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RIVER AND THE CRITICAL POINT AT INFINITY IN THE PLANAR QUADRATIC DIFFERENTIAL SYSTEMS

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Abstract: In this paper, quadratic polynomial differential systems of the form: $X' = P(X,Y)$, $Y' = Q(X,Y)$ defined in R^2 are discussed. We will see that, for some quasi homogeneous vector fields, the river phenomenon can exist. We will give thus a sufficient conditions for their existence by studying the critical points at infinity.

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1. Introduction and Statement of the Main Result

The present work treats the survey of the river phenomenon for the quadratic differential system:

$$X' = \frac{dX}{dt} = P(X,Y) \ \ ; \ \ Y' = \frac{dY}{dt} = Q(X,Y), \tag{1}$$

where the dependent variables X and Y and the independent variable (the time) t are all real, and $P, Q \in R[X,Y]$. As usual $R[X,Y]$ denotes the ring of polynomials in the variables X and Y with real coefficients. In what follows, all mentioned functions are in $R[X,Y]$ and all constants are real. We say

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that $m = \max\{\deg P, \deg Q\}$ is the degree of the polynomial systems. The polynomial systems of degree 2 will be called quadratic systems, and, in what follows P and Q are as $degP = degQ = 2$.

Let us note that when the polynomials P and Q are the generalized, i.e., of the form:

$$P(X,Y) = \sum a_{ij} X^i Y^j\,, \quad Q(X,Y) = \sum b_{ij} X^i Y^j,$$

where i, j are in Z^2, the techniques of the nonstandard analysis (see [4], [5]) are often used to obtain information on the dynamics of these systems.

Specially in the search of the rivers, these techniques permit to transform the system (1) into a fast-slow system

$$x^{'} = p(x,y,\varepsilon)\,, \quad y^{'} = q(x,y,\varepsilon)/\varepsilon^c\,, \tag{2}$$

where p, q are real polynomials in x and y in R, $\varepsilon > 0$ small and $c > 0$.

As usual, this transformation is obtained by making to the system (1) the change of variables (macroscope)

$$x = \varepsilon X\,, \quad y = \varepsilon^r Y\,, \tag{3}$$

where $\varepsilon > 0$ small and $r \neq 0$. This action permits to restore the points X, Y from infinity toward a limited region and to study the regular trajectories belonging to the so-called slow curve defined by

$$C := \{(x,y) : q(x,y,0) = 0\}\,. \tag{4}$$

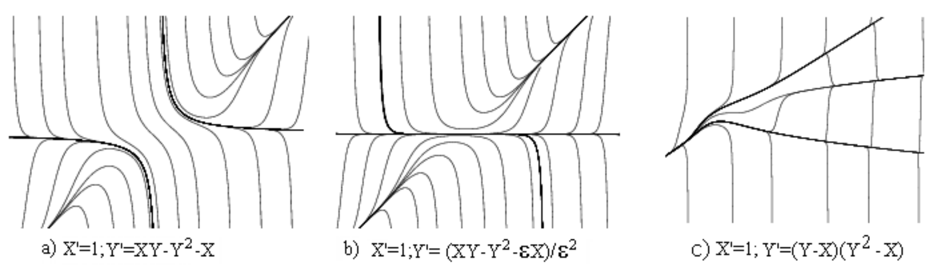

Figure 1: Example of polynomial differential systems with regular river. Note that the system (b) defined by $X^{'} = 1$; $Y^{'} = (XY - Y^2 - \varepsilon X)/\varepsilon^2$, is the system deduced from (a) by the macroscope ($x = \varepsilon X$, $y = \varepsilon Y$) represented here for $\varepsilon \simeq 0$. The slow curve of this system is the set defined by $\{(X,Y) : XY - Y^2\}$ and the nonconstant portion of the slow curve is defined by $\{(X,Y) : Y - X = 0\}$.

Let us note that the choice of the fine macroscope (and therefore r) that leads to a fast slow field ($c > 0$) is given by simple exam of the Newton's polygon of Q (Theorem 2) and when this exam gives $c < 0$, it is sufficient to change X in Y and to study the river in Y (see [4]). An interesting question is to know what occurs when we obtain $c = 0$. It is the case for the quasi homogeneous polynomial vector field, and when one sees the example of Figure 2 below, one clearly sees that there is presence of river although $c = 0$. In that follows, we are going to be interested in this case.

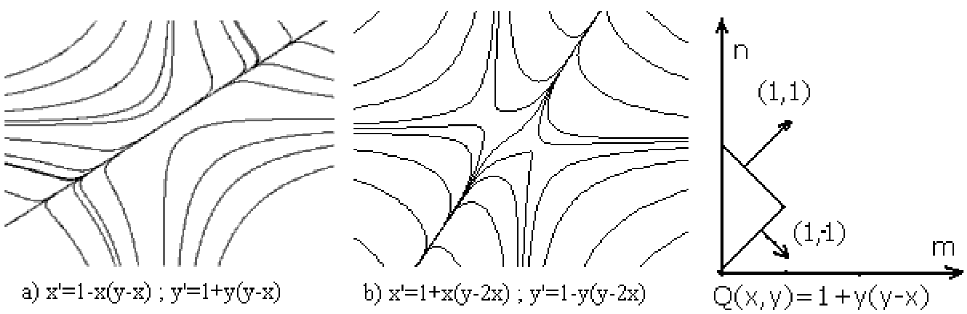

Figure 2: Examples of quadratic system with river phenomenon, and their Newton polygon.

We are concerned with the real quadratic system of the form:

$$\begin{cases} X' = P_0(X,Y) + P_1(X,Y) + P_2(X,Y) = P(X,Y)\,, \\ Y' = Q_0(X,Y) + Q_1(X,Y) + Q_2(X,Y) = Q(X,Y)\,, \end{cases} \tag{5}$$

where P_i (respectively Q_i) is the sum of terms in X and Y of degree i of P (respectively Q) and $gcd(P,Q)$ =constant. Instead of restoring the points X, Y from infinity toward a limited region and to study the regular trajectories belonging to the slow curve we use the compactification of the planar polynomial vector field by the change of variable $Z = 1/X$, $U = Y/X$ and list all possible critical points "at infinity" on the line $Z = 0$.

More explicitly the systems above can be written in the form:

$$(S)\begin{cases} X' = a_{00} + a_{10}X + a_{01}Y + a_{20}X^2 + a_{11}XY + a_{20}Y^2 = P(X,Y)\,, \\ Y' = b_{00} + b_{10}X + b_{01}Y + b_{20}X^2 + b_{11}XY + b_{20}Y^2 = Q(X,Y)\,. \end{cases} \tag{6}$$

This paper is organised as follow. The new dynamical system is deduced in Section 2. In Section 3, we recall some preliminary results given in [1], [4], and in Section 4 we apply these results for giving a sufficient condition for the existence of the river phenomenon in this critical case.

2. Compactification of the Real Planar Vector Field

Let us consider the change of variables: $P : R^2 - \{X = 0\} \longrightarrow R^2 - \{0\}$, $(X, Y) \longrightarrow (Z, U)$, where $Z = 1/X, U = Y/X$ (i.e. $X = 1/Z; Y = U/Z$).

That is transforms the plane in the following way easy to check

— it takes line with slope m into line with slope b: $Y = mX + b \longrightarrow U = bZ + m$

— it takes curve with the form $Y = cX^s + o(X^s)$ when $X \longrightarrow \infty$ into curve with the form $U = cZ^{1-s} + o(Z^{1-s})$ when $Z \longrightarrow 0^+$.

Making in the system (S) the change P one obtains the new system:

$$\begin{cases} Z' = -Z^2 P(1/Z, U/Z)\,, \\ U' = ZQ(1/Z, U/Z) - ZUP(1/Z, U/Z)\,, \end{cases} \tag{7}$$

which is equivalent, while multiplying by Z, to the system.

$$\begin{cases} Z' = -Z^3 P_0(1, U) - Z^2 P_1(1, U) - ZP_2(1, U)\,, \\ U' = Z^2[Q_0(1, U) - UP_0(1, U)] + Z[Q_1(1, U) - UP_1(1, U)] \\ \qquad\qquad + [Q_2(1, U) - UP_2(1, U)]\,, \end{cases} \tag{8}$$

represents the vector field of (1) near the line at infinity $L_\infty = \{Z = 0\}$.

In the neighbourhood of $Z = 0$ an equilibrium point satisfy the equation:

$$[Q_2(1, U) - UP_2(1, U)] = 0\,.$$

Or, while returning to the equation (6) one obtains

$$XQ_2(X, Y) - YP_2(X, Y) = 0\,,$$

where $P_2(X, Y)$ and $Q_2(X, Y)$ are the homogeneous parts of degree 2 of $P(X, Y)$ and $Q(X, Y)$ respectively.

3. Preliminaries

In this section we will briefly recall some basic definitions given by the methods of nonstandard analysis in [4], and [5].

A real number x is *infinitesimal,* denoted by $x \simeq 0$, if $| x |< a$ for all standard positive real number a, *limited* if $| x |< a$ for some standard a, *appreciable* if it is limited and not infinitesimal, and *unlimited* and denoted by $x \simeq \pm\infty$ if it is not limited. Let (E, d) be a standard metric space. Two points x and y in E are *infinitely close*, denoted by $x \simeq y$, if $d(x, y) \simeq 0$. The element x is *nearstandard* in E if there exists a standard $x_0 \in E$ such that $x \simeq x_0$. Note that a real number is nearstandard in R if and only if it is limited. The point x_0 is called the *standard part* of x (it is unique) and is also denoted $^\circ x$. A vector x in R^d, d standard, is *infinitesimal* (respectively *limited*, *unlimited*) if $\mid x \mid$ is infinitesimal (respectively limited, unlimited), where $\mid . \mid$ is a standard norm in R^d.

Let us recall that a trajectory $\varphi(x)$ of (1) is *regular* if $f'_y(x, \varphi(x)) \neq 0$ for positive appreciable x. A *r-macroscope* is a map M_r: $R^2 \to R^2$ defined by

$$M_r(X, Y) = (\varepsilon X, \varepsilon^r Y)\ ,\ r \in R,\ \varepsilon \in R^*_+, \text{and } \varepsilon \simeq 0. \tag{9}$$

Consider the differential equation

$$dY/dX = F(X,Y)\,, \tag{10}$$

where F is a standard rational function defined as $F(X,Y) = Q(X,Y)/P(X,Y)$ and P, Q are two polynomials of degree $d > 1$. One supposes that P and Q have not a common nonconstant polynomial factor. One applies the macroscope (9) to this equation and one obtains the new ordinary differential equation

$$\varepsilon^c dy/dx = f(x,y,\varepsilon) = \frac{q(x,y,\varepsilon)}{p(x,y,\varepsilon)}\ . \tag{11}$$

Definition 1. (Regular River) Let $\Phi(X)$ be a standard solution of (10) defined for any $X > 0$. The function $\Phi(X)$ is a river of type (k,r) of (10) in $X = +\infty$ if and only if there exist some real (standard) r and k, $k \neq 0$, such that:

a) Equation (10) changes under the r-macroscope into a fast-slow equation of the form (11), with f nearly standard function nonequivalent to 0.

b) For any x appreciable positive, $^\circ f(x, kx^r) = 0$ and $^\circ f'_y(x, kx^r) \neq 0$.

c) For any x appreciable positive, $\varepsilon^r \Phi(x/\varepsilon)/x^r \simeq k$.

If $^\circ f'_y(x, \Phi(x)) > 0$ this trajectory is called repulsive, otherwise if $^\circ f'_y(x, \Phi(x)) < 0$ it is called attractive.

This definition was used in [4] to introduce a first approach to rivers, and it is easy to see that the phenomenon is related directly to the properties of the fast-slow fields.

The following theorem allows us, by studying the algebraic structure of the second member of (10), to give an algorithm in the aim to determine the rivers when the second member is a polynomial function.

Notation. Let $P(X,Y) = \sum a_{m_i,n_i} X^{m_i} Y^{n_i}$, where $i \in I \subset Z$, be a generalized polynomial. Let $r \in Q$.

a) We define the $r-degree$ of the polynomial P(X,Y) and we note $r\partial^\circ P$ the degree in λ of $P(\lambda X, \lambda^r Y)$. We have thus $r\partial^\circ P = \max\{m_i + rn_i, i \in I \subset Z\}$.

b) We denote by $P_r(X,Y)$ the coefficient of the polynomial which composes $P(\lambda X, \lambda^r Y)$ and has the higher degree in λ.

Let $P(X,Y)$ and $Q(X,Y)$ be two generalized polynomials and $F(X,Y)$ the rational fraction defined by $F(X,Y) = Q(X,Y)/P(X,Y)$. Let $r \in Q$. We define:

(i) $r\partial^\circ F = r\partial^\circ Q - r\partial^\circ P$.

(ii) $F_r = Q_r/P_r$.

(iii) We define the Newton's polygon $N(P)$ associated to the polynomial $P(X,Y) = \sum a_{m_i,n_i} X^{m_i} Y^{n_i}$ as the convex hull of the set $E(P) = \{(m_i,n_i), a_{m_i,n_i} \neq 0\}$.

Theorem 2. (see [4]) *Let k and r two real (standard) numbers, $k \neq 0$. The equation (10) admits a river of type (k,r) in $X = +\infty$ if and only if:*

(a) $(1,r)$ is orthogonal to $N(Q)$.

(b) $r\partial^\circ F + 1 - r > 0$.

(c) $F_r(1,k) = 0$.

(d) $Q_r)'_y . P_r(1,k) \neq 0$.

Indeed, if one refers to equation (10), this theorem permits, by plotting the Newton's polygon, to find the r-macroscope which allows us to obtain a fast-slow equation like (11) and to study the non-singular solutions (condition (d)) of equation $F_r(1,k) = 0$.

Finally, we note that the conditions (b) impose to c to be positive. However, as depicted in the example of Figure 2, $c := r\partial^\circ F + 1 - r = 0$, but the river phenomenon does exist. This is the so-called *pseudo-singular* rivers. In the following, we will interest for this case and we begin by recalling some results given in [1].

Definition 3. (Generalized River, see [1]) Let $\Phi(X)$ a standard solution of (10) defined for $X > 0$.

Φ is said a generalized river repulsive of (10) if Φ is a trajectory with polynomial growth and $\lim_{X \to +\infty} X F'_y(X, \Phi(X)) = +\infty$.

Φ is said a generalized river attractive of (10) if Φ is a trajectory with polynomial growth and $\lim_{X\to+\infty} XF_y^{'}(X, \Phi(X)) = -\infty$.

This definition generalizes the concept of river and in the case of a rational differential equations, the difference between river and generalized river is that for the generalized rivers one can have $c := r\partial^{\circ}F + 1 - r = 0$, and it is what one is going to use here.

4. Main Results

Definition 4. (Macroscopic Singularity) Let k and r be two (standard) real numbers, $k \neq 0$, $r \neq 0$. It is said that $\varphi(x) = kx^r$ is a macroscopic singularity of type (k,r) for the system (10) if the conditions bellow are satisfied:

1) the r-macroscope transforms the equation (10) into an equation of the form: $x^{'} = p(x, y, \varepsilon)$; $y^{'} = q(x, y, \varepsilon)$.

2) $\forall x$ $p(x, kx^r, 0) = 0$ and $q(x, kx^r, 0) = 0$.

3) $\forall$x appreciable $f_y^{'}(x, kx^r, 0) \neq 0$ where $f(x, y, \varepsilon) = p(x, y, \varepsilon)/q(x, y, \varepsilon)$.

Proposition 5. *For the quadratic polynomials differential system, all common factor of the homogeneous parts of degree 2 of $P(X, Y)$ and $Q(X, Y)$ represents a macroscopique singularity.*

Proof. It is sufficient to see that so P_2 and Q_2 are not identically null, while drawing the polygon of Newton $N(Q)$, the only r that is orthogonal to $N(Q)$ is $r = 1$. While applying the macroscope $x = \varepsilon X$, $y = \varepsilon Y$, one obtains $Q_r = Q_2$ and $P_r = P_2$. As P_2 and Q_2 have a common factor (by hypothesis), this common factor represents the macroscopic singularity. □

Proposition 6. *For the quadratic polynomials differential system, all macroscopic singularity is a root of $XQ_2(X, Y) - YP_2(X, Y) = 0$, where $P_2(X, Y)$ and $Q_2(X, Y)$ are the homogeneous parts of degree 2 of $P(X, Y)$ and $Q(X, Y)$, respectively.*

Proof. Since P_2 and Q_2 are homogeneous, this macroscopic singularity will represent the common factor of P_2 and Q_2, and as they are invariant under the linear macroscope $x = \epsilon X$, $y = \epsilon Y$, this singularity will be solution of $XQ_2(X, Y) - YP_2(X, Y) = 0$. □

Example. Let us consider the differential system:

$$\begin{cases} X^{'} = 1 - X(Y - X)\,, \\ Y^{'} = 1 + Y(Y - X)\,. \end{cases} \tag{12}$$

To determine the rivers of this differential equation (example (a) of the Figure 2), one can proceed by using Theorem 2.

For this equation $Q(X,Y) = (1+X(Y-X))$ and $P(X,Y) = (1-X(Y-X))$. As it is seen in Figure 2, its Newton polygon $N(Q)$ gives rise to $r1 = -1$ and $r2 = 1$. If one takes $r1 = -1$ the macroscope associated ($x = \varepsilon X$, $y = \varepsilon^{-1} Y$) reduced the system (12) with the system:

$$x^{'} = x^2 + (1 - xy)\varepsilon^2\,,$$
$$y^{'} = (1 - yx) + y\varepsilon^2\,,$$

which is equivalent to the system:

$$x^{'} = x^2 + (1 - xy)\varepsilon^2\,,$$
$$y^{'} = (1 - yx) + y\varepsilon^2.$$

This system is invariant under the associated macroscope ($u = \varepsilon x$, $v = \varepsilon^{-1} y$) and does not present any macroscopic singularity. It is thus seen that only $r2 = 1$ is interesting. One chooses $r2 = 1$ and under the corresponding macroscope ($x = \varepsilon X$, $y = \varepsilon Y$) one obtain the new system:

$$x^{'} = -x(y - x) + \varepsilon^2\,,$$
$$y^{'} = y(y - x) + \varepsilon^2\,,$$

which has the straight line ($y = x$) as macroscopic singularity and is equivalent to the system: $x^{'} = -x(y-x)$, $y^{'} = y(y-x)$. As the line $(y-x)$ is the common factor of this system, in accordance with the definition (regular river) we cannot discuss of river in this case. However, we shall show that $\varphi(x,y) = y - x = 0$ is a particular solution of this system which define the river phenomenon.

Indeed, let

$$V \;:= (1 - X(Y - X))\frac{\partial}{\partial X} + (1 + Y(Y - X))\frac{\partial}{\partial Y}\,, \tag{13}$$

be the vector field associated to the system (12). We recall that an algebraic curve $\varphi(x,y) = 0$ is an algebraic curve of the system (13) if and only if there exist a polynomial $k = k(x,y)$ satisfying

$$V(\varphi) \;= ((1 - X(Y - X))\frac{\partial \varphi}{\partial X} + (1 + Y(Y - X))\frac{\partial \varphi}{\partial Y} = k\varphi\,. \tag{14}$$

Then it is obvious that $\varphi(X,Y) = Y - X = 0$ verify this equation and the cofactor k is defined by $k(X,Y) = (X+Y)$.

Let $F(X,Y) = (1+Y(Y-X))/(1-Y(Y-X))$, then

$F'_y(X,X) = 2X$ is positive for all $X > 0$ and $\lim\limits_{X\to+\infty} XF_y^{'}(X,\,\Phi(X)) = +\infty$.

It results from the Definition 3 that $\Phi(X) = X$ is river (repulsive) for the differential system (12), i.e. for this example, the macroscopic singularity $Y - X = 0$ define (under the condition $\lim\limits_{X\to+\infty} XF_y^{'}(X,\,\Phi(X)) = +\infty$) the river.

Consider now the system

$$\begin{cases} X^{'} = a_{11}X + a_{12}Y + X(b_{11}X + b_{12}Y)\,, \\ Y^{'} = a_{21}X + a_{22}Y + Y(b_{21}X + b_{22}Y)\,, \end{cases} \tag{15}$$

where a_{ij}, b_{ij} $(i,j = 1,2)$ are constants parameters. The system (15) appears in plasma physics, mathematics, chemistry and population evolution system (see [3], [2], [6]). For example, the Lotka-Volterra system can be deduced from this system by choosing $a_{ij} = 0$ $i,j = 1,2$.

For more convenient, the system (15) can be put onto another form which does not contain the linear cross terms $a_{12}Y$ and $a_{21}X$. To do this, let us introduce in (15) the linear transformation

$$\begin{cases} X = x - \dfrac{a_{12}}{b_{12}}\,, \\ Y = y - \dfrac{a_{21}}{b_{21}}\,, \end{cases} \tag{16}$$

provided that $b_{12} \neq 0$ and $b_{21} \neq 0$. Then the system (15) becomes

$$\begin{cases} x^{'} = d_1 + x(a_1 + b_{11}x + b_{12}y)\,, \\ y^{'} = d_2 + y(a_2 + b_{21}x + b_{22}y)\,, \end{cases} \tag{17}$$

which becomes the Lotka-Volterra system if $d_1 = d_2 = 0$. The relation between the new and the old parameters are

$$\begin{aligned} d_1 &= \frac{a_{12}}{b_{12}}\left(\frac{a_{12}b_{11}}{b_{12}} - a_{11}\right)\,, \\ d_2 &= \frac{a_{21}}{b_{21}}\left(\frac{a_{21}b_{22}}{b_{21}} - a_{22}\right)\,, \\ a_1 &= a_{11} - \frac{2b_{11}a_{12}}{b_{12}} - \frac{b_{12}a_{21}}{b_{21}}\,, \\ a_2 &= a_{22} - \frac{2b_{22}a_{21}}{b_{21}} - \frac{b_{21}a_{12}}{b_{12}}\,. \end{aligned} \tag{18}$$

Proof. By deriving relation (16) one obtains:

$$x' = a_{11}\left(x - \frac{a_{12}}{b_{12}}\right) + a_{12}\left(y - \frac{a_{21}}{b_{21}}\right) + \left(x - \frac{a_{12}}{b_{12}}\right)\left(b_{11}\left(x - \frac{a_{12}}{b_{12}}\right) + b_{12}\left(y - \frac{a_{21}}{b_{21}}\right)\right)$$

$$\Leftrightarrow x' = a_{11}x - \frac{a_{11}a_{12}}{b_{12}} + b_{11}x^2 - 2\frac{b_{11}a_{12}}{b_{12}}x + b_{12}yx - \frac{b_{12}a_{21}}{b_{21}}x + \frac{a_{12}^2}{b_{12}^2}b_{11}$$

$$\Leftrightarrow x' = \frac{a_{12}}{b_{12}}\left(\frac{a_{12}}{b_{12}}b_{11} - a_{11}\right) + x\left(a_{11} - \frac{2b_{11}a_{12}}{b_{12}} - \frac{b_{12}a_{21}}{b_{21}} + b_{11}x + b_{12}y\right).$$

The same calculus with the second variable y leads to the following equation:

$$y' = a_{21}\left(x - \frac{a_{12}}{b_{12}}\right) + a_{22}\left(y - \frac{a_{21}}{b_{21}}\right) + \left(y - \frac{a_{21}}{b_{21}}\right)\left(b_{21}\left(x - \frac{a_{12}}{b_{12}}\right) + b_{22}\left(y - \frac{a_{21}}{b_{21}}\right)\right)$$

$$\Leftrightarrow y' = a_{22}y - \frac{a_{22}a_{21}}{b_{21}} + b_{21}yx - \frac{b_{21}a_{12}}{b_{12}}y + b_{22}y^2 - 2\frac{b_{22}a_{21}}{b_{21}}y + \frac{a_{21}^2}{b_{21}^2}b_{22}$$

$$\Leftrightarrow y' = \frac{a_{21}}{b_{21}}\left(\frac{a_{21}}{b_{21}}b_{22} - a_{22}\right) + y\left(a_{22} - \frac{2b_{22}a_{21}}{b_{21}} - \frac{b_{21}a_{12}}{b_{12}} + b_{21}x + b_{22}y\right). \qquad \square$$

Consider now the system

$$\begin{cases} x' = d_1 + x(b_{11}x + b_{12}y), \\ y' = d_2 + y(b_{21}x + b_{22}y). \end{cases} \tag{19}$$

It is clear that this system is equivalent to the system

$$\begin{cases} x' = 1, \\ y' = \dfrac{d_2 + y(b_{21}x + b_{22}y)}{d_1 + x(b_{11}x + b_{12}y)}. \end{cases} \tag{20}$$

Theorem 7. *If the constant parameters in system (19) verify:*

1) $b_{11} = -b_{21}; b_{12} = -b_{22}$. *2)* $d_1b_{21} + d_2b_{22} = 0$. *3)* $d_1b_{21} \neq 0$.

Then the system (19) admits a river in $X = +\infty$. *This river is attractive if* $d_2b_{12} < 0$, *and repulsive if* $d_2b_{12} > 0$.

Proof. The proof of this theorem is based on the fact that if one put $\varphi(x,y) = b_{11}x + b_{12}y$, then $\varphi(x,y) = 0$ represents a macroscopic singularity of equation (19) which is also an algebraic curve of this system. The condition $d_2b_{12} < 0$ allows us when one put $y = -\frac{b_{11}}{b_{12}}x$ to have $\lim_{x\to+\infty} xf_y'(x,y) = -\infty$

where $\frac{dy}{dx} = \frac{d_2+y(b_{21}x+b_{22}y)}{d_1+x(b_{11}x+b_{12}y)} = f(x,y)$ for all $(x,y) \in \{(x,y) : \varphi(x,y) = 0\}$.

As $\varphi(x,y)$ is polynomial, in accordance with Definition 2, it follows that $\varphi(x,y)$ is a solution for the previous system (19). $\square$

Observation. One could hope to say that a macroscopic singularity is sufficient to determine the river. It is not the case, since when one sees the next one example, the macroscopic singularity is solution of the differential equation but does not verify the condition of attraction or repulsion as it is mentioned in Definition 3.

$$\begin{cases} x' = d_1 + x(b_{11}x + b_{12}y)\,, \\ y' = d_2 + y(b_{11}x + b_{12}y)\,. \end{cases} \tag{21}$$

One observes that when the condition $d_1b_{11} + d_2b_{12} = 0$ is satisfied, then $\varphi(x,y) = b_{11}x + b_{12}y$ is an algebraic curve of the system (21) which is also a macroscopic singularity. However, as it is obvious to see, it is not a critical point at infinity since $XQ_2(X,Y) - YP_2(X,Y) \equiv 0$. In this case one cannot talk about attractivity or repulsively since the field is trivial. Also, if one puts

$$\frac{dy}{dx} = \frac{d_2 + y\,(b_{11}x + b_{22}y)}{d_1 + x\,(b_{11}x + b_{22}y)} = f(x,y)\,,$$

one clearly sees that f_y'cancels itself along the curve $\varphi(x,y) = b_{11}x + b_{12}y = 0$, and thus $\lim\limits_{x\to+\infty} xf_y'\,(x,\,y) = 0$, where $y = -\frac{b_{11}}{b_{22}}x$.

5. Conclusion

We have discussed here a class of differential system for which we cannot applied the result given in [1] and [4]. It is part of answer to an open problem in the situation where the Newton diagram does not lead to a fast-slow field. This result can nevertheless enable us to build river phenomena by giving birth to the macroscpic singularities, and to see the conditions in which these singularities will be attractive or repulsive.

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