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FUNCTIONALS ON NORMED SEQUENCE SPACES AND EXPONENTIAL INSTABILITY OF LINEAR SKEW-PRODUCT SEMIFLOWS

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Abstract: The aim of this paper is to give necessary and sufficient conditions for uniform exponential instability of linear skew-product semiflows in terms of functionals on normed sequence spaces. We obtain the versions of some results due to Datko, Pazy, Neerven and Zabczyk for the case of instability of linear skew-product semiflows.

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1. Introduction

In recent years, a significant progress has been made in the study of the asymptotic behavior of evolution equations, by treating them using the theory of linear skew-product semiflows. Thus, an important list of well-known results for the cases of C_0-semigroups and evolution operators has been extended for linear skew-product semiflows. One of the most remarkable result in stability theory of linear evolution operators in Banach spaces has been obtained by Datko in [3]. An extension of this result has been given by Pazy in [12]. An impor-

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tant generalization of Datko's result was proved by Zabczyk in [15]. A new and interesting idea has been presented by Neerven in [11], where an unified treatment of the preceding results is given and the exponential stability of C_0-semigroups has been characterized in terms of functionals on Banach function spaces. Some generalizations of these results for the case of linear skew-product semiflows have been presented in [1], [2], [3], [5], [6], [7] and [14].

In this paper we shall present characterizations for uniform exponential instability of linear skew-product semiflows in the spirit of Neerven's approach. Thus, we obtain the versions of the theorems due to Datko, Pazy, Zabczyk and Neerven for the case of uniform exponential instability of linear skew-product semiflows. As consequences, we obtain some results presented in [9].

2. Linear Skew-Product Semiflows

Let X be a Banach space, let Θ be a metric space and let $E = \Theta \times X$. We denote by $\mathcal{B}(X)$ the Banach algebra of all bounded linear operators from X into itself.

Definition 2.1. A continuous mapping $\sigma : \mathbb{R}_+ \times \Theta \to \Theta$ is called a *semiflow* on Θ, if it has the following properties:

(i) $\sigma(0, \theta) = \theta$, for all $\theta \in \Theta$;

(ii) $\sigma(t + s, \theta) = \sigma(t, \sigma(s, \theta))$, for all $t, s \geq 0$ and $\theta \in \Theta$.

Definition 2.2. An application $C : \mathbb{R}_+ \times \Theta \to \mathcal{B}(X)$ is called a *cocycle* on $E = \Theta \times X$, if it satisfies the following conditions:

(c_1) $C(0, \theta) = I$, the identity operator on X, for all $\theta \in \Theta$;

(c_2) $C(t + s, \theta) = C(t, \sigma(s, \theta))C(s, \theta)$, for all $(t, s, \theta) \in \mathbb{R}_+^2 \times \Theta$ (the cocycle identity);

(c_3) there are $M, \omega > 0$ such that $\|C(t, \theta)\| \leq Me^{\omega t}$, for all $(t, \theta) \in \mathbb{R}_+ \times \Theta$.

(c_4) for every $(\theta, x) \in \Theta \times X$, the function $C(\cdot, \theta)x$ is continuous.

Definition 2.3. A pair $S = (C, \sigma)$, where C is a cocycle on $E = \Theta \times X$ and σ is a semiflow on Θ is called a *linear skew-product semiflow* on E.

Examples of linear skew-product semiflows can be found in [1], [2], [3] and [14].

Definition 2.4. A linear skew-product semiflow $S = (C, \sigma)$ on E is said to be:

(i) *injective*, if for every $(t, \theta) \in \mathbb{R}_+ \times \Theta$ the linear operator $C(t, \theta)$ is injective;

(ii) *uniformly exponentially instable*, if there are $N, \nu > 0$ such that

$$\|C(t, \theta)x\| \geq Ne^{\nu t}\|x\|, \quad \text{for all} \quad (t, \theta, x) \in \mathbb{R}_+ \times E.$$

An example of uniformly exponentially instable linear skew-product semiflow is given in [9]. We have:

Proposition 2.1. *A linear skew-product semiflow $S = (C, \sigma)$ on $E = \Theta \times X$ is uniformly exponentially instable if and only if there exists a nondecreasing sequence $f : \mathbb{N} \to \mathbb{R}_+^*$ with $\lim_{n\to\infty} f(n) = \infty$ and*

$$\|C(n, \theta)x\| \geq f(n)\|x\|, \quad \textit{for all} \quad (n, \theta, x) \in \mathbb{N} \times E.$$

Proof. *Necessity.* It is obvious.

Sufficiency. Let $t \geq 0$ and $n \in \mathbb{N}$ such that $n \leq t < n + 1$. Then, for all $(t, \theta, x) \in \mathbb{R}_+ \times E$ we have that

$$\begin{aligned} \|C(t, \theta)x\| &= \|C(t - n, \sigma(n, \theta))C(n, \theta)x\| \geq f(t - n)\|C(n, \theta)x\| \\ &\geq f(0)\|C(n, \theta)x\| \geq f(0)f(1)\|C(n - 1, \theta)x\| \geq \cdots \geq f(0)f(1)^n\|x\| \\ &= f(0)e^{\nu n}\|x\| \geq Ne^{\nu t}\|x\|, \end{aligned} \quad (2.1)$$

where $\nu = \ln f(1) > 0$ and $N = f(0)/f(1) > 0$, which shows that S is uniformly exponentially instable. □

3. Main Results

Let $\mathcal{S}(\mathbb{R})$ be the set of all real sequences. By $\mathcal{S}^+(\mathbb{R})$ we denote the set of all $s \in \mathcal{S}(\mathbb{R})$ with $s(n) \geq 0$, for all $n \in \mathbb{N}$.

Let $\mathcal{F}$ be the set of all functions $F : \mathcal{S}^+(\mathbb{R}) \to [0, \infty]$ with the properties:

(f_1) if $s_1, s_2 \in \mathcal{S}^+(\mathbb{R})$ with $s_1 \leq s_2$ then $F(s_1) \leq F(s_2)$;

(f_2) there exists $\alpha > 0$ such that $F(c\chi_{\{n\}}) \geq \alpha c$, for all $c > 0$ and $n \in \mathbb{N}$;

(f_3) there exists $f \in \mathcal{S}^+(\mathbb{R}_+)$ with $\lim_{n\to\infty} f(n) = \infty$ such that $F(c\chi_{\{0,\ldots,n\}}) \geq f(n)$, for all $c > 0$ and $n \in \mathbb{N}$.

Here χ_A denotes the characteristic function of the set A.

For every injective linear skew-product semiflow $S = (C, \sigma)$ on $E = \Theta \times X$ and every $x \in X$ with $\|x\| = 1$, we associate the following sequences:

$$c_{x,\theta}(n) = \frac{1}{\|C(n,\theta)x\|}, \quad c^m_{x,\theta}(n) = c_{x,\theta}(m+n), \quad t^m_{x,\theta}(n) = \frac{c^m_{x,\theta}(n)}{c_{x,\theta}(m)},$$

for all $m, n \in \mathbb{N}$.

Remark 3.1. If the linear skew-product semiflow $S = (C, \sigma)$ is uniformly exponentially instable, then there exists $F \in \mathcal{F}$ with the properties:

(i) $\sup\limits_{\substack{\theta\in\Theta\\ \|x\|=1}} F(c_{x,\theta}) < \infty$;

(ii) there exists $N > 0$ such that $F(c^m_{x,\theta}) \leq N c_{x,\theta}(m)$, for all $m \in \mathbb{N}$ and $x \in X$ with $\|x\| = 1$;

(iii) $\sup\limits_{\substack{\|x\|=1\\ (m,\theta)\in\mathbb{N}\times\Theta}} F(c^m_{x,\theta}) < \infty$;

(iv) $\sup\limits_{\substack{\|x\|=1\\ (m,\theta)\in\mathbb{N}\times\Theta}} F(t^m_{x,\theta}) < \infty$.

Indeed, if we consider the function $F : \mathcal{S}^+(\mathbb{R}) \to [0, \infty]$ defined by

$$F(s) = \sum_{n=0}^{\infty} s(n),$$

then it is easy to verify that the uniform exponential instability property of S implies the conditions (i), (ii), (iii) and (iv).

The main result of this paper is:

Proposition 3.1. *An injective linear skew-product semiflow $S = (C, \sigma)$ on $E = \Theta \times X$ is uniformly exponentially instable if and only if there is $F \in \mathcal{F}$ such that*

$$\sup_{\substack{\theta\in\Theta\\ \|x\|=1}} F(c_{x,\theta}) < \infty.$$

Proof. Necessity. results from Remark 3.1.

Sufficiency. Let $M = \sup\limits_{\substack{\theta\in\Theta\\ \|x\|=1}} F(c_{x,\theta})$. Using the inequality

$$c_{x,\theta} \geq c_{x,\theta}(n)\chi_{\{n\}}, \quad \text{for all} \quad n \in \mathbb{N} \quad \text{and} \quad (\theta, x) \in E \quad \text{with} \quad \|x\| = 1,$$

we deduce that

$$M \geq F(c_{x,\theta}) \geq F(c_{x,\theta}(n)\chi_{\{n\}}) \geq \alpha c_{x,\theta}(n),$$

and hence

$$M\|C(n,\theta)x\| \geq \alpha\|x\|, \quad \text{for all} \quad n \in \mathbb{N} \quad \text{and} \quad (\theta, x) \in E \quad \text{with} \quad \|x\| = 1.$$

This implies that

$$\alpha\|C(k,\theta)x\| \leq M\|C(n,\theta)x\|, \quad \text{for all} \quad n, k \in \mathbb{N}, \quad k \leq n \quad \text{and} \quad (\theta, x) \in E$$

with $\|x\| = 1$.

It follows that

$$c_{x,\theta} \geq \frac{\alpha c_{x,\theta}(n)}{M}\chi_{\{0,\ldots,n\}}$$

and hence

$$c_{x,\theta} = \sum_{k=0}^{\infty} c_{x,\theta}(k)\chi_{\{k\}} \geq \sum_{k=0}^{n} c_{x,\theta}(k)\chi_{\{k\}} \geq \frac{\alpha c_{x,\theta}(n)}{M}\sum_{k=0}^{n}\chi_{\{k\}} = \frac{\alpha c_{x,\theta}(n)}{M}\chi_{\{0,\ldots,n\}}.$$

Then by condition (f_3) it results that

$$M \geq F(c_{x,\theta}) \geq F(\frac{\alpha c_{x,\theta}(n)}{M}\chi_{\{0,\ldots,n\}}) \geq \frac{\alpha f(n) c_{x,\theta}(n)}{M}$$

and hence

$$\|C(n,\theta)x\| \geq \frac{\alpha}{M^2}f(n), \quad \text{for all} \quad n \in \mathbb{N} \quad \text{and all} \quad (\theta, x) \in E,$$

with $\|x\| = 1$.

By Proposition 2.1 it results that S is uniformly exponentially instable. □

Corollary 3.1. *An injective linear skew-product semiflow $S = (C, \sigma)$ on $E = \Theta \times X$ is uniformly exponentially instable if and only if there exist $N > 0$ and $F \in \mathcal{F}$ such that*

$$F(c_{x,\theta}^{m}) \leq N c_{x,\theta}(m),$$

for all $m \in \mathbb{N}$, $(\theta, x) \in E$ with $\|x\| = 1$.

Proof. Necessity. It results from Remark 3.1.

Sufficiency. We observe that

$$\sup_{\substack{\theta\in\Theta\\ \|x\|=1}} F(c_{x,\theta}) = \sup_{\substack{\theta\in\Theta\\ \|x\|=1}} F(c^0_{x,\theta}) \le N c_{x,\theta}(0) = N < \infty$$

and by Proposition 3.1 it follows that S is uniformly exponentially instable. □

Corollary 3.2. *An injective linear skew-product semiflow $S = (C, \sigma)$ on $E = \Theta \times X$ is uniformly exponentially instable if and only if there exists $F \in \mathcal{F}$ such that*

$$\sup_{\substack{\|x\|=1\\ (m,\theta)\in\mathbb{N}\times\Theta}} F(c^m_{x,\theta}) < \infty.$$

Proof. Necessity. It results from Remark 3.1.

Sufficiency. It results from Proposition 3.1 taking into account that

$$\sup_{\substack{\theta\in\Theta\\ \|x\|=1}} F(c_{x,\theta}) = \sup_{\substack{\theta\in\Theta\\ \|x\|=1}} F(c^0_{x,\theta}) \le \sup_{\substack{\|x\|=1\\ (m,\theta)\in\mathbb{N}\times\Theta}} F(c^m_{x,\theta}) < \infty. \qquad \square$$

Similarly, we obtain the following result.

Corollary 3.3. *An injective linear skew-product semiflow $S = (C, \sigma)$ on $E = \Theta \times X$ is uniformly exponentially instable if and only if there exists $F \in \mathcal{F}$ such that*

$$\sup_{\substack{\|x\|=1\\ (m,\theta)\in\mathbb{N}\times\Theta}} F(t^m_{x,\theta}) < \infty.$$

Remark 3.2. The preceding results are discrete versions of a Neerven's theorem (see [11]) for the case of instability property.

We shall denote by Φ the set of all nondecreasing functions $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\varphi(0) = 0$ and $\varphi(t) > 0$, for all $t > 0$.

Corollary 3.4. *An injective linear skew-product semiflow $S = (C, \sigma)$ on $E = \Theta \times X$ is uniformly exponentially instable if and only if there exists $\varphi \in \Phi$ such that*

$$\sup_{\substack{\theta\in\Theta\\ \|x\|=1}} \sum_{n=0}^{\infty} \varphi(c_{x,\theta}(n)) < \infty.$$

Proof. Necessity. It is trivial for $\varphi(t) = t$.
Sufficiency. It results from Proposition 3.1 for

$$F(s) = \sum_{n=0}^{\infty} \varphi(s(n)).$$ □

Remark 3.3. The preceding corollary extends a Zabczyk's Theorem ([15]) for the case of exponential instability.

For the particular case $\varphi(t) = t^p$ we obtain:

Corollary 3.5. *An injective linear skew-product semiflow $S = (C, \sigma)$ on $E = \Theta \times X$ is uniformly exponentially instable if and only if there is $p \in [1, \infty)$ such that*

$$\sup_{\substack{\theta \in \Theta \\ \|x\|=1}} \sum_{n=0}^{\infty} [c_{x,\theta}(n)]^p < \infty.$$

Remark 3.4. Corollary 3.5 is a discrete version of Datko's Theorem (see [4]) for the case of exponential instability. It can be also considered as a variant for the exponential instability of a theorem proved by Przyluski and Rolewicz in [13] for the case of exponential stability.

Let $\mathcal{B}(\mathbb{N})$ be the set of all normed sequence spaces B (see [10]) with the properties:

(i) $\chi_{\{0,\ldots,n\}} \in B$, for all $n \in \mathbb{N}$;

(ii) $\lim_{n\to\infty} |\chi_{\{0,\ldots,n\}}|_B = \infty$;

(iii) there exists $\alpha > 0$ such that $|\chi_{\{n\}}|_B \geq \alpha$, for all $n \in \mathbb{N}$.

Corollary 3.6. *An injective linear skew-product semiflow $S = (C, \sigma)$ on $E = \Theta \times X$ is uniformly exponentially instable if and only if there exists a normed sequence space $B \in \mathcal{B}(\mathbb{N})$ such that for every $\theta \in \Theta$ and $x \in X$ with $\|x\| = 1$, we have that $c_{x,\theta} \in B$ and*

$$\sup_{\substack{\theta \in \Theta \\ \|x\|=1}} |c_{x,\theta}|_B < \infty.$$

Proof. Necessity. It is immediate for $B = l^1$.
Sufficiency. Let $F : \mathcal{S}^+(\mathbb{R}) \to [0, \infty]$ be the function defined by

$$F(s) = \sup_{n \in \mathbb{N}} |s \cdot \chi_{\{0,\ldots,n\}}|_B.$$

It is easy to see that $F \in \mathcal{F}$ and

$$c_{x,\theta}\chi_{\{0,\ldots,n\}} \leq c_{x,\theta}, \quad \text{for all} \quad n \in \mathbb{N} \quad \text{and} \quad (\theta, x) \in E \quad \text{with} \quad \|x\| = 1.$$

Then

$$\sup_{\substack{\theta\in\Theta \\ \|x\|=1}} F(c_{x,\theta}) \leq \sup_{\substack{\theta\in\Theta \\ \|x\|=1}} |c_{x,\theta}|_B < \infty.$$

By Proposition 3.1 it results that S is uniformly exponentially instable. □

Remark 3.5. Corollary 3.6 is a discrete variant for exponential instability of Theorem 3.1.5 from [10].

As a particular case, for the Banach sequence space

$$B = \{s \in \mathcal{S}^+(\mathbb{R}) : \beta s \in l^p\}$$

we obtain:

Corollary 3.7. *An injective linear skew-product semiflow $S = (C, \sigma)$ on $E = \Theta \times X$ is uniformly exponentially instable if and only if there are $p \in [1, \infty)$ and $\beta \in \mathcal{S}^+(\mathbb{R})$ with $\beta > 0$ and $\sum_{n=0}^{\infty} \beta(n) = \infty$ such that*

$$\sup_{\substack{\theta\in\Theta \\ \|x\|=1}} \sum_{n=0}^{\infty} \beta^p(n)[c_{x,\theta}(n)]^p < \infty.$$

Remark 3.6. The preceding corollary is an extension of Corollary 3.1.6. from [10].

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