International Journal of Pure and Applied Mathematics
Volume 30 No. 1 2006, 109-117

ON A PARAMETERIZATION OF THE POINCARE GROUP

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Abstract: In this paper the Poincare group is studied. Namely for a given vector of velocity and given direction and angle of space rotation, the corresponding matrix of the Poincare group is found. In Theorem 1 two functionally independent invariant scalars under the transformations of similarity are found, and there does not exist another invariant scalar which is functionally independent from them.

AMS Subject Classification: 83A05
Key Words: Poincare group, Lorentz transformations, space rotations

1. Introduction

In a Minkowski space $M^4 = R^{3,1}$ with coordinates (x^1, x^2, x^3, x^4), where $x^4 = ct$ is the time coordinate and c is the light velocity the metric has the form $g_{ij} = \text{diag}(-1, -1, -1, 1)$ and hence the infinitesimal distance between two neighboring points is given by

Received: June 8, 2006

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$$ds^2 = (dx^4)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2. \tag{1.1}$$

The set of linear transformations which preserve this metric is called *the Poincare group*, which we shall further denote by G. Notable elements of G are the Lorentz transformations, which represent motions with velocity (v_x, v_y, v_z), given by matrices of the form

$$L = \begin{bmatrix} 1+\frac{1}{\nu}V_1^2 & \frac{1}{\nu}V_1V_2 & \frac{1}{\nu}V_1V_3 & -V_1 \\ \frac{1}{\nu}V_1V_2 & 1+\frac{1}{\nu}V_2^2 & \frac{1}{\nu}V_2V_3 & -V_2 \\ \frac{1}{\nu}V_1V_3 & \frac{1}{\nu}V_2V_3 & 1+\frac{1}{\nu}V_3^2 & -V_3 \\ -V_1 & -V_2 & -V_3 & V_4 \end{bmatrix}, \tag{1.2}$$

where $V_1 = \beta v_x/c, V_2 = \beta v_y/c, V_3 = \beta v_z/c, V_4 = \beta$, and we denoted $\beta = 1/\gamma,\ \gamma = \sqrt{1-\frac{v^2}{c^2}}$ and $\nu = 1+\beta$.

Another important subset of G is the set of space rotations; e.g., a space rotation of angle α about the direction of the unit vector (a, b, c) is given by

$$R = \begin{bmatrix} a^2(1-\cos\alpha)+\cos\alpha & ab(1-\cos\alpha)-c\sin\alpha & ac(1-\cos\alpha)+b\sin\alpha & 0 \\ ab(1-\cos\alpha)+c\sin\alpha & b^2(1-\cos\alpha)+\cos\alpha & bc(1-\cos\alpha)-a\sin\alpha & 0 \\ ac(1-\cos\alpha)-b\sin\alpha & bc(1-\cos\alpha)+a\sin\alpha & c^2(1-\cos\alpha)+\cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{1.3}$$

Notice that while the set of space rotations (1.3) is a subgroup of G, the set (1.2) is not.

In this paper we give a parameterization of an arbitrary transformation in G, by joining a pair consisting of a 3-vector of velocity and a 3-vector of angular rotation.

2. The Structure of the Poincare Group

The following problem naturally appears. *Let be given simultaneously a vector of velocity $\vec{v}$ and a rotation of angle α about the unit direction (a, b, c). How can we determine the corresponding matrix in G?* Notice that there are many possible ways to parameterize a subset of G by 6 parameters (dim $G = 6$), but not each of them is a solution of our problem. For example, two such parameterizations can be obtained by the matrix products LR and RL, where L and R are given by (1.3) and (1.4) respectively. Indeed, the matrix product LR represents an element of G such that first we have the space rotation, and then according to the rotated system we have a Lorentz transformation of velocity $\vec{v}$, but not according to our initial system. The matrix product RL represents an element of G such that first we have a Lorentz transformation, and then according to the moving system we have rotation, but not with respect to our initial system. In order to find a solution, we refer to [1], where is given a general parameterization of $SO(4)$ using 6 parameters P_x, P_y, P_z, Q_x, Q_y, and Q_z. In order to obtain accordingly a general parameterization of G, we multiply by i (the imaginary unit) the fourth column and with $-i$ the fourth row. Hence we obtain the parameterization of a subset of G provided by $(A_{ij})_{i,j=\overline{1,4}}$ with the matrix entries

$$
\begin{aligned}
A_{11} &= (1 + P_x^2 - P_y^2 - P_z^2 + Q_x^2 - Q_y^2 - Q_z^2 + S^2)/\lambda, \\
A_{22} &= (1 - P_x^2 + P_y^2 - P_z^2 - Q_x^2 + Q_y^2 - Q_z^2 + S^2)/\lambda, \\
A_{33} &= (1 - P_x^2 - P_y^2 + P_z^2 - Q_x^2 - Q_y^2 + Q_z^2 + S^2)/\lambda, \\
A_{44} &= (1 + P_x^2 + P_y^2 + P_z^2 + Q_x^2 + Q_y^2 + Q_z^2 + S^2)/\lambda, \\
A_{12} &= 2(P_z + P_xP_y + Q_xQ_y - SQ_z)/\lambda, \\
A_{23} &= 2(P_x + P_yP_z + Q_yQ_z - SQ_x)/\lambda, \\
A_{31} &= 2(P_y + P_zP_x + Q_zQ_x - SQ_y)/\lambda, \\
A_{21} &= 2(-P_z + P_xP_y + Q_xQ_y + SQ_z)/\lambda, \\
A_{32} &= 2(-P_x + P_yP_z + Q_yQ_z + SQ_x)/\lambda, \\
A_{13} &= 2(-P_y + P_zP_x + Q_zQ_x + SQ_y)/\lambda, \\
A_{14} &= 2(-Q_x - SP_x + P_yQ_z - P_zQ_y)/\lambda, \\
A_{24} &= 2(-Q_y - SP_y + P_zQ_x - P_xQ_z)/\lambda, \\
A_{34} &= 2(-Q_z - SP_z + P_xQ_y - P_yQ_x)/\lambda, \\
A_{41} &= 2(-Q_x - SP_x - P_yQ_z + P_zQ_y)/\lambda, \\
A_{42} &= 2(-Q_y - SP_y - P_zQ_x + P_xQ_z)/\lambda, \\
A_{43} &= 2(-Q_z - SP_z - P_xQ_y + P_yQ_x)/\lambda,
\end{aligned}
\tag{2.1}
$$

where $\lambda = 1 + P_x^2 + P_y^2 + P_z^2 - Q_x^2 - Q_y^2 - Q_z^2 - S^2$, and $S = P_xQ_x + P_yQ_y + P_zQ_z$.

Further we are going to find the relationship between the parameters P_x, P_y, P_z, Q_x, Q_y, Q_z, and the given values (v_x, v_y, v_z), (a, b, c), α. We solve this problem assuming that P_x, P_y, and P_z are functions of a, b, c, and α, while Q_x, Q_y, and Q_z are functions of v_x, v_y, and v_z. If we put $Q_x = Q_y = Q_z = 0$ in (2.1), and solve the matrix equality $A = R$, where R is given by (1.3), we obtain the following unique solution

$$P_x = -a\tan\frac{\alpha}{2}, \quad P_y = -b\tan\frac{\alpha}{2}, \quad P_z = -c\tan\frac{\alpha}{2}. \tag{2.2}$$

Further if we put $P_x = P_y = P_z = 0$ in (2.1), and solve the matrix equality $A = L$, where L is given by (1.2), we obtain the following unique solution

$$Q_x = \frac{1}{1+\gamma}\frac{v_x}{c},\ Q_y = \frac{1}{1+\gamma}\frac{v_y}{c},\ Q_z = \frac{1}{1+\gamma}\frac{v_z}{c}. \tag{2.3}$$

The following theorem shows that the real number

$$\begin{aligned}\lambda(A) &= \lambda(v_x, v_y, v_z, a, b, c, \alpha) \\ &= 1 + \tan^2(\frac{\alpha}{2}) - \frac{\frac{v^2}{c^2}}{(1+\gamma)^2} - \frac{(av_x + bv_y + cv_z)^2}{c^2}\frac{\tan^2\frac{\alpha}{2}}{(1+\gamma)^2}\end{aligned}$$

appears to be an invariant scalar.

Theorem 1. *The scalar λ is an invariant under the matrix mapping $A \mapsto BAB^{-1}$, i.e.*

$$\lambda(BAB^{-1}) = \lambda(A), \tag{2.4}$$

for each $A, B \in G$, and the expressions

$$P_x^2 + P_y^2 + P_z^2 - Q_x^2 - Q_y^2 - Q_z^2 = \tan^2(\frac{\alpha}{2}) - \frac{v^2/c^2}{(1+\gamma)^2}$$

and

$$S = P_xQ_x + P_yQ_y + P_zQ_z = -\frac{av_x + bv_y + cv_z}{c}\frac{\tan\frac{\alpha}{2}}{1+\gamma}$$

are invariant scalars as well.

Remark. For each parameters $v_x, v_y, v_z, a, b, c, \alpha$ which determine the matrix A, we should find the parameters v_x', v_y', v_z', a', b', c', α' for the matrix BAB^{-1} and should prove that the corresponding scalar expressions are the same for both sets of parameters. But the parameters v_x', v_y', v_z', a', b', c', α' cannot be uniquely determined for a given matrix in the Poincare group, since the mapping $\psi : (v_x, v_y, v_z, a, b, c, \alpha) \mapsto A$ is generally just a local (not global) homeomorphism. Thus the previous theorem can be conceived in the following manner. Let we have any path in the Poincare group G, which connects the matrices A and BAB^{-1}. According to the local homeomorphism ψ, we have uniquely determined the values $v_x, v_y, v_z, a, b, c, \alpha$ along the curve, and now Theorem 1 states that the considered scalars are the same for the matrices A and BAB^{-1}. It is sufficient to prove the theorem for the special cases when the matrix B is given by (1.2) and (1.3). Computer test of the theorem according to the previous remark is given in the next section.

Since the Lie algebra of the Poincare group has only 2 scalars invariant under the similar transformations, which are functionally independent, the Poincare group does not posses more than 2 such scalars. So Theorem 1 cannot be generalized for existence of more functionally independent invariant scalars.

Proof. First let us consider the structure of the matrix A given by (2.1). A straight calculation shows that the matrix A can be written in the form

$$A = [A^{(0)} + A^{(1)} + A^{(2)} + A^{(3)} + A^{(4)}]/\lambda\,,$$

where

$$A^{(0)} = I, \quad A^{(1)} = \begin{bmatrix} 0 & 2P_z & -2P_y & -2Q_x \\ -2P_z & 0 & 2P_x & -2Q_y \\ 2P_y & -2P_x & 0 & -2Q_z \\ -2Q_x & -2Q_y & -2Q_z & 0 \end{bmatrix},$$

$$A^{(2)} = \frac{1}{2}A^{(1)}A^{(1)} + (P_x^2 + P_y^2 + P_z^2 - Q_x^2 - Q_y^2 - Q_z^2)I,$$

$$A^{(3)} = S \begin{bmatrix} 0 & -2Q_z & 2Q_y & -2P_x \\ 2Q_z & 0 & -2Q_x & -2P_y \\ -2Q_y & 2Q_x & 0 & -2P_z \\ -2P_x & -2P_y & -2P_z & 0 \end{bmatrix}, \quad A^{(4)} = S^2 I,$$

with

$$\lambda = 1 + P_x^2 + P_y^2 + P_z^2 - Q_x^2 - Q_y^2 - Q_z^2 - S^2, \;\; S = P_xQ_x + P_yQ_y + P_zQ_z.$$

Now assume that $A^{(1)}$ is a tensor with respect to the matrix transformations from the group G. Hence it follows that $P_x^2 + P_y^2 + P_z^2 - Q_x^2 - Q_y^2 - Q_z^2$ and $S = P_xQ_x + P_yQ_y + P_zQ_z$ are scalars with respect to these transformations, and λ is a scalar as well. Using that $P_x^2 + P_y^2 + P_z^2 - Q_x^2 - Q_y^2 - Q_z^2$ is a scalar, we just obtain that $A^{(2)}$ is also a tensor, and using that S is a scalar, it follows that $A^{(3)}$ and $A^{(4)}$ are tensors, and A is a tensor too. Because the local homeomorphism associates $(P_x, P_y, P_z, Q_x, Q_y, Q_z)$ - i.e. $A^{(1)}$ to the matrix A, if A is a tensor, it follows that $A^{(1)}$ is a tensor. Hence

$$P_x^2 + P_y^2 + P_z^2 - Q_x^2 - Q_y^2 - Q_z^2, \quad S = P_xQ_x + P_yQ_y + P_zQ_z,$$

and λ behave as scalars according to the considered transformations so the proof will be completed. Thus we have only to prove that A is a tensor. Let us choose arbitrary orthonormal vectors $U^i_{(1)}$, $U^i_{(2)}$, $U^i_{(3)}$, and $U^i_{(4)}$. If A is an element of the Poincare group, considered as a linear transformation, then for each α and β,

$$A_{ij}U^i_{(\alpha)}U^j_{(\beta)} = C_{(\alpha)(\beta)}$$

determines 4^2 scalars, and hence A must be a tensor. As a consequence of the previous discussion we obtain that the tensor A locally is induced by the tensor $A^{(1)}$ and conversely. □

According to the proof of Theorem 1, we have the following

Corollary. *The matrix $A^{(1)}$, where $P_x, P_y, P_z, Q_x, Q_y, Q_z$ are given by (2.2) and (2.3) determines a tensor.*

According to Theorem 1 we have a "total angle of rotation" θ in four dimensions. Indeed, if $\lambda \geq 1$, then θ is given by $\lambda = 1 + \tan^2(\frac{\theta}{2})$, and hence θ can be expressed as a function of $v_x, v_y, v_z, a, b, c, \alpha$. If $\lambda < 1$, then θ is as large as the imaginary angle which represents a motion with velocity u, such that $\lambda = 1 - \frac{\frac{u^2}{c^2}}{(1+\gamma)^2}$. Hence this "total velocity u" can be expressed as a function of $v_x, v_y, v_z, a, b, c, \alpha$.

One of the applications of the previous results can improve the calculation of the de Sitter geodetic precession [3], for higher order post Newtonian approximations. In [2] are studied the equations of motion in a field of inertial forces via the Poincare group, referring in more details to the case of rotating systems.

3. Numerical Techniques for Viewing the Arguments of BAB^{-1}

Although the complete proof of the Theorem 1 is given, the values of the arguments for the matrix BAB^{-1} are not obvious. For that reason, we used computer calculations. Namely, in the equality (2.4) $\lambda(BAB^{-1})$ requires a procedure that produces the six arguments $a, b, \alpha, v_x, v_y, v_z$ from an arbitrary matrix in G. We used numerical techniques for finding solutions of the system of nonlinear equations (2.1).

We consider two special cases. In the first one, we take the matrix L presented in (1.2), while the second case is for the matrix R in (1.3). Since we are describing the process for both cases, we will further refer these matrices as B.

Firstly, we obtain a matrix A, whose elements are given in (2.1) by assigning random values for the six variables $a, b, \alpha, v_x, v_y, v_z$. We do the same for B (for the variables v_x, v_y, v_z and a, b, α in L and R respectively). We calculate $\lambda(A)$ and BAB^{-1}. Next, using numerical methods we provide values for six corresponding unknown variables referring BAB^{-1}.

Then we calculate $\lambda(BAB^{-1})$ and estimate $|\lambda(BAB^{-1}) - \lambda(A)|$. Since the numerical methods give roots that may not be solutions of the given system, we will consider only the cases in which the solution accuracy is of order 10^{-7}. The following algorithm is based on the previously described procedure.

read n *// n is the number of attempts*
for $i \leftarrow 1$ **to** n
    $a \leftarrow$ rand(); $b \leftarrow$ rand()$\cdot\sqrt{1-a^2}$; $c \leftarrow \sqrt{1-a^2-b^2}$; $\alpha \leftarrow \pi$ rand();
    $v_x \leftarrow$ rand(); $v_y \leftarrow$ rand()$\cdot\sqrt{1-v_x^2}$; $v_z \leftarrow$ rand()$\cdot\sqrt{1-v_x^2-v_y^2}$;
    // rand() gives a random number in [0,1]
    $v^2 \leftarrow v_x^2 + v_y^2 + v_z^2$;
    calculate $\lambda(A)$ and A;
    // as in the formula for $\lambda(A)$ *and in (2.1) respectively*
    // making the matrix B
    if B stands for L **then** v_x, v_y, v_z take new random values
        in the same manner as above;
    if B stands for R **then** a, b, α take new random values
        in the same manner as above;
    calculate B; *// as in (1.3) or (1.4)*
    $T \leftarrow BAB^{-1}$;
    solve the system (2.1) for $a, b, \alpha, v_x, v_y, v_z$ referring T;
    if $|\lambda(BAB^{-1}) - \lambda(A)| < 0.00001$ **then** *success = success+1*
        else *fail = fail+1;*
end for
write *success, fail;*

The algorithm above is designed to run in the *Matlab 7* environment, using the function $fsolve()$ from the Optimization Toolbox. This function uses the Gauss-Newton method based on the nonlinear least-squares algorithms. The algorithm returns a point where the residue is small. However, if the Jacobian of the system is close to singular, the algorithm might converge to a point that is not close to a solution of the system of equations. In the actual script file we specified the use of large-scale optimization. Large-scale algorithm is a subspace trust region method and is based on the interior-reflective Newton method.

Our algorithm counts those attempts when $|\lambda(BAB^{-1}) - \lambda(A)| < 10^{-5}$, as being successful. Each of these attempts corresponds to a numerical solution of accuracy of order $< 10^{-7}$ regarding the returned value of the objective function used by $fsolve()$ (the objective function, is actually the system (2.1) referring T). Also, every numerical solution provides a satisfactory small difference between $\lambda(A)$ and $\lambda(BAB^{-1})$. Some points of convergence that are not solutions of the given system (the returned value of the objective function used by $fsolve()$ at the found root is large: $> 10^{-2}$), may give a difference between the obtained values for $\lambda(A)$ and $\lambda(BAB^{-1})$ of order 10^{-4}. With an intention of ignoring such cases, we set the control limit to 10^{-5} (under a value of order 10^{-4}). Observing the results we note that for all numerical solutions the control limit 0.00001 is not exceeded and in most cases, $|\lambda(BAB^{-1}) - \lambda(A)| < 10^{-13}$.

Although many of the solutions of the system are complex numbers, the imaginary part is very small. This suggests that we actually obtained solutions which we can consider as real. This is satisfactory enough, because $|\lambda(BAB^{-1}) - \lambda(A)|$ is acceptably small for them as for the real solutions, in the sense of the previous discussion.

Running the program, we obtained nearly 33% successful attempts for the matrix of Lorenz transformations and nearly 75% for the matrix of space rotations. This percentage discrepancy of the successful attempts is apparently due to the compoundness of the matrix of Lorenz transformations as well as the accepted approximation of order 10^{-20} regarding the complete expressions in the components of that matrix, all of which may be obstacles for the numerical method.

We conclude that whenever a solution is obtained, $|\lambda(BAB^{-1}) - \lambda(A)|$ is small and vice versa: for small $|\lambda(BAB^{-1}) - \lambda(A)|$, there is no case when the method converged to a root that is not a solution of the system.

More than 10000 attempts have been made and the obtained results made a strong indication that the given equality stands.

The readers interested in trying the actual script files for both cases, the handle function used in $fsolve()$ and interested in suggestions prior to executing may contact the authors.

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