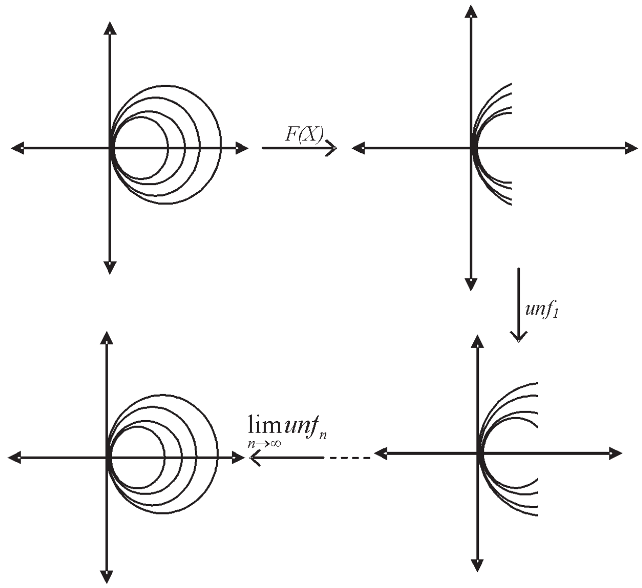

Figure 7:

the origin is a base point. Then the product of the $\overline{r}_n$ is a homomorphism $r : \pi_1(F(X)) \to \prod_{\infty} Z$ to the direct product of infinite number of copies of Z and clearly, we can prove that r is onto, it follows from $\prod_{\infty} Z$ uncountable that $\pi_1(F(X))$ is uncountable. Since $\overline{F}(\pi_1(X)) = \pi_1(F(X))$ it follows that $\overline{F}(\pi_1(X))$ is uncountable. □

Theorem 4. *Let X be the subspace of R^2 that is the union of the circles C_n of radius $\frac{1}{n}$ and centre $(\frac{1}{n}, 0)$ for $n = 1, 2, \ldots.$ And $F : X \to X$ is a folding such that $F(C_n) \neq C_n$ $\forall n$. Then there are unfoldings* unf $: F(X) \subset X \to X$ *such that $\pi_1(\lim_{n\to\infty}(\mathrm{unf}_n(F(X)))$ is uncountable*

Proof. Let $F : X \to X$ be a folding such that $F(C_n) \neq C_n$ $\forall n$, i.e. folding by cut. Then, we can define a sequence of unfoldings $\mathrm{unf}_1 : F(X) \to X_1, F(X) \subseteq X_1 \subseteq X, \mathrm{unf}_2 : X_1 \to X_2, X_1 \subseteq X_2 \subseteq X, \ldots, \mathrm{unf}_n : X_{n-1} \to X_n, X_{n-1} \subseteq X_n \subseteq X$, and so $\lim_{n\to\infty}(\mathrm{unf}_n(F(X)) = X$, as in Figure 7, thus $\pi_1(\lim_{n\to\infty}(\mathrm{unf}_n(F(X)))$ is uncountable. □

Theorem 5. *Let $X \subset R^3$ be the union of the circles C_n of radius $\frac{1}{n}$ and centred at $(\frac{1}{n}, 0)$ for $n = 1, 2, \ldots.$Then there are foldings $F_n : X \to X$ and*

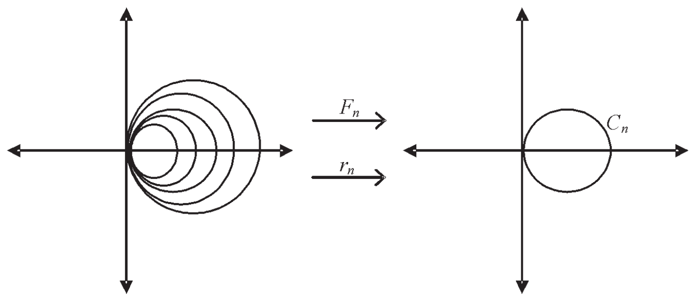

Figure 8:

retractions $r_n : X \to C_n$ *such that* $\pi_1(F_n(X)) = \pi_1(r_n(X)) \approx Z$.

Proof. Let $F_n : X \to X$ be a folding such that $F_n(C_m) = C_n \ \forall m = 1, 2, ...$then $F_n(X) = C_n$ as in Figure 8 and so $\pi_1(F_n(X)) \approx Z$. Also, consider the retractions $r_n : X \to C_n$, which collapsing all C_i except C_n to the origin and so $r_n(X) = C_n$, thus $\pi_1(r_n(X)) \approx Z$. Therefore $\pi_1(F_n(X)) = \pi_1(r_n(X)) \approx Z$. □

Theorem 6. *Let* $X \subset R^3$ *be the union of the spheres* $S_n^2 - \{a_1, a_2\}, a_1, a_2 \notin \bigcap_{n=1}^{\infty} S_n^2$ *of radii* $\frac{1}{n}$ *and centred at* $(\frac{1}{n}, 0, 0), n = 1, 2, ...$ *Then there are foldings* $F_n : X \to X$ *and retractions* $r_n : X \to S_n^2 - \{a_1, a_2\}$ *such that* $\pi_1(F_n(X)) = \pi_1(r_n(X)) \approx Z$.

Proof. Let $F_n : X \to X$ be a folding such that $F_n(S_m^2 - \{a_1, a_2\}) = S_n^2 - \{a_1, a_2\}, m = 1, 2, ...$then $F_n(X) = S_n^2 - \{a_1, a_2\}$ and so $\pi_1(F_n(X)) = \pi_1(S_n^2 - \{a_1, a_2\})$. Since S_n^1 is a deformation retract of $S_n^2 - \{a_1, a_2\}$ it follows that $\pi_1(S_n^2 - \{a_1, a_2\}) \approx \pi_1(S_n^1)$. Hence $\pi_1(F_n(X)) \approx Z$. Also, consider the retractons $r_n : X \to S_n^2 - \{a_1, a_2\}$ which collapsing all $S_i^2 - \{a_1, a_2\}$ exept $S_n^2 - \{a_1, a_2\}$ to the origin, $\pi_1(r_n(X)) = \pi_1(S_n^2 - \{a_1, a_2\})$ and so $\pi_1(r(X)) \approx Z$. Therefore, $\pi_1(F_n(X)) = \pi_1(r_n(X)) \approx Z$. □

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