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ALMOST PRIMITIVE AND TEST ELEMENTS OF FREE COLOR LIE SUPERALGEBRAS

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Abstract: Let L be a finitely generated free color Lie superalgebra. We obtain new almost primitive and test elements of L.

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1. Introduction

Test elements of free groups have been of interest for many years. Examples of such elements were given in [1], [2]. In [8] E.C. Turner proved that test elements for monomorphisms of free groups of finite rank are exactly the elements with maximal rank (an element of the free group has maximal rank if it does not belong to any free factor of the group). The analogs of these result for free Lie algebras and free color Lie superalgebras were proved by A.A. Mikhalev and A.A. Zolotykh in [6] and by A.A. Mikhalev and J.T. Yu in [3].

Let F be the free group and φ be an endomorphism of F. A test element in a free group F is an element f with the property that if $\varphi(f) = f$ then φ must be an automorphism. An almost primitive element is an element of a free group F which is not primitive in F but which is primitive in any proper subgroup of F containing it.

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Almost primitive and test elements for free Lie algebras were found by A.A. Mikhalev and J.T. Yu, see [4]. In this article we consider the case of free color Lie superalgebras.

2. Preliminaries

Let G be an Abelian group, K a field, char$K \neq 2$, K^* the multiplicative group of K, $\varepsilon : G \times G \to K^*$ a skew symetric bilinear form for which

$$\varepsilon(g,g) = \pm 1, \quad G_- = \{g \in G : \varepsilon(g,g) = -1\}.$$

A graded algebra $R = \underset{g\in G}{\oplus} R_g$ over K is color Lie superalgebra if;

$$\begin{aligned} [x,y] &= -\varepsilon(d(x),d(y))[y,x], \\ [x,[y,z]] &= [[x,y],z] + \varepsilon(d(x),d(y))[y,[x,z]], \\ [v,[v,v]] &= 0, \end{aligned}$$

with $d(v) \in G_{-1}$ for $G-$homogeneous elements $x,y,z,v \in R$, where $d(a) = g$ if $a \in R_g$.

If H is $G-$graded associative algebra over K then $[H]$ denotes the color Lie superalgebra with operation $[,]$, where $[u,v] = uv - \varepsilon(d(u),d(v))vu$ for $G-$homogeneous elements $u,v \in H$.

Let $X = \underset{g\in G}{\cup} X_g$ be a $G-$graded set, $d(x) = g$ for $x \in X_g$ and let $A(X)$ be the free $G-$graded associative $K-$algebra, $L(X)$ is the free color Lie superalgebra with the set X of free generators.

Let X be a finite set and $L = L(X)$, be a free color Lie superalgebra on X. For $u \in L$, by $d(u)$ we denote the usual degree of u. Consider a weight function $\mu : X \to N$, where N is the set of positive integers. Let $\Gamma(X)$ be the free groupoid of nonassosociative monomials in the alphabet X, $S(X)$ the free semigroup of associative words in X, and

$$\sim : \Gamma(X) \to S(X)$$

the bracket removing homomorphism. We set $\mu(x_1,...,x_n) = \sum_{i=1}^{n} \mu(x_i)$ for $x_1,...,x_n \in X$ and $\mu(u) = \mu(\widetilde{u})$ for $u \in \Gamma(X)$.

Definition 1. Let $L(X)$ be the free color Lie superalgebra generated by the set X. We say that a set $Y \subseteq L(X)$ is independent if Y is a free generating set for the subalgebra it generates. In other words, there are non-trivial relations between elements of Y.

Let l be the usual length function on the free associative algebra A. For element $a \in A$, by a^0 we denote the sum of monomials of the highest degree in a with respect to the function l.

Definition 2. A set $Y \subseteq L$ is called reduced if for every $y \in Y$, the element y^0 does not belong to the subalgebra of L generated by the set $\{u^0 : u \in Y, u \neq y\}$.

Then we have a Shirshov's result [7].

Theorem 3. *Every reduced subset of L is an independent set.*

Definition 4. Let $S = \{s_\alpha : \alpha \in I\}$ be a subset of L. A mapping $\theta : S \to S' \subseteq L$ is an elementary transformation of S if either θ is an nondegenerate linear transformation of S, or there is $\beta \in I$ such that $\theta(s_\alpha) = s_\alpha$ for all $\alpha \in I$, $\alpha \neq \beta$ and

$$\theta(s_\beta) = s_\beta + f(\{s_\alpha : \alpha \neq \beta\}.$$

Definition 5. An element u of $L(X)$ is said to be primitive if it is an element of some set of free generators of the algebra $L(X)$.

Definition 6. An almost primitive element of the free color Lie superalgebra $L = L(X)$ is an element which is not primitive in L but which is primitive in any proper subalgebra of L containing it.

Lemma 7. *Let K be a field,* $\operatorname{char} K \neq 2$ *and let u be an almost primitive element of $F(X)$ whose component of degree 1 is equal to zero. Then u is a test element of $F(X)$.*

3. Main Theorem

For $a \in L(X)$ by ada and Ada we denote the operators of left and right multiplication, respectively

$$(ada)(b) = [a, b] \text{ and } (b)Ad(a) = [b, a] \text{ for } b \in L(X).$$

Theorem 8. *Let K be a field,* $\operatorname{char} K \neq 2$ *and $X = \{x, y, z\}$. Then the element $u = [x, y] + (x)(Adz)^k$ $(k \geq 2)$ is an almost primitive element of the free color Lie superalgebra $L(X)$ over K. It is also a test element of $L(X)$.*

Proof. Suppose that the element u belongs to a finitely generated subalgebra H of $L(X)$. Let $x > y > z$ and let $\{h_1, ..., h_m\}$ be a reduced set of free generators of H. Then the leading part u^0 of u is polynomial in $h_1^0, ..., h_m^0$. It is clear that $u^0 = (x)(Adz)^k$. If in the expression of u^0 we have a nonzero linear summand h_i^0 for some i, then $u^0 = \alpha h_i^0 + f(\{h_j^0 : j \neq i\})$, $0 \neq \alpha \in K$; therefore $u = \alpha h_i + f^{'}(\{h_j : j \neq i\})$ and u is a primitive element of H. Otherwise $z \in H$.

Now we consider the generalized degree function μ given by $\mu(x) = 1$, $\mu(y) = N > k$, $\mu(z) = 1$. Let $\{h_1^{'}, ..., h_m^{'}\}$ be a reduced set of free generators of the subalgebra H with respect to μ. We have $\hat{u} = [x, y]$ and $\hat{u}$ is a polynomial in $\hat{h}_1^{'}, ..., \hat{h}_m^{'}$. If $\hat{u}$ has a nonzero linear summand $\hat{h}_j$ for some j, then

$$\hat{u} = \beta \hat{h}_j + g(\{\hat{h}_i : i \neq j\}) \ , 0 \neq \beta \in K;$$

therefore $u = \beta h_j + g^{'}(\{h_i : i \neq j\})$ and u is a primitive element of H. Otherwise $x \in H$. If $x, z \in H$, then $u^{'} = u - (x)(Adz)^k = [x, y] \in H$. If $(u^{'})^0$ has a linear summand in the expression in $h_1^0, ..., h_m^0$, then u is a primitive element of H. Otherwise $y \in H$ and $H = L(X)$. Thus u is a almost primitive element of $L(X)$. Since the component of degree one of u is zero by Lemma 7 u is a test element of $L(X)$.

Theorem 9. *The element $u = (x)(Ady)^k + (x)(Adz)^l$, where k and l are positive integers is an almost primitive element of $L(x, y, z)$ if and only if min$\{k, l\} = 1$ and max$\{k, l\} \geq 2$.*

Proof. If $k = l = 1$ then $u = [x, y + z]$ and u is not a primitive element of the proper subalgebra of $L(x, y, z)$ generated by x and $y + z$. Suppose that $k, l > 1$. Let $u_1 = (x)(Ady)^{k-1}$, $u_2 = (x)(Adz)^{l-1}$, $u_3 = y$, $u_4 = z$. Then $U = \{u_1, u_2, u_3, u_4\}$ is a reduced set, $u = [u_1, u_2] + [u_3, u_4]$, $H = L(u_1, u_2, u_3, u_4) \neq L(x, y, z)$ and u is not a primitive element of H. Hence u is not an almost primitive element of $L(x, y, z)$. Using Theorem 8 we complete the proof. □

Theorem 10. *If $k, l \geq 2$ and $k \neq 1$ then the element $u_{k,l}(x, y) = (adx)^k(y) + (x)(Ady)^l$ is an almost primitive element of $L(x, y)$.*

Proof. We may suppose that $k > l$. In this case $u_{k,l}^0(x, y) = (adx)^k(y)$. Suppose that the element u belongs to a finitely generated subalgebra H of $L(X)$. Let $\{h_1, ..., h_m\}$ be a reduced set of free generators of H. Then the leading part u^0 of u is a polynomial in $h_1^0, ..., h_m^0$. As in the proof of Theorem 8 if u is not a primitive element of H then $x \in H$. In the case, we consider the generalized degree function μ given by $\mu(x) = 1$, $\mu(y) = N > k$. Let $\{h_1^{'}, ..., h_m^{'}\}$ be a reduced set of free generators of the subalgebra H with respect to μ. We have $\hat{u} = (x)(Ady)^l$. Again either u is a primitive element of H or $y \in H$. If $y \in H$ then $H = L(X, Y)$. This completes the proof. □

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