GLOBAL EXISTENCE FOR A QUASILINEAR HYPERBOLIC EQUATION IN A NONCYLINDRICAL DOMAIN

J. Ferreira¹ ⁵, C.A. Raposo², M.L. Santos³

¹,²Departamento de Matemática
Universidade Federal de São João del-Rei
Praça Frei Orlando, 170, CEP: 36307-352
São João del-Rei, MG, BRAZIL
¹e-mail: jf@ufsj.edu.br
²e-mail: raposo@ufsj.edu.br
³Departamento de Matemática
Universidade Federal do Pará
CEP: 66075-110 Belém, PA, BRAZIL
and
IESAM-Instituto de Estudos Superiores da Amazônia
Av. Governador José Malcher
1148, Belém, PA, BRAZIL
e-mail: ls@ufpa.br

Abstract: We study the existence of a weak global solution of the mixed problem to the quasilinear hyperbolic equation

\[ u_{tt} - \text{div} (|\nabla u|^{p-2} \nabla u) - \Delta u_t = f(t, x) \quad (1) \]

is a noncylindrical domain. Our proof is based on a penalty argument by J.L. Lions and Galerkin approximations.

AMS Subject Classification: 35L70
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1. Introduction

Let Ω be a bounded domain in \( \mathbb{R}^n \) with smooth boundary \( \partial \Omega \). Given \( T > 0 \)
let $Q = \Omega \times (0, T)$ be a standard cylindrical domain in $\mathbb{R}^{n+1}$ and let $\hat{Q} \subset Q$ be noncylindrical domain in $\mathbb{R}^{n+1}$ with lateral boundary $\hat{\Sigma}$ that will be precised later. We deal with existence of weak solutions of the mixed problem to the quasilinear hyperbolic equation
\[
\begin{cases}
  u_{tt} - \text{div} \left( |\nabla u|^{p-2} \nabla u \right) - \Delta u_t = f & \text{in } \hat{Q}, \\
  u = 0 & \text{on } \hat{\Sigma}, \\
  u(x, 0) = u_0, \quad u_t(x, 0) = u_1 & \text{in } \Omega_0,
\end{cases}
\]
where $\text{div}, \nabla$ and $\Delta$ are respectively the divergence, gradient and the Laplacian operators with respect to the variable $x$ (in the sense of distributions), $p \geq 2$, and $\Omega_0$ is the “basis” of $\hat{Q}$.

Semilinear evolution equations in noncylindrical domains have been considered by several authors. For work in this subject we refer the reader to e.g. [2], [3], [4], [5], [6], [9], [10], [11], [12] and the references therein. It should be noted that none of the above referred papers deals with quasilinear hyperbolic equations in noncylindrical domains. On the other hand, a related problem in cylindrical domains was done by D.D. Ang and A.P.N. Dinh [1]. They considered weak solutions of a wave equation with strong damping involving a quasilinear operator defined in $H^1_0(\Omega)$.

Our analysis is based upon the penalty method introduced by J.L. Lions in [7] and assume a monotonicity condition on the nondylindrical domain $\hat{Q}$ (see condition (2.1) below). We also exploit the fact that $Au = \text{div} \left( |\nabla u|^{p-2} \nabla u \right)$ is a bounded, monotone and hemicontinuous operator from $W^{1,p}_0(\Omega)$ to $W^{-1,p'}(\Omega)$.

The precise statement of our existence theorem is given in Section 2 together with some notations. The proof is presented in Section 3.

\section{2. Notations and Statement of the Main Result}

Let us fix some notations on the function spaces that will be considered. We denote by $\langle \cdot, \cdot \rangle$ and $| \cdot |$ the inner-product and the norm of $L^2(\Omega)$ respectively and by $\langle \cdot, \cdot \rangle$, the duality pairing between $W^{1,p}_0(\Omega)$ and $W^{-1,p'}(\Omega)$. When $p = 2$ we use the notation $H^1_0(\Omega) = W^{1,2}_0(\Omega)$. If $T > 0$ is given and $X$ is Banach space with norm $\| \cdot \|_X$, we denote by $L^p(0, T; X)$, $1 \leq p \leq \infty$, the Banach space of the measurable $X$–valued functions $u : (0, T) \to X$ with $\|u(t)\|_X \in L^p(0, T)$. In this space we consider the norm
\[
\|u\|_{L^p(0, T; X)} = \left( \int_0^T \|u(t)\|^p_X \, dt \right)^{\frac{1}{p}}.
\]
if \( 1 \leq p < \infty \) and for \( p = \infty \) then we assume

\[
\|u\|_{L^\infty(0,T;X)} = \text{ess sup}_{0 < t < T} \|u(t)\|_X.
\] (2.1)

Besides, as usual, we write \( u' \) for the derivative of \( u \) with respect to \( t \) and sometimes we write \( u(t) \) instead \( u(\cdot, t) \).

Next we precise the notations and the hypotheses on the noncylindrical domain \( \hat{Q} \subset Q = \Omega \times (0,T) \). Let us set \( \Omega_s = \hat{Q} \cap (\mathbb{R}^n \times \{s\}) \), where \( 0 < s < T \), \( \Omega_0 = \text{int}(\hat{Q} \cap (\mathbb{R}^n \times \{0\})) \) that is assumed to be non empty, \( \Omega_T = \text{int}(\hat{Q} \cap (\mathbb{R}^n \times \{T\})) \), \( \Gamma_s = \partial \Omega_s \), \( \Sigma = \bigcup_{0 < t < T} \Gamma_t \) the lateral boundary.

We assume that \( \hat{Q} \) is increasing in time, that is,

\[
\Omega_t^* \subset \Omega_s^* \quad \text{if} \quad t < s,
\] (2.2)

where \( \Omega_t^* \) is the projection of \( \Omega_t \) on \( \Omega \). Moreover, we assume the following regularity property:

\[
\text{if} \quad u \in W^{1,p}_0(\Omega) \quad \text{and} \quad u|_{\Omega \setminus \Omega_t^*} = 0 \quad \text{then} \quad u|_{\Omega_t^*} \in H^{1/2}_{0}(\Omega_t^*).
\] (2.3)

Some function spaces are in order. We consider

\[
L^q(0,T;L^p(\Omega_t)) = \{ u \in L^q(0,T;L^p(\Omega)) : u|_Q \setminus \hat{Q} = 0 \},
\]
i \leq p, q \leq \infty, with the norm defined by

\[
\|u\|_{L^q(0,T;L^p(\Omega_t))} = \left( \int_0^T \|u(t)\|_{L^p(\Omega_t^*)}^q \, dt \right)^{\frac{1}{q}}
\]

if \( 1 \leq q < \infty \) and

\[
\|u\|_{L^\infty(0,T;L^p(\Omega_t))} = \text{ess sup}_{0 < t < T} \|u(t)\|_{L^p(\Omega_t^*)}.
\]

We also consider the space \( L^q(0,T;W^{1,p}_0(\Omega_t)) \) with an obvious meaning. Then as a consequence of the regularity property (2.3), we have that \( L^q(0,T;L^p(\Omega_t)) \) and \( L^q(0,T;W^{1,p}_0(\Omega_t)) \) are closed subspaces of \( L^q(0,T;L^p(\Omega)) \) and \( L^q(0,T;W^{1,p}_0(\Omega)) \) respectively.

Now we are in a position to state our existence result.

**Theorem 2.1.** Assume that (2.2)-(2.3) hold. Then given \( u_0 \in W^{1,p}_0(\Omega_0) \), \( u_1 \in L^2(\Omega_0) \) and \( f \in L^2(0,T;L^2(\Omega_t)) \), there exists a function \( u : Q \to \mathbb{R} \) such that

\[
u \in L^\infty(0,T;W^{1,p}_0(\Omega_t)),
\] (2.4)
\[ u' \in L^\infty(0, T; L^2(\Omega_t)) \cap L^2(0, T; H^1_0(\Omega_t)) \] (2.5)
\[ u(0) = u_0 \quad \text{and} \quad u'(0) = u_1 \quad \text{a.e. in} \quad \Omega_0^+, \] (2.6)
\[ u'' - \text{div}(|\nabla u|^{p-2}\nabla u) - \Delta u' = f \quad \text{in} \quad L^2(0, T; W^{-1,p'}(\Omega_t)). \] (2.7)

We observe that initial conditions (2.6) make sense since (2.4), (2.5) and (2.7) imply that \( u' \) is weakly continuous from \([0, T]\) to \( W^{-1,p'}(\Omega_0^+) \).

3. Proof of Theorem 2.1

The proof of Theorem 2.1 will be done by using the Faedo-Galerkin method. First we find a solution of a penalized problem on the cylinder \( Q \) and then we show that its restriction to the noncylindrical domain \( \hat{Q} \) is in fact a weak solution of the original problem.

To this end, let \( \tilde{u}_0 \in W^{1,p}_0(\Omega) \), \( \tilde{u}_1 \in L^2(\Omega) \) and \( \tilde{f} \in L^2(Q) \) be the extensions, by zero outside of \( \Omega_0 \), of \( u_0 \), \( u_1 \) and \( f \) respectively. Let \( M : Q \to \mathbb{R} \) be the “penalty” function defined by

\[ M(x, t) = \begin{cases} 0, & \text{if} \quad (x, t) \in \hat{Q} \cup \Omega_0, \\ 1 & \text{otherwise}. \end{cases} \]

Accordingly, we first prove the following theorem.

**Theorem 3.1.** Suppose the hypotheses of Theorem 2.1 hold. Then for each \( \epsilon > 0 \) there exists a function \( u_\epsilon : Q = \Omega \times (0, T) \to \mathbb{R} \) such that

\[ u_\epsilon \in L^\infty(0, T; W^{1,p}_0(\Omega)), \] (3.8)
\[ u_\epsilon' \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1_0(\Omega)), \] (3.9)
\[ u_\epsilon(0) = \tilde{u}_0 \quad \text{and} \quad u_\epsilon'(0) \quad \text{a.e. in} \quad \Omega, \] (3.10)
\[ u_\epsilon'' - \text{div}(|\nabla u_\epsilon|^{p-2}\nabla u_\epsilon) - \Delta u_\epsilon' + \frac{1}{\epsilon} M u_\epsilon' = \tilde{f} \quad \text{in} \quad L^2(0, T; W^{-1,p'}(\Omega)). \] (3.11)

The proof of Theorem 3.1 will be done in several steps. We begin with the:

**Step 1: Penalized Approximated Problem.** Let us fix an integer \( r \) greater than \( 1 + \frac{n}{2} - \frac{n}{p} \) so that \( H^r_0(\Omega) \hookrightarrow W^{1,p}_0(\Omega) \) continuously. Let \( \{w_1, w_2, w_3, \cdots\} \) be an orthonormal “Galerkin” basis of \( H^r_0(\Omega) \) and for each \( m \in \mathbb{N} \) let \( V_m \) be the subspace spanned by \( \{w_1, w_2, \cdots, w_m\} \). For a given \( \epsilon > 0 \) we consider function

\[ u_m(t) = \sum_{j=1}^{m} g_{mj}(t)w_j, \]
where \( g_{em_j}(t) \) are the solutions of the ODE system
\[
\begin{align*}
(u''_{em}(t), w_j) &+ \langle Au_{em}(t), w_l \rangle + \langle \nabla u'_{em}(t), \nabla w_j \rangle \\
+ \frac{1}{\epsilon} (M(t)u'_{em}(t), w_j)(\tilde{f}(t), w_j), \\
&u_{em}(0) = u_{0m} \quad \text{and} \quad u'_{em}(0) = u'_{1m},
\end{align*}
\]
(3.12)
with \( u_{0m} \to \tilde{u}_0 \) strongly in \( W^{1,p}_0(\Omega) \) and \( u_{1m} \to \tilde{u}_1 \) strongly in \( L^2(\Omega) \). As it is well-known, the system (3.12)-(3.13) has a local solution \( u_{em}(t) \) defined in some interval \( [0, t_{em}) \), \( 0 < t_{em} < T \) (see e.g. [8]).

Step 2: A Priori Estimates I. From the approximated equation (3.12) we get
\[
\frac{1}{2}|u'_{em}(t)|^2 + \frac{1}{p}||u_{em}(t)||_{W^{1,p}_0(\Omega)}^p + \int_0^t ||u'_{em}(s)||_{H^1_0}^2 \, ds + \frac{1}{\epsilon} \int_0^t |M(s)u'_{em}(s)|^2 \, ds \\
\leq \frac{1}{2}|u_{0m}|^2 + \frac{1}{p}||u_{1em}||_{W^{1,p}_0(\Omega)}^p + \frac{1}{2}||\tilde{f}||_{L^2(\Omega)} + \frac{1}{2} \int_0^t |u'_{em}(s)|^2 \, ds.
\]
Then by the Gronwall’s Lemma
\[
\frac{1}{2}|u'_{em}(t)|^2 + \frac{1}{p}||u_{em}(t)||_{W^{1,p}_0(\Omega)}^p \\
+ \frac{1}{\epsilon} \int_0^t |M(s)u'_{em}(s)|^2 \, ds + \int_0^t ||u'_{em}(s)||_{H^1_0}^2 \, ds \leq C,
\]
for some constant \( C > 0 \) independently of \( \epsilon, m \) and \( t \). So we can extend the approximated solutions \( u_{em}(t) \) to the whole interval \( [0, T] \). Besides we get
\[
\begin{align*}
(u_{em}) & \quad \text{is bounded in} \quad L^\infty(0, T; W^{1,p}_0(\Omega)), \\
(u'_{em}) & \quad \text{is bounded in} \quad L^\infty(0, T; L^2(\Omega)), \\
(u''_{em}) & \quad \text{is bounded in} \quad L^2(0, T; H^1_0(\Omega)), \\
(\frac{1}{\sqrt{\epsilon}} Mu'_{em}) & \quad \text{is bounded in} \quad L^2(0, T; L^2(\Omega)), \\
(u_{em}(T)) & \quad \text{is bounded in} \quad W^{1,p}_0(\Omega), \\
(u'_{em}(T)) & \quad \text{is bounded in} \quad L^2(\Omega).
\end{align*}
\]
(3.14) (3.15) (3.16) (3.17) (3.18) (3.19)

Further \( Au = -\text{div}(|\nabla u|^{p-2}\nabla u) \) is a bounded operator from \( W^{1,p}_0(\Omega) \) to \( W^{-1,p'}(\Omega) \), it follows from (3.14) that
\[
\begin{align*}
(Au_{em}) & \quad \text{is bounded in} \quad L^\infty(0, T; W^{-1,p'}(\Omega)).
\end{align*}
\]
(3.20)
Step 3: A Priori Estimates II. Now we are going to obtain an estimate for $u''_{em}$. This will be done with a standard projection argument. Let us define the projection operator $P_m : H^1_0(\Omega) \rightarrow V_m$ by

$$P_m[h] = \sum_{j=1}^m ((h, w_j))w_j, \quad h \in H^1_0(\Omega),$$

where $((\cdot, \cdot))$ denotes the inner-product in $H^1_0(\Omega)$. Let $P_{m}^* \in L(H^{-r}(\Omega), H^{-r}(\Omega))$ be the self-adjoint extension of $P_m$. Then since $P_{m}^*[h] = P_m[h] = h$ for all $h \in V_m$, we conclude from the approximated equation that

$$(u''(t), v) = (P_{m}^*[\tilde{f}(t)], v) - (P_{m}^*[Au_{em}(t)], v) + (P_{m}^*[\triangle u_{em}(t)], v)$$

$$- \frac{1}{\epsilon}(P_{m}^*[M(t)u_{em}(t)], v)$$

for all $v \in V_m$. So by a density argument we have from (3.16), (3.17) and (3.20) that

$$(u''_{em}) \text{ is bounded in } L^2(0,T; H^{-r}(\Omega)).$$ (3.21)

Step 4: Passage to the Limit. We now pass limit (for $m \rightarrow \infty$) on the penalized approximated problem. First we observe that from (3.14) and (3.16), going to a subsequence if necessary, there exists a function $u_{\epsilon}$ such that

$$u_{em} \rightharpoonup u_{\epsilon} \text{ weakly star in } L^\infty(0,T; W^{1,p}_0(\Omega)),$$ (3.22)

$$u'_{em} \rightharpoonup u'_{\epsilon} \text{ weakly in } L^2(0,T; H^1_0(\Omega)),$$ (3.23)

and from (3.20) there exists $\chi_{\epsilon} \in W^{-1,p'}(\Omega)$ such that

$$Au_{em} \rightharpoonup \chi_{\epsilon} \text{ weakly star in } L^\infty(0,T; W^{-1,p'}_0(\Omega)).$$ (3.24)

Moreover, (3.18), (3.19) and (3.21) imply that

$$u_{em}(T) \rightharpoonup u_{\epsilon}(T) \text{ weakly in } W^{1,p}_0(\Omega),$$ (3.25)

$$u'_{em}(T) \rightharpoonup u'_{\epsilon}(T) \text{ weakly in } L^2(\Omega).$$ (3.26)

Next, by applying the Aubin-Lions Compactness Lemma (see e. g. [8]), we infer from (3.14)-(3.15) an (3.16)-(3.21), that

$$u_{em} \rightarrow u_{\epsilon} \text{ strongly in } L^2(0,T; L^2(\Omega))$$ (3.27)

and

$$u'_{em} \rightarrow u'_{\epsilon} \text{ strongly in } L^2(0,T; L^2(\Omega)).$$ (3.28)
respectively. Then since $M \in L^\infty(0,T;L^\infty(\Omega))$, we also get
\[ \frac{1}{\epsilon} M u_{\epsilon m}' \to \frac{1}{\epsilon} M u'_\epsilon \text{ strongly in } L^2(0,T;L^2(\Omega)). \] (3.29)

The above convergences allow us easily to pass the limit in the approximated equation (3.12). Therefore we obtain
\[ \frac{d}{dt} (u'_\epsilon(t),v) + \langle \chi_\epsilon(t), v \rangle + (\nabla u'_\epsilon(t), \nabla v) + \frac{1}{\epsilon} (M(t)u'_\epsilon(t), v) = (\tilde{f}(t), v), \] (3.30)
for all $v \in W^{1,p}_0(\Omega)$ in the sense of distributions. Moreover, from a standard argument it follows that (3.10) holds.

**Step 5: The Monotonicity Argument.** To complete the proof of Theorem 3.1 we must show that $\chi_\epsilon = Au_\epsilon$. To this end we do an analysis on the monotonicity property of $A : W^{1,p}_0(\Omega) \to W^{-1,p}(\Omega)$. It suffices to show that
\[ \int_0^T \langle \chi_\epsilon(t) - Av, u_\epsilon - v \rangle \, dt \geq 0 \] (3.31)
for all $v \in W^{1,p}_0(\Omega)$, since from hemicontinuity of $A$ we obtain
\[ \chi_\epsilon = Au_\epsilon. \]

To show (3.31) we begin by noting that $\int_0^T \langle Au_{\epsilon m}(t) - Av, u_{\epsilon m} - v \rangle \, dt \geq 0$, for all $v \in W^{1,p}_0(\Omega)$. Then using the convergences (3.22) and (3.24) we get
\[ \limsup_{m \to \infty} \int_0^T \langle Au_{\epsilon m}(t), u_{\epsilon m}(t) \rangle \, dt - \int_0^T \langle \chi_\epsilon(t), v \rangle \, dt - \int_0^T \langle Av, u_{\epsilon m}(t) - v \rangle dt \geq 0. \] (3.32)

Now working on the approximated problem we have
\[ \int_0^T \langle Au_{\epsilon m}(t), u_{\epsilon m}(t) \rangle \, dt = (u'_{\epsilon m}(0), u_{\epsilon m}(0)) - (u'_{\epsilon m}(T)) + \int_0^T |u'_{\epsilon m}(t)|^2 \, dt + \frac{1}{2} \|u_{\epsilon m}(0)\|^2_{H_0^1} - \frac{1}{2} \|u_{\epsilon m}(T)\|^2_{H_0^1} - \frac{1}{\epsilon} \int_0^T (M(t)u'_{\epsilon m}(t), u_{\epsilon m}(t)) \, dt + \int_0^T (\tilde{f}(t), u_{\epsilon m}(t)) \, dt. \]
In order to pass the limit in \( \int_0^T \langle Au_{em}(t), u_{em}(t) \rangle \, dt \), we remark that from the initial conditions (3.13) it follows that

\[
(u'_{em}(0), u_{em}(0)) \to (\bar{u}_1, \bar{u}_0) \quad \text{and} \quad \|u_{em}(0)\|_{H^1_0} \to \|\bar{u}_0\|_{H^1_0}.
\]

In addition, using (3.25) and the fact that \( p \geq 2 \), we have that \( u_{em}(T) \to u_\epsilon(T) \) weakly in \( H^1_0(\Omega) \). Thus the weak lower semicontinuity of the norm gives

\[
\|u_{em}(T)\|_{H^1_0} \leq \liminf_{m \to \infty} \|u_\epsilon(T)\|_{H^1_0}.
\]

From the compact inclusion of \( W^{1,p}_0(\Omega) \) in \( L^2(\Omega) \), we also get from (3.25) that \( u_{em}(T) \to u_\epsilon(T) \) strongly in \( L^2(\Omega) \). Hence

\[
(u'_{em}(T), u_{em}(T)) \to (u'_\epsilon(T), u_\epsilon(T)).
\]

Therefore (3.28), (3.29) and the above limits imply that

\[
\limsup_{m \to \infty} \int_0^T \langle Au_{em}(t), u_{em}(t) \rangle \, dt \leq (u'_\epsilon(0), u_\epsilon(0)) - (u'_\epsilon(T), u_\epsilon(T))
\]

\[
+ \int_0^T |u'_\epsilon(t)|^2 \, dt + \frac{1}{2} ||\bar{u}_0||_{H^1_0}^2 - \frac{1}{2} ||u_\epsilon(T)||_{H^1_0}^2 - \frac{1}{\epsilon} \int_0^T (M(t)u'_\epsilon(t), u_\epsilon(t))dt
\]

\[
+ \int_0^1 (f(t), u_\epsilon(t))dt. \quad (3.33)
\]

On the other hand, with a density argument, we conclude from the approximated problem that

\[
- \int_0^T \langle \chi_\epsilon(t), u_\epsilon(t) \rangle dt \leq (u'_\epsilon(T), u_\epsilon(T)) - (u'_\epsilon(0), u_\epsilon(0)) - \int_0^T |u'_\epsilon(t)|^2dt
\]

\[
+ \frac{1}{2} ||u_\epsilon(T)||_{H^1_0}^2 - \frac{1}{2} ||\bar{u}_0||_{H^1_0}^2 + \frac{1}{\epsilon} \int_0^T (M(t)u'_\epsilon(t), u_\epsilon(t))dt
\]

\[
- \int_0^T (f(t), u_\epsilon(t))dt. \quad (3.34)
\]

Then combining (3.32), (3.33) and (3.34) we get (3.31). The proof of Theorem 3.1 is complete.

**Proof of Theorem 2.1.** Now we show that as \( \epsilon \to 0 \), \( u_\epsilon \) converges to a function \( w \) whose restriction to \( \hat{Q} \) is the desired weak solution of our problem.
Let us remark that since the estimates obtained in Steps 2 and 3 do not depend on $\epsilon$, going to a subsequence if necessary, there exists a function $w$ such that
\[
\begin{align*}
  u_\epsilon &\to w \quad \text{weakly star in } L^\infty(0,T;W^{1,p}_0(\Omega)), \\
  u_\epsilon &\to w \quad \text{strongly in } L^2(0,T;L^2(\Omega)), \\
  u'_\epsilon &\to w' \quad \text{weakly in } L^2(0,T;H^1_0(\Omega)), \\
  u'_\epsilon &\to w' \quad \text{strongly in } L^2(0,T;L^2(\Omega)),
\end{align*}
\]
and for some function $\Theta$
\[
\frac{1}{\epsilon} Mu'_\epsilon \to \Theta \quad \text{strongly in } L^2(0,T;L^2(\Omega)).
\]

Then, with the same argument used before, we infer that
\[
\begin{align*}
  w'' - \text{div} \left( |\nabla w|^{p-2} \nabla w \right) - \Delta w' + \Theta \tilde{f} &\in L^2(0,T;W^{-1,p'}(\Omega)), \\
  w(0) = \tilde{u}_0 \quad \text{and } w'(0) = \tilde{u}_t \quad \text{a.e. in } \Omega.
\end{align*}
\]

On the other hand, from (3.17), there exists $C > 0$ such that
\[
\int_Q M(x,t)u'_\epsilon(x,t)^2 dx dt \leq C \epsilon
\]
and hence $Mw' = 0$ a.e. in $Q$. Then by the definition of $M$, $w' = 0$ a.e. in $Q \setminus \hat{Q}$. This implies that $w$ restricted to $Q \setminus \hat{Q}$ is a constant function with respect to $t$, and since $w(0) = \tilde{u}_0 = 0$ a.e. in $\Omega \setminus \Omega^*_0$, we get from the assumption (2.1) that $w = 0$ a.e. in $Q \setminus \hat{Q}$. Therefore setting $u = w|_{\hat{Q}}$ we see that $u$ satisfies (2.3) and (2.4). Besides, noting that $M = 0$ in $\hat{Q}$ we obtain from finally (3.35) that
\[
  u'' - \text{div} \left( |\nabla u|^{p-2} \nabla u \right) - \Delta u' = f \quad \text{in } L^2(0,T;W^{-1,p'}(\Omega_t)).
\]
The condition (2.6) follows from (3.36), and hence the proof of Theorem 2.1 is complete.

\[\square\]

4. Final Comments

The method explored in this work can be used to solve problems with the differential equations
\[
  u_{tt} - \text{div} \left( |\nabla u|^{p-2} \nabla u \right) - \Delta u_t + F(u) = f,
\]
(4.37)
where $F$ is a continuous functions such that $sF(s) \geq 0$ for everg real $s$, and

$$
\begin{align*}
    u_{tt} - \text{div} \left( |\nabla u|^{p-2} \nabla u \right) - \triangle v_t &= 0, \\
    v_{tt} - \text{div} \left( |\nabla v|^{p-2} \nabla v \right) - \triangle u_t &= 0.
\end{align*}
$$

(4.38)

Now we would like to mention that the problem (1) as well as the corre-
sponding problem for the equations (4.37)-(4.38) in a bounded domain with
moving boundary related with the uniqueness of weak solutions are interesting
open problem.

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