EVEN AUTOMORPHISMS OF TREES AND INDUCTIVE ALGEBRAS

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Abstract: Let $(\pi, \mathcal{H})$ be a unitary representation of a locally compact group $G$. A commutative subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$ is called $\pi$–inductive when $\pi(g)\mathcal{A}\pi(g^{-1}) = \mathcal{A}$ for all $g$. The classification of maximal inductive algebras sheds light on the possible realizations of $\pi$ on function spaces. In this paper we deal with the automorphism group of a locally finite homogeneous tree and its principal series spherical representations. We show that for some exceptional representations there exists just one inductive algebra besides those known. Finally, we generalize the main results to the subgroup of even automorphisms of the tree.

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1. Introduction. Inductive Algebras

Let $G$ be a separable locally compact group and $\pi : G \to \mathcal{B}(\mathcal{H})$ a unitary representation of $G$ on a Hilbert space $\mathcal{H}$. A self–adjoint, weakly closed commutative subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$ is called, according to Mackey’s formulation, a system of imprimitivity for $\pi$ when

$$\pi(g^{-1})\mathcal{A}\pi(g) = \mathcal{A}$$

for every $g \in G$. Given such an algebra, Mackey’s Classical Imprimitivity Theorem ([6, Theorem 3.10]) says that there exist a $G$–space $X$, a measure $\mu$
on $X$, a cocycle $A$ and a Hilbert space $\tilde{\mathcal{H}}$ such that:

— $\pi$ is equivalent to the representation $\pi_X$ on $L^2(X, d\mu, \tilde{\mathcal{H}})$ defined in the following way:

$$\left(\pi_X(g)f\right)(x) = A(g, x)\left(\frac{d\mu(g^{-1}x)}{d\mu(x)}\right)^{1/2} f(g^{-1}x).$$

(1.2)

— Through the equivalence, $A$ corresponds to $L^\infty(X, d\mu)$, regarded as an algebra of multiplication operators on $L^2(X, d\mu, \tilde{\mathcal{H}})$.

Hence the study of systems of imprimitivity for $\pi$ can lead to progress in the classification of realizations of $\pi$ on function spaces. More generally, a (non necessarily self–adjoint) subalgebra of $\mathcal{B}(\mathcal{H})$ is called a $\pi$–inductive algebra when it is commutative and stable through conjugation by $\pi$. Realizations of $\pi$ on spaces of functions on $X$ (not necessarily $L^2$–spaces) can give rise to $\pi$–inductive algebras of multipliers (not necessarily $L^\infty$ algebras). Conversely, one can identify possible realizations of $\pi$ through a systematic search for $\pi$–inductive algebras.

By Mackey’s Theorem, replacing a system of imprimitivity by a smaller one gives a quotient of the first realization. So one usually restricts to identifying systems of imprimitivity of a given representation which are maximal with respect to inclusion. Accordingly, we shall restrict here to identification of maximal inductive algebras.

This approach to realizations was proposed by Tim Steger. No precise correspondence between realizations and inductive algebras has yet been proved. Moreover, one should make sure that restriction to maximal algebras causes no loss of generality. Still, in Vemuri and Steger’s work about matrix groups, this method precisely recovered the classical realizations. So, when the inductive algebra approach fails, it can be conjectured that there exist somewhat unnatural realizations. On the other hand, detection of unexpected maximal inductive algebras for other representations could lead to the discovery of hitherto unknown realizations associated with them.

We summarize Steger and Vemuri’s work, pointing out the occurrences of nonself–adjoint inductive algebras. In [11], Vemuri studied the standard representation of the Heisenberg group. Maximal inductive algebras for this representation turn out to be self–adjoint, hence imprimitivity systems. For the three–dimensional Heisenberg group Vemuri proves that the following list of realizations is complete: the standard one on $L^2(\mathbb{R}^2)$, those obtained from it through Fourier transform and (more generally) through the metaplectic group, and those on $L^2((\mathbb{R}/\mathbb{Z})^2)$. 
Representations of $SL(2, \mathbb{R})$ (see [5, Chapter II, 5-6]) were dealt with by Steger and Vemuri in [10]. For the principal series there exist two different realizations on $L^2(\partial D)$, where $D$ is the unit disc. The two maximal inductive algebras can both be identified, in the corresponding realizations, with the self–adjoint multiplier space $L^\infty(\partial D)$.

The discrete series, instead, has a natural realization on an $L^2$–space of holomorphic functions on $D$. Here the only maximal inductive algebra is the multiplier algebra $H^\infty(D)$. Finally, for the complementary series there exist exactly two different, reciprocally adjoint maximal inductive algebras. The two corresponding realizations are, respectively, a Sobolev space on the circle $\partial D$ and its dual.

Now there is a close analogy (see [1], [2]) between the principal series of $SL(2, \mathbb{R})$ and the spherical principal series representations of the full automorphism group of a homogeneous tree. These representations are realized on the boundary $\Omega$ of the tree, and the role of $L^2(\partial D)$ is played now by $L^2(\Omega)$, on which $L^\infty(\Omega)$ acts by multiplication. In our previous paper [9] we classified maximal inductive algebras for such representations. As in Steger and Vemuri’s work, our results suggest that the list of realizations is complete, by ruling out existence of unexpected maximal inductive algebras.

In this paper we prove further results about the principal series representations. For the midpoint representation we show that there exists exactly one more maximal inductive algebra. This algebra is still self–adjoint but, unlike the previous examples, it is not maximal Abelian. Moreover, for general representations, we show that restricting to the subgroup of even automorphisms does not bring in other maximal inductive algebras.

We refer the reader to [9] for a review of possible extensions of the inductive algebra approach to complementary and cuspidal series of the full automorphism group and to the principal series of $PGL(2, \mathbb{F})$, where $\mathbb{F}$ is a nonarchimedean local field.

2. The Automorphism Group of a Homogeneous Tree

A tree is a connected graph without circuits. This means that, given $x, y$ in the set $\mathcal{X}$ of vertices, one can choose in a unique way $n \in \mathbb{N}$ and $x_0, x_2, \ldots, x_n \in \mathcal{X}$ so that $x_0 = x, x_n = y, (x_j, x_{j+1})$ is an edge and $x_j \neq x_{j+2}$. Then $(x_0, x_1, x_2, \ldots, x_n)$ is called a chain of length $n$ connecting $x_0$ with $x_n$. As a generalization one obtains infinite and doubly infinite chains by allowing in-
dexes to vary in \( \mathbb{N} \) and \( \mathbb{Z} \) respectively. Doubly infinite chains are also called geodesics.

Given \( x \neq y \in \mathcal{X} \), let \( d(x, y) \) be the length of the shortest chain connecting them. Then \( d \) is a distance on \( \mathcal{X} \). When a reference vertex \( O \) is fixed, we will often put \( |x| := d(x, O) \). Given \( x \in \mathcal{X}, n \in \mathbb{N}, \) we let \( B(x, n) := \{ y \in \mathcal{X} : d(y, x) \leq n \} \) (the ball centered in \( x \) with radius \( N \)) and \( \partial B(x, n) := \{ y \in \mathcal{X} : d(y, x) = n \} \) (the boundary of the ball).

By definition, an automorphism of \( \mathcal{X} \) is a distance–preserving bijection of the set of vertices. Let \( G \) be the full group of automorphisms of \( \mathcal{X} \). Then \( G \) is a locally compact topological group with respect to the topology generated by the sets

\[
U_F(g) := \{ h \in G : h = g \text{ on } F \}
\]

with \( F \) varying among the finite subsets of \( \mathcal{X} \).

**Definition 1.** For \( A \subset \mathcal{X} \) we let \( K_A \) be the subgroup of automorphisms fixing \( A \) pointwise. In particular, for a vertex \( x \), \( K_x \) is the subgroup of automorphisms leaving \( x \) fixed.

If a reference vertex \( O \) is fixed, \( \mathcal{X} \) can be identified with \( G/K_O \). As a consequence, complex–valued functions on \( \mathcal{X} \) can be lifted to right–\( K_O \)–invariant functions on \( G \). It suffices to put \( f(g) := f(x) \) whenever \( gO = x \). Call a function \( f : \mathcal{X} \to \mathbb{C} \) radial when \( f(x) \) only depends on \( |x| \): then radial functions correspond to bi–\( K_O \)–invariant functions on \( G \).

### 3. The Boundary of \( \mathcal{X} \) and the Principal Spherical Series of \( G \)

Two infinite chains \( (x_j), (y_j) \) are declared equivalent when, up to a shift of indexes, \( x_j = y_j \) infinitely often. The resulting quotient space is called the boundary of \( \mathcal{X} \) and is denoted by \( \Omega \). Note that \( \Omega \) is a \( G \)–space in a natural way. For \( x \neq y \in \mathcal{X} \) we let \( \mathcal{X}(x, y) \) be the set of vertices which can be reached by chains starting in \( x \) and having \([x, y]\) as a subchain. We also let \( \Omega(x, y) \) be the set of limit points of infinite chains contained in \( \mathcal{X}(x, y) \). The sets \( \Omega(x, y) \) make up a basis for a totally disconnected compact topology on \( \Omega \). We will represent locally constant functions on \( \Omega \) as in Figure 1 below. The function displayed there has value \( \eta \) on \( \Omega(O, x) \), \( \theta \) on \( \Omega(O, y) \) and \( \varepsilon \) on \( \Omega(x, O) \sim \Omega(O, y) \). When no value is specified for some subset of \( \Omega \), it will be understood that the function is zero there.
Definition 2. Let
\[
\begin{align*}
    \nu(\Omega) &= 1 \\
    \nu(\Omega(O, x)) &= \frac{q^{|x|} - 1}{q + 1} \quad \text{for } x \neq O.
\end{align*}
\]  

(3.1)

Then \( \nu \) can be extended to a Borel probability measure on \( \Omega \). For \( g \in G \) put \( \nu_g(A) := \nu(g^{-1}A) \) and take \( K_O \) as in Definition 1. Then \( \nu_k = \nu \) for \( k \in K_O \). For general \( g, \nu_g \neq \nu \); still \( \nu_g \equiv \nu \) and the Radó–Nikodým derivative \( d\nu_g/d\nu \) takes on finitely many values on \( \Omega \). This derivative is called the Poisson kernel of \( \nu \) relative to \( g \) and is denoted by \( P(g, \cdot) \). As in Section 2, we put \( P(x, \omega) := P(g, \omega) \) for \( x = gO \in X \).

Now let \( f \in L^2(\Omega, \nu) \). For \( g \in G, z \in \mathbb{C} \) put
\[
(\pi_z(g)f)(\omega) = P^z(g, \omega)f(g^{-1}\omega) \quad \nu\text{-a.e.},
\]  

(3.2)

where \( P \) is the Poisson kernel defined above. It is easy to see that \( \pi_z \) is a representation of \( G \) on \( L^2(\Omega, \nu) \). Also, \( \pi_z \) is unitary if and only if \( z = 1/2 + is \) with \( s \in \mathbb{R} \). In this case we will put \( \pi_s := \pi_{1/2+is} \) for short. It is shown in [3, II.3] that the mapping
\[
P^{1/2+is}(g, \cdot) \mapsto P^{1/2-is}(g, \cdot) \quad \text{for every } g \in G
\]  

(3.3)

extends to a unitary operator \( J_s \) that intertwines \( \pi_s \) with \( \pi_{-s} \).

For \( \phi \in L^\infty(\Omega, \nu) \) let \( M_\phi \) act on \( L^2(\Omega, \nu) \) through pointwise multiplication by \( \phi \). Then for every \( s \) the algebra
\[
\mathcal{M} := \{M_\phi : \phi \in L^\infty(\Omega, \nu)\}
\]  

(3.4)

is maximal commutative (see for example [8, Proposition 4.7.6]) and \( \pi_s \)-inductive, hence maximal \( \pi_s \)-inductive. By the intertwining properties of \( J_s, J_s^{-1}\mathcal{M}J_s \) is also \( \pi_s \)-inductive. As shown in [9], \( J_s^{-1}\mathcal{M}J_s \neq \mathcal{M} \) when \( \text{Re} q^is \neq 0 \) and \( \text{Im} q^is \neq 0 \). The family \( \{\pi_s : s \in \mathbb{R}\} \) is called the principal spherical series of \( G \). In fact \((G, K_O)\) is a Gelfand pair and the diagonal coefficients of the representations above with respect to the constant \( 1 \) are bi–\( K_O \)-invariant and normalized to 1 in \( O \). Moreover they are eigenfunctions of the Laplace operator.
so that they correspond to spherical functions in the classical sense. It follows that every $\pi_s$ is irreducible. Throughout this paper, as in [9], it will be understood that $\text{Im} q^{is} \neq 0$.

The correspondence between eigenvectors of $L$ and diagonal coefficients can be made more precise. Indeed, given $\phi$ such that $L\phi = \lambda\phi$, by [3, II.1.2] there exist a finitely additive measure $m$, defined on the subalgebra of Borel subsets of $\Omega$ generated by the sets $\Omega(O, x)$, and $z \in \mathbb{C}$ such that

$$\lambda = \frac{q^{1-z} + q^z}{q+1}$$

and

$$\phi(x) = (\mathcal{P}_z m)(x) := \int_{\Omega} P^z(x, \omega) dm(\omega)$$

for every $x$. We say that $\phi$ is the *Poisson transform* of $m$. When regarded as a function on $X$, $\phi$ is radial if and only if $m$ is a multiple of $\nu$ ([3, II.2.1]). Finally, the normalization condition for spherical functions and the positive-definiteness condition force $m = \nu$ and either $\text{Re} z = 1/2$ or $z \in ([0, 1] \sim \{1/2\}) \times \{0, \pi/(\log q)\}$. In the second case $\pi_z$ is said to belong to the complementary series, and acts unitarily on a different completion of the space of locally constant functions on $\Omega$. In this paper we shall only deal with principal series representations.

4. The Measure $\mu$ on the Boundary

The probability measure $\nu$ defined in Section 3 is invariant through the subgroup $K_O$. For classification purposes it is convenient to consider instead the subgroup $K'$, which stabilizes an edge $\{O, O'\}$, and replace $\nu$ by a $K'$–invariant measure $\mu$. It suffices to put

$$\mu(\Omega(O, x)) = \frac{1}{2} q^{-d(x, \{O, O'\})} \quad \text{for } x \notin \{O, O'\}$$

and extend $\mu$ to the Borel $\sigma$–algebra in the usual way. Accordingly, a slightly different representation of $G$ can be defined as follows.
**Definition 3.** We put $\mathcal{H} := L^2(\Omega, d\nu)$, $\mathcal{H}' := L^2(\Omega, d\mu)$ and, for $f \in \mathcal{H}'$,

$(\pi'_s(g)f)(\omega) := P^{1/2+is}(g, \omega)f(g^{-1}\omega)$ \quad $\mu$–a.e.

where $P$ is defined as in Section 3, but starting from the new measure $\mu$.

It is easy to see that $\mu \cong \nu$ and that the operator

$J : \mathcal{H}' \longrightarrow \mathcal{H}$

$$f \mapsto f \cdot \left(\frac{d\mu}{d\nu}\right)^{1/2+is}$$

intertwines $\pi'_s$ with $\pi_s$. The explicit description of $J$ will be given in Section 8.

In this case the second example of maximal inductive algebra is provided by $I^{-1}_{sM}I_s$, where $I_s$ is a unitary intertwiner of $\pi'_s$ and $\pi'_{-s}$ defined as in (3.3) in terms of the new measure.

### 5. A Martingale Decomposition of $\mathcal{H}'$

**Definition 4.** Let again $\mathcal{H}' := L^2(\Omega, \mu)$ and for $n \geq 1$ let $\mathcal{M}_n \subset \mathcal{H}'$ be generated by

$$\left\{1_{\Omega(O,x)} : d(x, \{O, O'\}) = n\right\}.$$ 

In other words, functions in $\mathcal{M}_n$ depend only of the first $n$ steps from $\{O, O'\}$. Moreover let

$$\begin{align*}
\mathcal{H}_{n+1} &= \mathcal{M}_{n+1} \oplus \mathcal{M}_n \quad (n \geq 1), \\
\mathcal{H}_0 &= C1, \\
\mathcal{K} &= C\mathcal{W} \quad \text{with} \quad W := 1_{\Omega(O,O')}-1_{\Omega(O',O)}, \\
\mathcal{H}_1 &= \mathcal{M}_1 \oplus (\mathcal{H}_0 \oplus \mathcal{K}).
\end{align*}$$

So for every $n \geq 1 \quad \mathcal{M}_n = \mathcal{H}_0 \oplus \mathcal{K} \oplus \bigoplus_{k=1}^n \mathcal{H}_k$. Since locally constant
functions are dense in $\mathcal{H}'$ we conclude that

$$\mathcal{H}' = \mathcal{H}_0 \oplus \mathcal{K} \oplus \bigoplus_{n \geq 1} \mathcal{H}_n.$$ (5.2)

Functions in $\mathcal{H}_n$ ($n \geq 1$) have mean zero on each half of the boundary, while $W$ is orthogonal to $\mathbf{1}$ but not to $\mathbf{1}_{\Omega(O,O')}$ and $\mathbf{1}_{\Omega(O',O)}$ (see Figure 2).

6. The Subspaces $\mathcal{H}_e$ and $\mathcal{H}_o$

From now until Section 9 inclusive, we shall suppose that $\text{Re } q^{is} = 0$.

**Definition 5.** An automorphism $g$ of $X$ is said to be **even** when $d(gx, x)$ is even for some $x \in X$, hence for every $x$. **Odd** automorphisms are defined similarly, and it is easy to see that the set $G^+$ of even automorphisms is a subgroup of index two.

Now consider $\mathcal{H} := L^2(\Omega, \nu)$ and $\pi_s$ as in Section 3.

**Theorem 6.** Let $\text{Re } q^{is} = 0$ and let $\mathcal{H}_e \subset \mathcal{H}$ be generated by the vectors $\pi_s(g)\mathbf{1}$ as $g$ varies in $G^+$. Define $\mathcal{H}_o$ analogously, starting from odd automorphisms. Then $\mathcal{H}_o \oplus \mathcal{H}_e = \mathcal{H}$.

**Proof.** By irreducibility it suffices to prove that $\mathcal{H}_o \bot \mathcal{H}_e$, that is $\langle \pi_s(g)\mathbf{1}, \mathbf{1} \rangle = 0$ for odd $g$. Indeed let $\psi_s := \mathcal{P}_{1/2+is}\nu$ be the Poisson Transform of $\nu$ as in (3.6). Then $\langle \pi_s(g)\mathbf{1}, \mathbf{1} \rangle = \psi_s(gO)$ for all $g$, and we reduce to proving that $\psi_s \equiv 0$ on vertices at odd distance from $O$. Now [3, II.1] shows that $\psi_s$ is an eigenfunction of the Laplace operator $L$. More precisely, given $y \in X$,

$$\langle L\psi_s(y) \rangle := \frac{1}{q+1} \sum_{\{w: d(w, y) = 1\}} \psi_s(w) = \frac{q^{1/2+is} + q^{1/2-is}}{q+1} \psi_s(y),$$ (6.1)

so that $L\psi_s = 0$ when $\text{Re } q^{is} = 0$. In other words the average of $\psi_s$ on the set of neighbors of any vertex is zero. But $\psi_s$ is also radial, hence identically zero on the vertices at odd distance from $O$. \hfill \square

**Remark 7.** While $\pi_s$, $\pi'_s$ are irreducible for all $s$, their restrictions to $G^+$ become reducible when $\text{Re } q^{is} = 0$ and in fact (see [7]) $\pi_s |_{G^+}$ decomposes into irreducibles exactly as shown in Theorem 6.

The following result naturally relates the direct sum decomposition of $\mathcal{H}$ given by Theorem 6 with the martingale decomposition (5.2) of $\mathcal{H}'$. 


Theorem 8. Let $f$ belong to the space $\mathcal{H}_n$ introduced in Definition 4. Suppose, moreover, that $\text{Supp } f \subseteq \Omega(O, x)$, where $d(x, O) = n - 1$. Then

$f \in \mathcal{H}_e$ if and only if $n$ is even,

$f \in \mathcal{H}_o$ if and only if $n$ is odd.

With a view to proving Theorem 8 we associate with every $v \in \mathcal{H}$ a function $\Phi(v)$ in the following way.

Definition 9. Given $v \in \mathcal{H}$, define $\Phi(v) : G \to \mathbb{C}$ by

$$(\Phi(v))(g) := \langle v, \pi_s(g)1 \rangle := \int_{\Omega} v(\omega)P^{1/2-\text{is}}(g, \omega) \, d\nu(\omega).$$

Since $\Phi(v)$ is right $K_O$–invariant, it determines a function on $\mathcal{X}$ which we also denote by $\Phi(v)$.

Remark 10. By definition, $v \in \mathcal{H}_e \ [v \in \mathcal{H}_o]$ if and only if $(\Phi(v))(x) = 0$ whenever $|x|$ is odd [even].

We collect some technical results about the mapping $v \mapsto \Phi(v)$.

Lemma 11. Given $n \geq 1$, let $x \in \mathcal{X}$ with $|x| = n - 1$. Let $f \in \mathcal{H}_n$ and $\text{Supp } f \subseteq \Omega(O, x)$. Then:

(i) $\text{Supp } \Phi(f) \subseteq \mathcal{X}(O, x) \sim \{x\}$

(ii) If $v \in \mathcal{X}(O, x)$ and $d(v, x) = 1$, $\Phi(f)$ is constant on the sets

$\mathcal{X}(O, v) \cap \partial B(O, N) : N \geq n$.

Proof. (i) Let $y \in (\mathcal{X} \sim \mathcal{X}(O, x)) \cup \{x\}$. The construction in [3, II.1] shows that the Poisson kernel $P(y, \cdot)$ is constant on the set $\Omega(O, x)$, hence on $\text{Supp } f$. Put $z := 1/2 + is$ for notational convenience. Then, for such $y$, we have

$$(\Phi(f))(y) = \int_{\Omega} f(\omega)P^z(y, \omega) \, d\nu(\omega) = (\text{const}) \times \int_{\Omega(O, x)} f(\omega) \, d\nu(\omega) = 0,$$

since $|x| = n - 1$ and $f \in \mathcal{H}_n$.

(ii) By hypothesis $|v| = n$. Since $f \in \mathcal{H}_n$ it must be constant on $\Omega(O, v)$, say $f|_{\Omega(O, v)} \equiv f_v$. Moreover, for $y \in \mathcal{X}(O, v)$, if $d(O, y) = N$, then $P(y, \cdot) \equiv q^{2n-N-2}$ on $\Omega(O, x) \sim \Omega(O, v)$. So, for any such $y$,

$$(\Phi(f))(y) = \int_{\Omega(O, v)} f(\omega)P^z(y, \omega) \, d\nu(\omega) + \int_{\Omega(O, v) \sim \Omega(O, v)} f(\omega)P^z(y, \omega) \, d\nu(\omega)$$
\[ f_v \int_{\Omega(O,v)} P^\tau(y, \omega) d\nu(\omega) + q^{(2n-N-2)\tau} \int_{\Omega(O,x)} f(\omega) d\nu(\omega) \]
\[ = f_v \int_{\Omega(O,v)} P^\tau(y, \omega) d\nu(\omega) - q^{(2n-N-2)\tau} f_v \nu(\Omega(O,v)), \]

since \( \int_{\Omega(O,x)} f d\nu = 0 \). Clearly the second summand is independent of \( y \), and so is the first (for fixed \( N \)) by symmetry.

**Lemma 12.** Given \( n \geq 1 \) there exists \( C_n \in \mathbb{C} \) with the following property. Let \( x, f \) be as in Lemma 11: then for every \( v \in B(x, 1) \cap B(O, n) \)
\[ (\Phi(f))(v) = C_n f_v, \tag{6.3} \]
where \( f_v \) is the value of \( f \) at all points of \( \Omega(x, v) \).

**Proof.** Given \( x, v \) as above, consider the Poisson kernel \( P(v, \cdot) \). Now [3, II.1], again, gives \( P(v, \cdot) \equiv q^{n-2} \) on \( \Omega(O, x) \sim \Omega(O, v) \) and \( P(v, \cdot) \equiv q^n \) on \( \Omega(O, v) \). Arguing as in the proof of Lemma 11 we find
\[ (\Phi(f))(v) = \left( q^{n-2} - q^{(n-2)\tau} \right) \nu(\Omega(O, v)) f_v = \left( q^{n-2} - q^{(n-2)\tau} \right) \frac{q^{1-n}}{q + 1} f_v. \]

**Proof of Theorem 8.** We are given \( x \in X \), with \( |x| = n - 1 \) and \( f \in \mathcal{H}_n \) with \( \text{Supp} f \subseteq \Omega(O, x) \). First of all, since \( \Phi(f) \) is a combination of Poisson kernels, (6.1) gives \( L \Phi(f) = 0 \), so that the average of \( \Phi(f) \) around every point is zero. Now, we apply Lemma 11 and Lemma 12. Starting from even \( n \) we find that \( (\Phi(f))(x) = 0 \) whenever \( |x| \) is odd, so \( f \in \mathcal{H}_e \) by Remark 10. The case of odd \( n \) is analogous.

7. The Inductive Algebra \( \mathcal{G} \)

Recall from Theorem 6 the definition of the subspaces \( \mathcal{H}_o, \mathcal{H}_e \), and let \( E_{\mathcal{H}_o}, E_{\mathcal{H}_e} \) be the corresponding projections. By the same theorem, \( E_{\mathcal{H}_o} + E_{\mathcal{H}_e} = I \).

**Definition 13.** We let \( \mathcal{G} \) be the two–dimensional commutative subalgebra of \( \mathcal{B}(\mathcal{H}) \) generated by \( E_{\mathcal{H}_o} \) and \( E_{\mathcal{H}_e} \).

We claim that \( \mathcal{G} \) is \( \pi_s \)-inductive. Indeed \( \pi_s(g) \) fixes both of \( E_{\mathcal{H}_o} \) and \( E_{\mathcal{H}_e} \) if \( g \) is even and interchanges them if \( g \) is odd. Since \( \mathcal{J} \), as constructed in Definition 4.2, intertwines \( \pi_s \) with \( \pi_s' \), it follows immediately that \( \mathcal{J}^{-1} \mathcal{G} \mathcal{J} \) is \( \pi_s' \)-inductive.

**Lemma 14.** When \( \text{Re} q^{is} = 0 \), there exists some maximal \( \pi_s \)-inductive
algebra besides $\mathcal{M}$ and $J_s^{-1}MJ_s$.

**Proof.** As in Section 4, let $K'$ be the stabilizer of the edge $\{O, O'\}$ and define a character $\chi$ on $K'$ as follows:

$$\chi(k) := \begin{cases} 1 & \text{if } kO = O, \\ -1 & \text{if } kO = O'. \end{cases} \quad (7.1)$$

Let $S := E_{H_{O'}} - E_{H_O} \in \mathcal{G}$. Since $\mathcal{J}$ intertwines $\pi'_s$ with $\pi_s$, it is obvious that $\tilde{S} := \mathcal{J}^{-1}S\mathcal{J}$ satisfies $\pi'_s(k)\tilde{S}\pi'_s(k^{-1}) = \chi(k)\tilde{S}$ for $k \in K'$. We say that $\tilde{S}$ transforms according to $\chi$ with respect to $\pi'_s$. But $\tilde{S} \notin \mathcal{CM}_W$, so, by [9, Theorem 3.2], $\tilde{S} \notin \mathcal{M}$. Hence $\mathcal{J}^{-1}\mathcal{G}\mathcal{J} \subset \mathcal{M}$, so $\mathcal{G} \not\subset \mathcal{M}$. Now recall the unitary intertwiner $J_s$ defined in (3.3). Since every Poisson kernel is a linear combination of functions of the form $1_{\Omega(O, x)}$, Theorem 8 gives $[J_s, S] = 0$. Since $S$ and $I$ generate $\mathcal{G}$ we have $J_s\mathcal{G}J_s^{-1} = \mathcal{G}$. We conclude that $\mathcal{G} \notin J_s^{-1}\mathcal{M}J_s$ as well. The proof is achieved by considering a maximal $\pi_s$–inductive algebra containing $\mathcal{G}$.

**Remark 15.** We will prove in Section 9 that in fact the only missing maximal inductive algebra is $\mathcal{G}$ itself. While existence of new maximal inductive algebras not containing $\mathcal{G}$ will be ruled out in the next section by a simple extension of the argument in [9], maximality of $\mathcal{G}$ will require a proof.

This contrasts with the situation studied in [9], where the relevant algebras are known to be maximal commutative. To see that $\mathcal{G}$ is not, consider a nontrivial closed subspace $F$ of $H_{O'}$, and let $T := E_F$. Then $T \notin \mathcal{G}$, $TE_{H_{O'}} = T = E_{H_{O'}}T$, and $TE_{H_O} = 0 = E_{H_O}T$. Hence $\mathcal{G} \oplus CT$ generates a commutative algebra.

### 8. Operators in $\mathcal{A}$ that Transform According to $\chi$

**Theorem 16.** Let $\text{Re} q^{is} = 0$, and let $\mathcal{A}$ be a maximal $\pi'_s$–inductive algebra. Then either $\mathcal{A} = \mathcal{M}$, or $\mathcal{A} = I_s^{-1}MI_s$, or $\mathcal{A} = \mathcal{J}^{-1}\mathcal{G}\mathcal{J}$.

**Proof.** We put $\pi' = \pi'_s$ for short. Recall from Lemma 14 the concept of an operator which transforms according to $\chi$.

The main classifications results of [9] depend on the value of $\text{Re} q^{is}$. However, part (i) of Theorem 3.2 and Theorem 4.1 in that paper do not require condition $\text{Re} q^{is} \neq 0$. So, in the present case, we still know that $\mathcal{A}$ contains a nontrivial operator $T$ that transforms according to $\chi$ with respect to $\pi'$. We claim that either $T \in \mathcal{CM}_W$, or $I_sTI_s^{-1} \in \mathcal{M}$, or $T$ is a nonscalar element of...
$J^{-1} \mathfrak{S} J$. Taking this for granted, we note that $I \in \mathcal{A}$ by maximality. So in the third case $\mathcal{A}$ contains $J^{-1} \mathfrak{S} J$. The first two cases are dealt with in [9, Section 3] and correspond to the first two algebras.

Now we prove the claim. Consider decomposition (5.2), choose bases for $\mathcal{H}_0$, $\mathcal{K}$, $\mathcal{H}_1$, ... to obtain a basis for $\mathcal{H}'$ and write $T$, accordingly, as an infinite block matrix of the form

$$
\begin{pmatrix}
0 & \lambda_{0\mathcal{K}} & \lambda_{1i_1} M_W i_1^2 \\
\lambda_{1i_0} & 0 & \lambda_{2i_2} M_W i_2^2 \\
\vdots & \ddots & \ddots
\end{pmatrix},
$$

where $i_k : \mathcal{H}_k \to \mathcal{H}'$ is the inclusion map. Obviously $T \in \mathbb{C} M_W$ if and only if

$$
\lambda_1 = \lambda_2 = \cdots = \lambda_{0\mathcal{K}} = \lambda_{1i_0}.
$$

The argument in [9, Section 3] shows that when $\text{Re} q^s$ is nonzero this is indeed the case, up to replacing possibly $T$ by $I_s T I_s^{-1}$.

Now we use relation

$$
\lambda_{0\mathcal{K}} \lambda_{1i_0} = \lambda_1^2
$$

from [9, Section 3], which is found by specializing (1.1) to matrix coefficients. This relation holds independently of $s$.

In order to find another relation we suppose from now on, for reference, that $q^s = i$ (the case of $q^s = -i$ is similar) and consider the matrix coefficients relative to $(1, v)$ and $(W, v)$ with $W$ as in (5.1) and $v \in \mathcal{H}_1$ such that $\text{Supp} v \subset \Omega(O, O')$. From (1.1) we get, for every $g$,

$$
\begin{align*}
\lambda_{0\mathcal{K}} \langle T \pi'(g) W, \pi'(g) v \rangle &= \lambda_1 \langle T \pi'(g) 1, \pi'(g) v \rangle, \\
\lambda_{1i_0} \langle T \pi'(g) 1, \pi'(g) v \rangle &= \lambda_1 \langle T \pi'(g) W, \pi'(g) v \rangle.
\end{align*}
$$

(8.2)

Now let $g$ be a translation one step to the left along a geodesic $(x_j)_j = -\infty$, with $x_0 = O$ and $x_1 = O'$. The Poisson kernel of $g$ can be computed by the method explained in [3, II.1]. Refer to Figure 3 for pictures of $\pi'(g) 1$ and $\pi'(g) W$. Choose $v \in \mathcal{H}_1$ as in Figure 4.

Since

$$
\begin{align*}
\pi'(g) 1 &= \frac{q-1}{2q} 1 - \frac{q-1+2i\sqrt{q}}{2q} W + \frac{1-i\sqrt{q}}{q} v_1, \\
\pi'(g) W &= \frac{q-1-2i\sqrt{q}}{2q} 1 - \frac{q-1}{2q} W + \frac{1+i\sqrt{q}}{q} v_1, \\
\pi'(g) v &= \frac{q-1}{2q} \left( q - i\sqrt{q} 1 + q + i\sqrt{q} W - \frac{2i\sqrt{q}}{(q-1)} v_1 \right),
\end{align*}
$$

with $v \in \mathcal{H}_1$ as in Figure 5, the second equation in (8.2) reads

$$
\lambda_{0\mathcal{K}}^2 (-q + 1 - 2i\sqrt{q})(1 + i\sqrt{q}) + \lambda_1^2 (q - i\sqrt{q})(q - 1).
$$
Note that $Tv_1 = -\lambda_1 v_1$ because $\text{Supp} v_1 \subset \Omega(O', O)$.

When $\lambda_1 = \lambda_{0K} = 0$ we get $\lambda_{K0} = \lambda_{0K} = 0$ from (8.2). Hence we can suppose $\lambda_1 \neq 0$ and use relation $\lambda_{K0} \lambda_{0K} = \lambda_1^2$ to rewrite (8.3) in terms of...
\[ y := \lambda_{0K}/\lambda_1 \text{ as follows:} \]
\[ (-q^2 + 3q - 3i\sqrt{q} + i\sqrt{q})y^3 + (q^2 - 3q - 3i\sqrt{q} + iq\sqrt{q})y^2 \]
\[ + (q^2 - 3q + 3i\sqrt{q} - iq\sqrt{q})y - q^2 + 3q - i\sqrt{q} + 3iq\sqrt{q} = 0. \]
\[ (8.4) \]

Two of the roots of (8.4), namely
\[ 1 \quad \text{and} \quad \frac{q - 1 - 2i\sqrt{q}}{q - 1 + 2i\sqrt{q}} = \frac{(q - 1)/2 - \mathrm{Im} q^{i\alpha}/q}{(q - 1)/2 + \mathrm{Im} q^{i\alpha}/q} \]
correspond to the two cases dealt with in [9, Section 3]. The third root is
\[ \frac{1 + i\sqrt{q}}{1 - i\sqrt{q}}. \]
\[ (8.5) \]

We will prove that when \( \lambda_{K0}/\lambda_1 \) attains the value (8.6), \( T \) is a scalar multiple of
\[ R := J^{-1} S J, \]
with \( S \) as in Definition 13. First of all, let us find the matrix of \( S \) with respect to the decomposition (5.2) of \( \mathcal{H}' \). Put \( \phi := -q \mathbf{1}_{\Omega(O',O)} + \mathbf{1}_{\Omega(O,O')} \).

Then
\[ W = 1 - \frac{q}{q + 1} - \frac{2}{q + 1} \phi. \]
\[ (8.7) \]

For \( k \geq 1 \) let \( \mathcal{H}_k^+ := \{ f \in \mathcal{H}_k : \text{Supp} f \subset \Omega(O, O') \} \) and \( \mathcal{H}_k^- := \{ f \in \mathcal{H}_k : \text{Supp} f \subset \Omega(O', O) \} \). By Theorem 8,
\[ \mathcal{H}_k^+ \subset \mathcal{H}_e, \quad \mathcal{H}_k^- \subset \mathcal{H}_o, \quad \mathcal{H}_k^+ \subset \mathcal{H}_o, \quad \mathcal{H}_k^- \subset \mathcal{H}_e \quad \text{for odd } k, \]
\[ \mathcal{H}_k^+ \subset \mathcal{H}_e, \quad \mathcal{H}_k^- \subset \mathcal{H}_o \quad \text{for even } k. \]
\[ (8.8) \]

From (8.7) and (8.8) we get, up to a multiplicative constant,
\[ S = \begin{pmatrix} 1 & 2 - 2q & I \mid \mathcal{H}_1^+ \\ 0 & 1 + q & -I \mid \mathcal{H}_1^- \\ 0 & -1 & \ddots \end{pmatrix}. \]
As for the matrix of $J$, recall from (4.2) that $J$ is just multiplication by $(d\mu/d\nu)^{1/2+i\pi s}$. Since $q^{1\pi s} = i$ we get, again up to scalar multiples,
\[
(d\mu/d\nu)^{1/2+i\pi s} = (1_{\Omega(O',O)} + i\sqrt{q}1_{\Omega(O,O')}).
\]

It follows that
\[
J^{-1} | H_0 \oplus K = \begin{pmatrix} 1 + i\sqrt{q} & 1 - i\sqrt{q} \\ 1 - i\sqrt{q} & 1 + i\sqrt{q} \end{pmatrix},
\]
\[
J^{-1} |_{\oplus_j H_j^+} = 2I,
\]
\[
J^{-1} |_{\oplus_j H_j^-} = 2i\sqrt{q}I
\]
and the coefficients of $J$ are found similarly. So
\[
R := J^{-1}S J = \begin{pmatrix} 0 & 1 + i\sqrt{q} \\ 1 - i\sqrt{q} & 0 \\ 1 + i\sqrt{q} & 1 - i\sqrt{q} \\ \vdots & \vdots \end{pmatrix}.
\]

Comparison with (8.6) shows that the first $2 \times 2$ blocks of the matrices of $T$ and $R$ are equal, and since both operators are scalar multiples of $M_W$ on $\oplus_{j \geq 1} H_j$ they commute. Put $\varepsilon(g) := (-1)^{|gO|}$. Then $\varepsilon(g) = \varepsilon(g^{-1})$ and $\pi'(g)R\pi'(g^{-1}) = \varepsilon(g)R$. Since $T$ satisfies (1.1) and $[R, T] = 0$ we get
\[
\pi'(g)RT\pi'(g^{-1})RT = \pi'(g)R\pi'(g^{-1})\pi'(g)T\pi'(g^{-1})TR
\]
\[
= \varepsilon(g)RT\pi'(g)T\pi'(g^{-1})R\pi'(g)\pi'(g^{-1})
\]
\[
= (\varepsilon(g))^2 RT\pi'(g)TR\pi'(g^{-1}) = RT\pi'(g)RT\pi'(g^{-1}).
\]
that is, $RT$ belongs to some $\pi'$-inductive algebra. But $RT$, as a product of elements of $\mathfrak{g}$, belongs to $\mathfrak{s}$. By Theorem 3.2, part (i) in [9] we have $RT \in CI$, hence $T \in CR^{-1} = CR$.

9. Maximality of $\mathfrak{g}$

In this section we conclude the classification of maximal inductive algebras for representations of the full automorphism group by proving maximality of the algebra $\mathfrak{g}$ introduced in Section 7. The main result (Lemma 19) extends
Theorem 4.1 in [9] to the case of nonscalar elements outside $\mathcal{G}$. The Hilbert space $\mathcal{H}'$ and the representation $\pi' = \pi'_\mathcal{G}$ are as in Definition 3.

Consider the edge $E := \{O, O', O''\}$, its pointwise stabilizer $K_E$ and finally $K_O$, the stabilizer of $O$ as in Definition 1. Also recall from Section 4 that $K'$ is the setwise stabilizer of $E$.

**Lemma 17.** Let $A \supseteq \mathcal{G}$ be maximal $\pi'$-inductive, and let $T \in A$ be $K_O$-invariant. Then $T \in \mathcal{G}$.

**Proof.** In fact only $K_E$-invariance of $T$ is required. Under this hypothesis, given $j \in G$ interchanging $O$ with $O'$, we find that

$$T_1 := T - \pi'(j)T\pi'(j^{-1}) \quad \text{transforms according to } \chi;$$
$$T_2 := T + \pi'(j)T\pi'(j^{-1}) \quad \text{is } K'-\text{invariant.}$$

Moreover $T_1, T_2 \in A$, so by Theorem 3.2 in [9] and Theorem 16 $T_2$ is scalar and $T_1$ is either a multiple of $M_W$, or of $I_s M_W I_s^{-1}$, or of $S$. Since $T_1$ must commute with $S$ the third case only is possible. Hence $T = \frac{1}{2}(T_1 + T_2) \in \mathcal{G}$. \qed

**Remark 18.** We have stated the lemma in a weaker form to point out the similarity with the case of $G^+$ (see Section 11).

**Lemma 19.** Let $A$ be maximal $\pi'$-inductive, and suppose that $A$ strictly contains $\mathcal{G}$. Then there exists in $A$ some $K_E$-invariant operator which is not $K_O$-invariant.

**Proof.** The following argument is adapted from [9, Section 4].

By Lemma 17 it suffices to prove that $A \sim \mathcal{G}$ contains some $K_E$-invariant operator. Remark first that $A \sim \mathcal{G}$ contains some operator fixed by pointwise stabilizers of nontrivial balls. Indeed, given a ball $B(O, n)$ as in Section 2, let $K_n$ the pointwise stabilizer of $B(O, n)$ and $dk$ its normalized Haar measure. Then, for $T \in A \sim \mathcal{G}$, the operators

$$T_n := \int_{K_n} \pi'(k^{-1})T\pi'(k) \, dk$$

belong to $A$ and weakly approach $T$ as $n \to \infty$. Since $\mathcal{G}$ is weakly closed, $T_n \notin \mathcal{G}$ for infinitely many $n$.

Now edges and nontrivial balls are complete subtrees of $\mathcal{X}$. This means that each of their vertices has either one neighbor or $q + 1$ neighbors in the subtree. So the proof can be achieved by showing that, when the relevant property holds for some finite complete subtree other than an edge, it is inherited by a strictly smaller one. Conjugating by $\pi'$ we can replace $S$ by any edge in $\mathcal{X}$.
or, equivalently, suppose that every subtree involved contains \(\{O, O'\}\). We fix some useful terminology, following [3, III.1].

**Definition 20.** A vertex in a complete subtree \(I\) is called *interior* or, respectively, a *boundary* vertex when each or, respectively, only one of their neighbors lies in \(I\).

The set of boundary vertices is denoted by \(\partial I\) as in the case of balls. Interior vertices of \(I\) having exactly \(q\) neighbors in \(\partial I\) are called *almost terminal*. For the complete subtree in Figure 7, vertex \(A\) is in the boundary, vertex \(B\) is almost terminal and vertex \(C\) is interior but not almost terminal.

\[
\begin{tabular}{ccc}
A & B & C \\
\end{tabular}
\]

**Figure 7:**

Now given \(I\), a finite complete subtree, let \(P \in \partial I\) and let \(P'\) be its unique neighbor in \(I\). Now recall from Section 3 the definition of subsets \(X(P', P)\) of \(X\) and subsets \(\Omega(P', P)\) of \(\Omega\). Let

\[
\begin{align*}
\mathcal{H}_P &:= \{f \in \mathcal{H} : \text{Supp} f \subseteq \Omega(P', P)\}, \\
\mathcal{M}_P^n &:= \left\{ \{1_{\Omega(P, S)} \in \mathcal{H}_P : S \in X(P', P), \ d(P, S) = n\} \right\} \ \text{for} \ n \geq 1, \\
\mathcal{H}_P^1 &:= \{f \in \mathcal{M}_P^1 : f \perp 1_{\Omega(P', P)}\}, \\
\mathcal{H}_P^n &:= \mathcal{M}_P^n \ominus \mathcal{M}_P^{n-1} \ \text{for} \ n > 1.
\end{align*}
\]

If \(T\) is a \(K_T\)–invariant operator it is easily seen that, for every \(P \in \partial I\), \(\mathcal{H}_P^n\) is \(T\) – invariant when \(n \geq 1\). Moreover, letting \(H_P\) be the subgroup

\[
\{g \in K_T : g \text{ fixes } X(P, P') \text{ pointwise}\}
\]

we see that \(\mathcal{H}_P^n\) is spanned by \(\{v \in \mathcal{H}_P : \pi'(h)v = -v \ \text{for some } h \in H_P\}\). It follows that \(T\) acts as a permutation on \(\{1_{\Omega(P', P)} : P \in \partial I\}\). Observe, finally, that

\[
\begin{align*}
\mathcal{H}_P^n &\subset \mathcal{H}_e \ \text{if} \ n \ \text{is even and} \ d(P, O) \ \text{is even or} \ n \ \text{is odd and} \ d(P, O) \ \text{is odd} \\
\mathcal{H}_P^n &\subset \mathcal{H}_o \ \text{if} \ n \ \text{is even and} \ d(P, O) \ \text{is odd or} \ n \ \text{is odd and} \ d(P, O) \ \text{is even}.
\end{align*}
\]
Hence, if $T \in \mathfrak{G}$, there exist $\gamma, \delta \in C$ such that, for every $P \in \partial I$,

$$T \mid_{\mathcal{H}_P} = \lambda_P \bigoplus_{n \geq 0} I_{\mathcal{H}_P^{2n}} \oplus \eta_P \bigoplus_{n \geq 0} I_{\mathcal{H}_P^{2n+1}} \quad (9.5)$$

with

$$(\lambda_P, \eta_P) = (\gamma, \delta) \quad \text{if } d(P, O) \text{ is even},$$

$$(\lambda_P, \eta_P) = (\delta, \gamma) \quad \text{if } d(P, O) \text{ is odd}. \quad (9.6)$$

Here $\mathcal{H}_P^0 := C_1\Omega(P', P)$.

Summing up, if $T$ is $K_I$–invariant and $T \not\in \mathfrak{G}$, one of the following holds:

— For every $P \in \partial I$, $T \mid_{\mathcal{H}_P}$ can be written as in (9.5), but there exists no pair $(\gamma, \delta)$ satisfying (9.6) for every $P$.

— For any $P \in \partial I$, $T(\mathcal{H}_P) \subseteq \mathcal{H}_P$, but for some $P$ the restriction $T \mid_{\mathcal{H}_P}$ is not of the form (9.5).

— For some $P, Q \in \partial I$, $P \neq Q$, $T(\mathcal{H}_P) \not\perp \mathcal{H}_Q$, hence $T(1_{\Omega(P', P)}) \not\perp 1_{\Omega(Q', Q)}$ as observed above.

In the first case we argue as follows. Given $T$ as in the hypothesis, take $Q \in \partial I$. Then

$$\mathcal{J} := I \sim (B(Q', 1) \cap \partial I) \quad (9.7)$$

is a complete subtree of $I$ and $Q' \in \partial \mathcal{J}$. So, for $H_{Q'}$ defined as in (9.3) but in terms of the new subtree, the operator

$$\tilde{T} := \int_{H_{Q'}} \pi'(h)T\pi'(h^{-1}) \, dh \quad (9.8)$$

is $K_{\mathcal{J}}$–invariant. Moreover $\tilde{T}(\mathcal{H}_{Q'}) \subseteq \mathcal{H}_{Q'}$ and

$$\tilde{T} \mid_{\mathcal{H}_{Q'}} = \tilde{\lambda}_Q \bigoplus_{n \geq 0} I_{\mathcal{H}_{Q'}^{2n}} \oplus \tilde{\eta}_Q \bigoplus_{n \geq 0} I_{\mathcal{H}_{Q'}^{2n+1}}, \quad (9.9)$$

where

$$\tilde{\lambda}_Q = \frac{1}{q} \sum_{R \in \partial I \cap B(Q', 1)} \lambda_R \quad \text{and} \quad \tilde{\eta}_Q = \frac{1}{q} \sum_{R \in \partial I \cap B(Q', 1)} \eta_R. \quad (9.10)$$

If $\tilde{T}$ constructed in (9.8) is in $\mathfrak{G}$, take $(\alpha, \beta) \in \{ (\lambda_R, \eta_R) : R \in \partial I \}$. Then, by (9.10), we have

$$\sum_{R \in \partial I \cap B(Q', 1)} = q\alpha \quad \text{and} \quad \sum_{R \in \partial I \cap B(Q', 1)} = q\beta. \quad (9.11)$$

Replace $T$ by its powers $T^n$. If $\tilde{T}^n$ also belongs to $\mathfrak{G}$ for every $n$ we get $T = \tilde{T}$. Repeat the procedure on boundary vertices at odd and even distance from $O$.
to deduce that the function 
\[ Q \mapsto (\lambda_Q, \eta_Q) \]
is constant on both halves of \( \partial I \) and satisfies condition (9.6). Indeed, given a complete subtree \( I \supseteq \{O, O'\} \), and \( P \in I \) almost terminal, vertices in \( B(P, 1) \cap \partial I \) all lie at equal distance from \( O \); they also lie at equal distance from \( O' \).

Hence \( T \) itself belongs to \( \mathfrak{G} \), a contradiction. We deduce that in this case \( I \) is not minimal among complete subtrees for which a stabilizer–invariant operator in \( A \sim \mathfrak{G} \) does exist.

The second case is easy. Indeed, given \( P \in \partial I \) as in the hypothesis, consider a proper complete subtree \( I' \) of \( I \) with \( P \in \partial I' \). Take \( M \in \partial I' \), \( M \neq P \), and put \( T' := \int_{H_M} \pi'(h)T\pi'(h^{-1}) \, dh \) with \( H_M \) as in (9.3). Then \( T' \) is \( K_{I'} \)-invariant and agrees with \( T \) on \( \mathcal{H}_P \). Hence \( T' \not\in \mathfrak{G} \) and we conclude as before.

We finally deal with the third case. This is very easy under the extra hypothesis that \( \text{three vertices at least in the subtree are almost terminal} \). Indeed, let \( P', Q', R' \) be almost terminal and different, and let \( P, Q, R \) be their respective boundary neighbors, with \( T(1_{\Omega(P', P)}) \not\perp 1_{\Omega(Q', Q)} \). Let
\[
\tilde{T}' := \int_{H_{R'}} \pi'(h)T\pi'(h^{-1}) \, dh
\]
with \( H_{R'} \) as in (9.3) relative to the complete subtree \( I' := I \sim (B(R', 1) \cap \partial I) \). Then \( \tilde{T}' \) is \( K_{I'} \)-invariant and agrees with \( T \) on \( \mathcal{H}_P \) and \( \mathcal{H}_Q \). In particular \( \tilde{T}'(1_{\Omega(P', P)}) \not\perp 1_{\Omega(Q', Q)} \), hence \( \tilde{T}' \not\in \mathfrak{G} \).

It remains to consider complete subtrees with only \( \text{one or two almost terminal vertices} \). Such subtrees are easily classified as follows (see Figures 8, 9):

(3.i) unit spheres;
(3.ii) pairs of unit spheres, with adjacent centers;
(3.iii) unions of unit spheres, with centers on a chain of length \( \geq 3 \).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure8.png}
\caption{Figure 8:}
\end{figure}
For subtrees of kind (3.ii) we recall that $T$ must act on the space spanned by $\{f_P := 1_{\Omega(O, P)} : P \in \partial T\}$. We relabel this set as $\{f_i : 1 \leq i \leq 2q\}$, with $i \leq q$ if and only if $\text{Supp} f_i \subset \Omega(O, O')$. Accordingly, when considering case (3.ii) we will only deal with the restrictions of the relevant operators to that finite–dimensional subspace.

Represent $T$, with respect to the basis $\{f_i\}$, as a matrix

$$A = \begin{pmatrix} B & C \\ D & E \end{pmatrix},$$

(9.12)

where $B, C, D, E$ are $q \times q$ blocks. Now, if either $B$ or $E$ is not diagonal (that is, $Tf_i \not\perp f_j$ for characteristic functions $f_i, f_j$ with support on the same half of the boundary), the inductive step can be performed just as in the first case of the lemma.

Otherwise consider $\tilde{T}$ as in (9.9), where the subgroup $H_{O'}$ is replaced by the intersection of $K_T$ and the pointwise stabilizer of $\mathcal{X}(O', O)$. The matrix of $\tilde{T}$ is

$$\tilde{A} = \begin{pmatrix} \tilde{B} & \tilde{C} \\ \tilde{D} & \tilde{E} \end{pmatrix},$$

(9.13)

where

$$\begin{align*}
\tilde{B} &= q^{-1}(\sum_{j=1}^{q} B_{jj})I_q, \\
\tilde{C}_{ij} &= q^{-1}\sum_{h=1}^{q} C_{hj} \quad \text{for every } i, \\
\tilde{E} &= E, \\
\tilde{D}_{ij} &= q^{-1}\sum_{h=1}^{q} D_{ih} \quad \text{for every } j,
\end{align*}$$

(9.14)

and $I_q$ is the identity operator on $\mathbb{C}^q$.

Switching $O$ and $O'$ in the definition of $\tilde{T}$ we obtain another operator $\tilde{T}' \in \mathcal{A}$. The corresponding matrix $\tilde{A}'$ can be obtained from $\tilde{A}$ by switching $C$ with $D$ and $B$ with $E$. Now suppose that both $\tilde{T}, \tilde{T}'$ belong to $\mathfrak{G}$. By symmetry, operators of $\mathfrak{G}$ are constant on each half of the basis. So $B = \lambda_1 I_q$ and $E = \lambda_2 I_q$ for some $\lambda_1, \lambda_2 \in \mathbb{C}$. Moreover, all rows and columns of $C$ and $D$ add up to

![Figure 9: Diagram](image)
zero. If the same procedure leads to an element of $G$ when applied to any power of $T$ we deduce, as in the first case, that $A$ is diagonal. This is a contradiction since we supposed $\langle Tf_i, f_j \rangle$ for some $i \neq j$.

Case (3.i) is similar to (3.ii) and slightly easier.

Case (3.iii) can be reduced to (3.ii) as follows. Let $(x_0, x_1, \ldots, x_n)$ be the central chain, and $P, Q$ be as in the hypothesis. If $P' = x_j$ with $0 < j < n$ and, say, $Q' = x_1$, then any $R \in B(x_n, 1) \sim \{x_{n-1}\}$ is a boundary vertex with $P' \neq P''$, $Q'$. If $Q' = x_k$ with $0 < k < n$ the reasoning is even simpler.

Finally, suppose that $\{P, Q\} = \{x_0, x_n\}$ and $H_R$ is $T$–invariant for every $R \in \partial I \sim \{P, Q\}$. Since $\bigoplus_{P \in \partial I} H_P$ is also invariant through $H_{x_0}$, $H_{x_n}$ and the group of inversions switching $x_0$ with $x_n$, we find the following reduction for $T$:

$$H = (H_P \oplus H_Q) \bigoplus_{R \neq P, Q} H_R$$

and the reasoning of part (3.ii) applies to operators

$$\int_{H_{P'}} \pi'(h)T\pi'(h^{-1}) \, dh \quad \text{and} \quad \int_{H_{Q'}} \pi'(h)T\pi'(h^{-1}) \, dh.$$ 

This concludes the proof of Lemma 19 and of maximality of $G$.

10. Restriction to Even Automorphisms

In the remainder of the paper we shall work in the following setting:

1. As in Definition 3, the Hilbert space $H$ is $L^2(\Omega, \nu)$, where $\nu$ is the $K_O$–invariant measure introduced in (3.1);

2. The real parameter $s$ is such that $\text{Re} \, q^i \neq 0$;

3. $\pi$ acts on $H$ as the restriction of $\pi_s$ (defined in Section 3) to the subgroup $G^+$ of even automorphisms;

4. Definition 4 and decomposition (5.1) will be reformulated in terms of $\nu$. More precisely, for $n \geq 1$ $M_n$ is now generated by $\{1_{\Omega(O, x)} : |x| = n\}$. Observe that in this case $M_0 = H_0 \oplus H_1$, so no analogue of subspace $K$ is required. It follows that

$$H = \bigoplus_{n \geq 0} H_n. \quad (10.1)$$

Note that, for $n \geq 0$, $\pi \mid_{K_O}$ acts on $H_n$, and denote by $\pi_n$ the corresponding
action. In particular, $\pi_0$ acts trivially on $C_1 = H_0$.

**Definition 21.** Let $A \subseteq B(H)$ be a $\pi$–inductive algebra. A linear mapping $J : H_1 \to A$ is called a $K_O$–mapping between $H_1$ and $A$ if

$$J(\pi_1(k)v) = \pi(k)J(v)\pi(k^{-1})$$

(10.2)

for all $v \in H_1$, $k \in K_O$.

The concept of a $K_O$–mapping defined on $H_1$ will be crucial in the classification of maximal $\pi$–inductive algebras. It is a multi–dimensional substitute of the concept of an operator transforming according to $\chi$ (as in Section 7 and in [9]) for the case of $G^+$ (which does not contain $K'$).

More precisely, we shall prove that existence of a nontrivial $K_O$–mapping forces (up to conjugation by $J_s$) $A \cap M \neq \{0\}$. When $A$ is maximal this means that $A = M$, because even translates of any nonzero vector in $H_1$ generate $H$.

Let $J : H_1 \to A$ be a $K_O$–mapping and for $k \in \mathbb{N}$ let $i_k : H_k \to H$ be the inclusion map, as in the proof of Theorem 16. Then $i_n^*J(v)i_m$ intertwines $\pi_n \otimes \pi_1$ with $\pi_m$ for every $v \in H_1$. This fact and the following lemma lead immediately to a partial classification result, which replaces Proposition 3.6 of [9].

For two unitary representations $\sigma, \tau$ of a group $H$, denote by $L_H(\sigma, \tau)$ and $c_H(\sigma, \tau)$ the space of unitary intertwiners and its dimension. In each occurrence, representations can be replaced by the spaces they act on.

**Lemma 22.** Let $n, m \geq 0$. Then

$$c_{K_O}(\pi_1 \otimes \pi_n, \pi_m) = \begin{cases} 1 & \text{if either } (n, m) = (0, 1) \text{ or } (n, m) = (1, 0), \\ 1 & \text{if } n = m \geq 1, \\ 0 & \text{otherwise}. \\ \end{cases}$$

(10.3)

**Proof.** Let again $O'$ be a neighbor of $O$, $\mathcal{E} := \{O, O'\}$ and $K_\mathcal{E}$ its pointwise stabilizer. Then $K_\mathcal{E} \leq K_O$ and

$$\pi \equiv \text{Ind}(K_{O'}, K_\mathcal{E}, 1)$$

(10.4)

on $H_0 \oplus H_1$, hence

$$L_{K_O}(H_n \otimes H_m, H_0 \oplus H_1) = L_{K_\mathcal{E}}(C, H_n \otimes H_m)$$

(10.5)

by Frobenius’ Reciprocity Theorem. We now recall (see [4, VII.2, Exercise 2]) that for any locally compact group $H$ and representations $\sigma_i : H \to \mathcal{B}(X_i)$
Then decomposition

$$L_H(X_1 \otimes X_2, X_3) \cong L_H(X_1, X_2' \otimes X_3)$$

(10.6)

provided that \( \dim(X_2) < \infty \). It is understood that \( H \) acts on the dual space \( X_2' \) through \( \sigma_2^* \), the dual representation of \( \sigma_2 \), defined by

$$\sigma_2^*(h)(\phi)(v) := \phi(\sigma_2(h)(v)) \quad (h \in H, \phi \in X_2', v \in X_2)$$

and the action of \( X_2' \otimes X_3 \) on \( X_2 \) is the linear extension of

$$\langle \phi_2 \otimes v_3, v_2 \rangle := \langle \phi_2, v_2 \rangle v_3 \quad (\phi_2 \in X_2', v_2 \in X_2, v_3 \in X_3).$$

Coming back to the case of \( K_O \) we note that, for the subrepresentations \( \pi_n \) of \( \pi \mid_{K_O} \), \( \pi_n^* = \pi_n \). Indeed \( \nu \) is \( K_O \)-invariant and identifying \( \mathcal{H}_n \) with \( (\mathcal{H}_n)' \) through the bilinear product \( (\phi, v) := \int_{\Omega} \phi \nu \, dv \) we find

$$\langle \pi(k)\phi, \pi(k) v \rangle = (\phi, v) = (\pi^*(k)\phi, \pi(k) v) \quad \text{for any } \phi, v \in \mathcal{H}_n, k \in K_O.$$  

Hence \( \pi \mid_{K_O} \) and \( \pi^* \mid_{K_O} \) coincide on \( \mathcal{H}_n \).

So in our case (10.6) gives \( L_{K_O}(\mathcal{H}_0 \oplus \mathcal{H}_1) \otimes \mathcal{H}_n, \mathcal{H}_m) \cong L_{K_O}(\mathcal{H}_n \otimes \mathcal{H}_m, \mathcal{H}_0 \oplus \mathcal{H}_1) \) and \( L_{K_\ell}(\mathcal{H}_n \otimes \mathcal{H}_m) \cong L_{K_\ell}(\mathcal{H}_n, \mathcal{H}_m) \). Keeping (10.5) in mind we finally get

$$c_{K_O}(\mathcal{H}_1 \otimes \mathcal{H}_n, \mathcal{H}_m) = c_{K_\ell}(\mathcal{H}_n, \mathcal{H}_m) - c_{K_O}(\mathcal{H}_0 \otimes \mathcal{H}_n, \mathcal{H}_m)$$

and the last summand clearly equals \( \delta_{nm} \), since the relevant restrictions of \( \pi \mid_{K_O} \) are irreducible and inequivalent.

Now we compute \( c_{K_\ell}(\mathcal{H}_n, \mathcal{H}_m) \). We can suppose \( n \geq m \). First of all, if \( n > 1 \), we let

$$\mathcal{H}_1 := \{ f \in \mathcal{H}_n : \text{Supp } f \subseteq \Omega(O, O') \}, \quad (10.7)$$

$$\mathcal{H}_2 := \{ f \in \mathcal{H}_n : \text{Supp } f \subseteq \Omega(O', O) \}. \quad (10.8)$$

Then decomposition \( \mathcal{H}_n = \mathcal{H}_1 + \mathcal{H}_2 \) reduces \( \pi_n \mid_{K_\ell} \). Moreover

$$\dim \mathcal{H}_1 = (q-1)q^{n-2} \quad \text{and} \quad \dim \mathcal{H}_2 = (q-1)q^{n-1}.$$  

So, when \( n, m \geq 2 \), \( \pi_n \mid_{K_\ell} \) and \( \pi_m \mid_{K_\ell} \) have no equivalent components unless \( n = m \), and consideration of dimensions gives

$$c_{K_\ell}(\mathcal{H}_n, \mathcal{H}_m) = \begin{cases} 
2 & \text{if } n = m, \\
0 & \text{if } n \neq m. 
\end{cases} \quad (10.9)$$

We now deal with \( \pi_1 \). A decomposition into irreducibles is \( \mathcal{H}_1 = \mathcal{H}_1^1 + \mathcal{H}_1^2 \), where

$$\mathcal{H}_1^1 := C(1 - (q+1)\mathbf{1}_{\Omega(O, O')}), \quad (10.10)$$

$$\mathcal{H}_1^2 := \{ f \in \mathcal{H}_1 : \text{Supp } f \subseteq \Omega(O', O) \}. \quad (10.11)$$
Now \( \text{dim } \mathcal{H}_1^q = q - 1 \), so comparison of dimensions, again, rules out equivalence between components of \( \mathcal{H}_1 \) and components of \( \mathcal{H}_n \) \((n > 1)\), except possibly \( \mathcal{H}_1^1 \) and \( \mathcal{H}_2^2 \). But for every \( v \) in a natural generating set for \( \mathcal{H}_1^1 \) we can find \( k \in K_E \) stabilizing \( \mathcal{H}_1^1 \) pointwise and such that \( \pi(k)v = -v \). Hence \( L_{K_E}(\mathcal{H}_1^1, \mathcal{H}_2^2) \) is trivial and \( c_{K_E}(\mathcal{H}_1, \mathcal{H}_n) = 0 \) for \( n \geq 2 \). On the other hand \( c_{K_E}(\mathcal{H}_1, \mathcal{H}_1) = 1 \).

Finally \( c_{K_E}(\mathcal{H}_0, \mathcal{H}_n) = \begin{cases} 1 & n = 0, 1, \\ 0 & n \geq 2. \end{cases} \)

**Corollary 23.** Let \( J : \mathcal{H}_1 \to A \) be a \( K_O \)-mapping between \( \mathcal{H}_1 \) and \( A \). Let \( \mathcal{H}_n, \mathcal{H}_m \) be two subspaces of decomposition (10.1), and for every \( k \geq 1 \) let \( i_k : \mathcal{H}_k \to \mathcal{H} \) be the inclusion map. Then

\[
i_n^*J(v)i_m = C_{n,m}i_n^*Mi_m \quad \text{for every } v \in \mathcal{H}_1
\]

with \( C_{n,m} \in \mathbb{C} \) independent of \( v \). Moreover \( C_{n,m} = 0 \) unless \( n = m \geq 1 \) or \( \{n, m\} = \{0, 1\} \).

**Proof.** This follows immediately from the lemma above, up to the simple verification that \( i_n^*Mi_m \in C_{K_O}(\pi_1 \otimes \pi_n, \pi_m) \) for every \( v \in \mathcal{H}_1 \). \(\square\)

In order to show that nontrivial \( K_O \)-mappings between \( \mathcal{H}_1 \) and \( A \) do exist when \( A \) is maximal, we will prove in the next section that \( K_O \)-invariant operators in \( A \) are scalar. Moreover, by maximality \( A \) contains some nonscalar \( K_E \)-invariant operator, where \( E = \{O, O'\} \) as before. The latter result is Theorem 4.3 in [9], which originally concerns the full automorphisms group but extends to \( G^+ \) with the same proof (relying on pointwise stabilizers, hence even automorphisms only).

Given this, let \( G^+ \) act on \( A \) by conjugation. Frobenius’ reciprocity gives

\[
1 + c_{K_O}(\mathcal{H}_1, A) = c_{K_O}(\mathcal{H}_0, A) + c_{K_O}(\mathcal{H}_1, A) = c_{K_O}(\mathcal{H}_0 \oplus \mathcal{H}_1, A) = c_{K_E}(\mathcal{H}_0, A) > 1.
\]

So the dimension \( c_{K_O}(\mathcal{H}_0, A) \) of the space of \( K_O \)-mappings in nonzero. \(\square\)

### 11. \( K_O \)-Invariant Elements of \( A \)

With a view to proving that a \( K_O \)-invariant operator \( T \) in \( A \) must be scalar, observe first that the restriction of \( \pi \) to \( K_O \) leaves each summand in (10.1) invariant, and every resulting subrepresentation is irreducible. As in Theorem
16, choose a basis for $\mathcal{H}$ made up of bases of the spaces of decomposition. Write $T$ as a diagonal block matrix:

$$T = \begin{pmatrix}
\lambda_0 & & \\
& \lambda_1 & \\
& & \lambda_2 \\
& & \\
& & & \ddots
\end{pmatrix}.$$  

Write $z := 1/2 + is$ and let $g \in G^+$ be an inversion with respect to a chain of length two in $\mathcal{X}(O, O')$ starting in $O$. We put $\lambda_0 = 0$ and deduce

$$\langle T\pi(g)1, \pi(g)v \rangle = 0 \quad \text{if and only if} \quad \lambda_2 = -\frac{q^4 + q^2 - q^{2z+2} + q^{2z+1}}{q^2(q^2 - 1)}\lambda_1$$

(see Figures 10, 11, 12 for pictures of the action of $g$ and the vectors involved).

The multiplying factor is nonzero and $\neq 1$ under our hypotheses about $s$ and $q$. So, up to replacing $T$ by $T - T^2$, the inner product above is nonzero when $\lambda_1 \neq 0$. This is a contradiction, since $1$ and $v$ are eigenvectors of $T$ with eigenvalues $\lambda_0, \lambda_1$, respectively.

We conclude that $\lambda_0 = \lambda_1$ in any case.

![Figure 10](image1)

![Figure 11](image2)

Given the same two-step translation, take $w \in \mathcal{H}_2$ as in Figure 11. Under
the hypothesis that $\lambda_1 = \lambda_0 = 0$ but $\lambda_2 \neq 0$ it is clear that
\[ \langle T\pi(g)v, \pi(g)w \rangle \neq 0 \]
since both vectors belong to $\mathcal{M}_2$ with nonzero projections along $w$.

Arguing as above we get $\lambda_2 = \lambda_1 = \lambda_0$ in any case.

Finally, for $w \in \mathcal{H}_3$ as in Figure 14 and $g$ an inversion switching $O$ with a vertex in $\mathcal{X}(O, O')$, at distance 4 from $O$, $P^2(g, \cdot)$ is as in Figure 13. So we get
\[ \langle T\pi(g)w, \pi(g)1 \rangle = \lambda_3 \frac{(q^{2z} - q^2)(q + q^{2z})(q - 1)}{q^{2z+4z}(q + 1)} \]
Keeping in mind that $q > 1$ and $\text{Re} q^i \neq 0$ we find that $\lambda_3 = 0$ in any case. A similar reasoning shows that $\lambda_n \equiv \lambda_2$ for all $n \geq 2$. 
12. The Range of a $K_O$–Mapping

Finally we show that, up to conjugation by $J_s$, nontrivial $K_O$–mappings between $H_1$ and $A$ whose existence was proved in Section 10 are of the form

$$J : H_1 \longrightarrow A, \quad v \mapsto M_v,$$

where $M_v(f) := vf$ for $f \in H$. So $M \cap A \neq \{0\}$ if a nontrivial $J$ exists, and we conclude that $M = A$ by maximality.

Let $\{x_1, x_2, \ldots, x_{q+1}\}$ be the neighbors of $O$ and $w_i := -q1_{\Omega(x_i)} + 1_{\Omega-\Omega(x_i)} \in H_1$.

The regular representation of $K_O$ acts on $\{w_i: 1 \leq i \leq q+1\}$ by permutations. It follows that for any $k \in \mathbb{N}$, $\sum_{j=1}^{q+1}(J(w_j))^k$ is $K_O$–invariant with respect to $\pi$, hence scalar by the result of Section 11. We know from Corollary 23 that for $1 \leq j \leq q+1$ and $k \in \mathbb{N}$,

$$J(w_j) = \begin{pmatrix} 0 & C_{01}i_0 M_{w_j} i_1^* \\ C_{10}i_1 M_{w_j} i_0 & C_{11}i_1 M_{w_j} i_1^* & C_{22}i_2 M_{w_j} i_2^* & 0 \\ & 0 & C_{33}i_3 M_{w_j} i_3^* & \ddots \end{pmatrix} \quad (12.1)$$

with inclusion operators $i_k$ defined as in Section 10 and coefficients $C_{hk}$ independent of $w_j$. So we reduce to proving that all nonzero coefficients are equal.

To simplify the computation of powers of $J(w_j)$ we will find a basic relation, analogous to (8.1), between coefficients $C_{01}, C_{10},$ and $C_{11}$.

Consider the five–dimensional subspace $W$ of locally constant functions in $H$ as in Figure 15. For such a function, $[a, b, c, d, e]$ will be called the 5–tuple of components. These are just the coordinates of the function if one chooses for $W$ a basis of characteristic functions in an obvious way.

A more useful basis for $W$ is $\{1, v_1, v_q, \phi, \xi\}$, where $v_1, v_q \in H_1$ are as in Figure 16 and finally $\phi \in H_2$, $\xi \in H_3$ have components $[0, 0, 1, 1 - q, 1 -$
$q$ and $[0, 0, 0, 1, 1 - q]$ respectively. Both $J(v_1)$ and $J(v_q)$ leave $W$ stable

$$
\begin{align*}
\begin{pmatrix}
0 & C_{01}q \\
C_{10} & C_{11}(1 - q) \\
C_{11} & -C_{22}q \\
-C_{33}q & 0
\end{pmatrix}
\end{align*}
$$

(12.2)

and

$$
\begin{align*}
\begin{pmatrix}
0 & 0 & C_{01}q(q - 1) & q + 1 \\
0 & 0 & C_{11}q + 1 & (q - 1) \\
C_{10} & C_{11} & C_{11}(2 - q) & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\end{align*}
$$

(12.3)

Since $\mathcal{A}$ is commutative we have $[J(v_1), J(v_q)] = 0$, which easily gives

$$C_{10}C_{01} = C_{11}^2.
$$

(12.4)

Now go back to operators (12.1). Keeping in mind (12.4) and the fact that $w_j \in \mathcal{H}_1$ for each $j$ and $\sum_j w_j^2$ is scalar, we see that all the blocks of $\sum_j (J(w_j))^2$
outside the diagonal cancel out. For any $k \geq 2$, likewise, we find
\[
\sum_{j=1}^{q+1} (J(w_j))^k = \begin{pmatrix}
C_{11}^{k} i_0 (M_{\sum_j w_j})^k_{i_0} & 0 \\
0 & C_{11}^{k} i_1 (M_{\sum_j w_j})^k_{i_1} \\
& & C_{22}^{k} i_2 (M_{\sum_j w_j})^k_{i_2} \\
& & 0 & C_{33}^{k} i_3 (M_{\sum_j w_j})^k_{i_3} \\
& & & & \ddots
\end{pmatrix}.
\] (12.5)

Since $q > 1$, $\sum_{j=1}^{q+1} w_j^k$ is nonzero for $k \geq 2$. We conclude that $C_{nn}^k = C_{11}^k$ for any $n,k \in \mathbb{N}$, hence $C_{nn} \equiv C_{11}$. So we only need to express $C_{01}, C_{10}$ in terms of $C_{11}$.

Indeed we will prove that, up to replacing $J(v)$ by $J_s^{-1} J(v) J_s$ for every $v \in \mathcal{H}_1$, $C_{11} = C_{01} = C_{10}$.

We start by computing the matrix of $J_s$. Since $\pi_s$ and $\pi_{-s}$ are irreducible, the unitary intertwiner $J_s$ is diagonal by Schur’s Lemma. Let its restriction to $\mathcal{H}_0 \oplus \mathcal{H}_1$ be decomposed as
\[
\begin{pmatrix}
\alpha_0 & 0 \\
0 & \alpha_1
\end{pmatrix},
\]
where $\alpha_0$ can be normalized to 1.

Writing explicitly $\pi_s(h), \pi_{-s}(h)$ for simple $h$, as in Section 11, one sees that
\[
\alpha_1 = \frac{q^2 - q^{2z}}{q(q^{2z} - 1)},
\] (12.6)
and computes
\[
J_s^{-1} J(v) J_s = \begin{pmatrix}
0 & \alpha_1 \\
\alpha_1^{-1} & 1
\end{pmatrix}
\]
on $\mathcal{H}_0 \oplus \mathcal{H}_1$. Now the case of $C_{01} = 0$ in (12.1) is easily handled. So, from now on, we renormalize $C_{01}$ to 1 and prove that $C_{11}$ equals either 1 or the inverse of coefficient $\alpha_1$ in (12.6).

Let $g$ be a step–two translation along a geodesic containing $O$, and $v_1, v_q \in \mathcal{H}_1$ as in Figure 16. Refer to Figure 17 for a picture of $\pi(g)1$, from which it is clear that $\pi(g)1$ belongs to $\mathcal{W}$. The same is true for $\pi(g)v_1$ and $\pi(g)v_q$, with components $[q^{-2z}, q^{-2z}, 1, q^{2z}, -q^{1+2z}]$ and $[(1-q)q^{-2z}, (1-q)q^{-2z}, 0]$ respectively. By inductivity of $A$, conjugates of $J(v_1)$ and $J(v_q)$ through even automorphism must commute. Keeping (12.2), (12.3) in mind we get
Figure 17: \( \pi(g)1 \) when \( g \) is a step–two translation along a geodesic containing \( O \)

\[
C_{10} \langle J(v_1)\pi(g)v_1, \pi(g)v_q \rangle = C_{11} \langle J(v_1)\pi(g)1, \pi(g)v_q \rangle, \quad (12.7)
\]

\[
C_{10} \langle J(v_1)\pi(g)v_q, \pi(g)v_q \rangle = \langle J(v_1)\pi(g)1, \pi(g)J(v_q)^*v_q \rangle, \quad (12.8)
\]

\[
\langle J(v_1)\pi(g)1, \pi(g)J(v_q)^*1 \rangle = C_{10} \langle J(v_1)\pi(g)v_q, \pi(g)1 \rangle. \quad (12.9)
\]

Under conditions (12.4) and \( C_{33} = C_{22} = C_{11} \), equations (12.7), (12.8), (12.8) become equations in \( C_{11} \)

\[
(q - 1)(q + q^{2z})C_{11} \left( C_{11} - 1 \right) \left( C_{11}(q^2 - q^{2z}) + q - q^{1+2z} \right)

\times \left( C_{11}(q^{2+2z} - q^2 - q^{1+2z} - q^{4z}) - q^2 - q^{1+4z} + q^{1+2z} \right)

= 0, \quad (12.10)
\]

\[
(q - 1)(q + q^{2z})C_{11} \left( C_{11} - 1 \right) \left( C_{11}(q^2 - q^{2z}) + q - q^{1+2z} \right)

\times \left( C_{11}(q^{4+2z} - 2q^4 - 2q^{3+2z} + q^2 - 2q^{2+4z} + q^{1+2z} + q^{4z}) + q^4 - q^{4+2z}

- q^{3+4z} + 2q^{3+2z} - q^3 + q^{2+4z} - q^2 + q^{1+4z} - q^{1+2z} \right) = 0, \quad (12.11)
\]

\[
(q - 1)(q + q^{2z})^2 \left( C_{11} - 1 \right) \left( C_{11}(q^2 - q^{2z}) + q - q^{1+2z} \right)

\times \left( C_{11}^2(q^2 - q^{2z}) + q - q^{1+2z} \right) = 0, \quad (12.12)
\]

respectively, and a long but elementary computation shows that when \( q > 1 \), \( \text{Re} q^s \neq 0 \), and \( \text{Im} q^s \neq 0 \) the only simultaneous solutions for (12.10), (12.11) and (12.12) are \( C_{11} = 1 \) and \( C_{11} = \alpha_1^{-1} \), as expected.

This concludes the classification of maximal \( \pi \)-inductive algebras.
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References


