International Journal of Pure and Applied Mathematics
Volume 30 No. 1 2006, 5-11

LOW RANK VECTOR BUNDLES AND REFLEXIVE SHEAVES ON $\mathbf{P}^3$ DEFINED OVER $\mathbb{F}_q$

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Abstract: Here for many Chern classes we prove the existence of rank 2 and rank 3 geometrically stable vector bundles and reflexive sheaves on $\mathbf{P}^3$ defined over $\mathbb{F}_q$ with q as low as we are able to do.

AMS Subject Classification: 14J60
Key Words: vector bundle over $\mathbf{P}^3$, stable vector bundle, reflexive sheaf, finite field

1. Introduction

Here for many Chern classes we prove the existence of rank 2 and rank 3 geometrically stable vector bundles and reflexive sheaves on $\mathbf{P}^3$ defined over $\mathbb{F}_q$ with q as low as we are able to do. We strat stating our results.

Theorem 1. *Fix a prime p, a p power q and integers c_1, c_2 such that $c_1^2 < 4c_2$. If either $c_1 \cdot c_2 \equiv 0 \pmod 2$, then set $\alpha := 0$. If $c_1 \cdot c_2 \equiv 1 \pmod 2$, then set $\alpha := 1$. Then there exists a rank two geometrically stable vector bundle E on $\mathbf{P}^3$ defined over $\mathbb{F}_q$ such that $c_1(E) = c_1$, $c_2(E) = c_2$ and with α as its Atiyah-Rees invariant. Furthermore, if $q \neq p$ we may also find E with cannot be defined over any $\mathbb{F}_{q'}$ with q' a p-power and $q' < q$.*

By [3], Remark 3.1.4, the integers c_1, c_2 and α covered in parts (a), (b) and (c) are the only numerical invariants allowed for rank 2 stable vector bundles on $\mathbf{P}^3(\bar{\mathbb{F}}_q)$.

Received: April 29, 2006

Remark 1. In the case $p = 2$ we do not claim that the Atiyah-Rees invariant α is constant in any connected family of rank 2 vector bundles on $\mathbf{P}^3(\bar{\mathbb{F}}_q)$. Our examples (lifted from [3]) may be obtained modulo 2 from examples over $\mathbb{Z}$ whose associated vector bundles over $\mathbb{C}$ have the prescribed Atiyah-Rees invariant α.

Now we consider the case of rank 2 reflexive sheaves on $\mathbf{P}^3$. Up to a twist by a line bundle we may assume $c_1 \in \{-1, 0\}$ and then use [4], Lemma 2.1, to get the Chern classes of the twisted twisted sheaf. In the following statement for all integers a, b we will write (a, b) for the open interval $\{x : \mathbb{Z} : a < x < b\}$.

Theorem 2. *Fix a prime p, a p-power q and integers c_1, c_2, c_3 such that $c_1 \in \{-1, 0\}$, $c_2 > 0$, $c_3 \equiv c_1 c_2 \pmod 2$, $0 \le c_3 \le c_2^2$ if $c_1 = -1$, $0 \le c_3 \le c_2 - c_2 + 2$ if $c_1 = 0$. Set $b(-1, c_2) := (-1 + \sqrt{4c_2 - 7})/2$ and $b(0, c_2) := \sqrt{c_2 - 2}$. If $c_1 = -1$ assume $c_3 \notin \cup_{r=1}^{b(-1,c_2)} (c_2^2 - 2rc_2 + 2(r+1)r, c_2^2 - 2(r-1)c_2)$. If $c_1 = 0$ assume $c_3 \notin \cup_{r=2}^{b(0,c_2)} (c_2^2 - (2r-1)c_2 + 2r^2, c_2^2 - (2r-3)c_2)$.*

(i) If $c_1 = -1$, $c_2 \ge 4$, and there are integers r, d such that $(c_2/2) + 1 \ge r \ge 1$, $0 \le d \le r(r-1)$ and $c_3 = c_2^2 - 2(r-1)c_2 + 2d$, then assume $q \ge (r^3 - 2r^2 + r - 1)r^2(r-1)^4$.

(ii) If $c_1 = -1$, c_2 is even, $c_2 \ge 2$, and $c_3 = c_2^2 - (2r-1)c_2 + 2d$ for some integers r, d such that $(c_2+1)/2 > r \ge 1$ and $0 \le d \le r^2$, then assume $q \ge (r^3 - 1)r^6$.

Then there exists a rank 2 geometrically stable reflexive sheaf on $\mathbf{P}^3$ defined over $\mathbb{F}_q$ such that $c_1(E) = c_1$, $c_2(E) = c_2$ and $c_3(E) = c_3$.

Now, following [7], we consider the case of rank 3 geometrically stable reflexive sheaves. Up to a twist it is sufficient to consider the case $c_1 \in \{-2, -1, 0\}$.

Theorem 3. *Fix a prime p, a q-power and integers $c_1 \in \{-2, -1, 0\}$ such that $c_2 \ge 6$ and $q \ge c_2^9$. Set $B(-2, c_2) = B(0, c_2) = \lfloor (-1 + \sqrt{4c_2 - 11})/2 \rfloor$ and $B(-1, c_2) = \lfloor -1 + \sqrt{c_2 - 2} \rfloor$.*

(i) If $c_1 = -1$ fix an integer c_3 such that $-c_2^2 + 3c_2 - 4 \le c_3 \le c_2^2 - 3c_2 + 2$, $c_3 \equiv 0 \pmod 2$ and $c_3 \notin \cup_{r=1}^{B(-1,c_2)} (c_2^2 - 2(r+1)c_2 + 2(r+1)^2, c_2 - 2rc_2)$;

(ii) If $c_1 = -2$ fix an integer c_3 such that $-c_2^2 \le c_3 \le c_2^2 - 2c_2 + 2$, $c_3 \equiv c_2 \pmod 3$ and $c_3 \notin \cup_{r=1}^{B(-2,c_2)} (c_2^2 - (2r+3)c_2 + 2(r+1)^2, c_2 - (2r+1)c_2 + 2r)$;

(iii) If $c_1 = 0$ fix an integer c_3 such that $-c_2^2 + c_2 \le c_3 \le c_2^2 - c_2$, $c_3 \equiv 0 \pmod 2$ and $c_3 \notin \cup_{r=1}^{B(0,c_2)} (c_2^2 - (2r+1)c_2 + 2(r+1)^2 - 2r, c_2 - (2r-1)c_2)$.

Then there exists a rank 3 geometrically stable reflexive sheaf on $\mathbf{P}^3$ defined over $\mathbb{F}_q$ such that $c_1(E) = c_1$, $c_2(E) = c_2$ and $c_3(E) = c_3$.

2. The Proofs

Remark 2. Let Y be any projective variety and F any coherent algebraic sheaf on Y, both defined over a field K. Since any field extension is flat, $\dim_K(H^0(Y,F)) = \dim_L(H^0(Y_L,F_L))$ for any field extension L of K ([2], III.9.3). The same observation is true for the dimension of the Ext-group of two sheaves on Y defined over K.

Remark 3. Fix an interger c_1, a locally Cohen-Macaulay curve in $\mathbf{P}^3$ defined over $\bar{\mathbb{F}}_q$ which is locallya complete intersection outside finitely many points of Y_{red} and $\beta \in H^0(Y,\omega_Y(4-c_1))$ which generates $\omega_Y(4-c_1)$ outside finitely many points of Y_{red}. By [4], Theorem 4.1, the pair (Y,β) uniquely determine a pair (G,s), where G is a rank 2 reflexive sheaf on $\mathbf{P}^3$ defined over $\bar{\mathbb{F}}_q$ and $s \in H^0(\mathbf{P}^3,G)$ induces an exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}^3} \to G \to \mathcal{I}_Y(c_1) \to 0\,. \tag{1}$$

Furthermore, $c_1(G) = c_1$, $c_2(G) = \deg(Y)$ and $c_3(Y) = 2p_a(Y) - 2 + d(4-c_1)$. Now assume Y defined over $\mathbb{F}_q$, $h^0(\mathbf{P}^3,\mathcal{I}_Y(c_1)) = 0$, and $h^0(Y,\omega_Y(4-c_1)) = 1$. Then the pair (G,s) is uniquely determined by Y and the integer $4-c_1$ and hence the sheaf G and the extension (1) is defined over $\mathbb{F}_q$. The sheaf G is locally free if and only if $\omega_Y \cong \mathcal{O}_Y(c_1-4)$ and β is a nowhere vanishing section of $\omega_Y(4-c_1)$. In the cases used in the proofs of Theorems 1, 2 and 3 we will always take β defined over $\mathbb{F}_q$ and hence we will always get a locally free G defined over $\mathbb{F}_q$.

Proof of Theorem 1. Up to a twist we may assume $c_1 \in \{-1.0\}$ ([3], p. 236).

(a) Here we assume $c_1 = 0$. By [3], p. 236, we have $c_1(F(m)) = c_1(F) + 2m$ and $c_2(F(m)) = c_2(F) + m \cdot c_1(F) + m^2$ for any $m \in \mathbb{Z}$ and any rank 2 vector bundle F on $\mathbf{P}^3$. Set $t := c_2 + 1$. Let $Y \subset \mathbf{P}^3$ any disjoint union of t lines. By [3], Example 3.1.2, and the inequality $t \geq 2$, Y uniquely determines the isomorphism class of a rank 2 stable vector bundle G on $\mathbf{P}^3$ such that $c_1(G) = 2$ and $c_2(G) = t$. Set $E := G(-1)$. We have $c_1(E) = 0$ and $c_2(E) = c_2$ ([3], p. 226). Let $G(1,3)$ denote the Grassmannian of all lines in $\mathbf{P}^3$. Notice that $\sharp(G(1,3)(\mathbb{F}_x)) = (x^4-1)(x^3-1)/(x^2-1)(x-1) = (x^2+1)(x^2+x+1)$ for all prime-powers x ([5], p. 65). Fix any $P \in \mathbf{P}^3(\mathbb{F}_{q^t}) \backslash \mathbf{P}^3(\mathbb{F}_{q^{t-1}})$. Set $A(P) :=$

$\{D \in G(1,3)(\mathbb{F}_{q^t})\backslash G(1,3)(\mathbb{F}_{q^{t-1}}) : P \in D\}$. Since $\sharp(\mathbf{P}^2(\mathbb{F}_{q^t})\backslash \mathbf{P}^2(\mathbb{F}_{q^{t-1}})) = q^2t + q^t - q^{2t-2} - q^{t-1}$, we have $sharp(A(P)) = q^2t + q^t - q^{2t-2} - q^{t-1}$. Notice that $\sharp(\mathbf{P}^3(\mathbb{F}_{q^t})\backslash \mathbf{P}^3(\mathbb{F}_{q^{t-1}})) = q^{3t} + q^{2t} + q^t - q^{3t-3} - q^{2t-2} - q^{t-1}$. Since $(q^{2t} + 1)(q^{2t} + q^t + 1) - (q^{2t-2} + 1)(q^{2t-2} + q^{t-1} + 1) > (q^2t + q^t - q^{2t-2} - q^{t-1})(q^{3t} + q^{2t} + q^t - q^{3t-3} - q^{2t-2} - q^{t-1})$, there is $D \in G(1,3)(\mathbb{F}_{q^t})\backslash G(1,3)(\mathbb{F}_{q^{t-1}})$ such that D contains no point of $\mathbf{P}^3(\mathbb{F}_{q^{t-1}})$. Let $G \cong \mathbb{Z}/t\mathbb{Z}$ be the Galois group of the field extension $\mathbb{F}_{q^t}/\mathbb{F}_q$. Set $Y := \cup_{g\in G} D$. A is a closed subscheme of $\mathbf{P}^3$ defined over $\mathbb{F}_q$ which splits into t different lines over $\mathbb{F}_{q^t}$. These lines are pairwise disjiont, because any $Q \in gD \cap hD$ for some $h, g \in G$ such that $h \neq g$ is defined over a proper subfield of $\mathbb{F}_{q^t}$, while $Y \cap \mathbf{P}^3(\mathbb{F}_{q^{t-1}}) = \emptyset$.

(b) Here we assume $c_1 = -1$, c_2 even and $c_2 \geq 2$. Set $r := c_2/2 + 1$. Hence $r \geq 2$. Let $Y \subset \mathbf{P}^3$ be the disjoint union of $2r$ smooth curves. Since $r \geq 2$, Y is contained in no plane. By [3], Example 3.1.3, and the inequality $r \geq 2$, Y uniquely determines the isomorphism class of a rank 2 stable vector bundle G on $\mathbf{P}^3$ such that $c_1(G) = 3$ and $c_2(G) = 2r$. Set $E := G(-1)$. We have $c_1(E) = -1$ and $c_2(E) = c_2$ ([3], p. 226). Take a plane $M \subset \mathbf{P}^3$ defined over $\mathbb{F}_{q^t}$, but not over $\mathbb{F}_{q^{t-1}}$. It is easy to find a smooth conic $A \subset M$ defined over $\mathbb{F}_{q^t}$, but not over $\mathbb{F}_{q^{t-1}}$, and such that $A \cap g(M) = \emptyset$ for all $g \in G\backslash\{\mathrm{Id}\}$. Set $Y := \cup_{g\in G} g(A)$.

(c) Here we assume $c_1 = -1$, $c_2 \geq 3$ and $\alpha = 1$. Set $r := c_2 + 1$. Hence $r \geq 4$. Let $Y \subset \mathbf{P}^3$ be any disjoint union of a nonsingular plane cubic curve and a nonsingular elliptic space of degree r. Since $r \geq 4$, the second curve does not lie in a plane. Hence we easily get $h^0(\mathbf{P}^3, \mathcal{I}_Y(2)) = 0$. By [3], Example 3.1.3, Y uniquely determines the isomorphism class of a rank 2 stable vector bundle G on $\mathbf{P}^3$ such that $c_1(G) = 4$, $c_2(G) = r + 3$ and $\alpha = 1$. Set $E := E(-2)$. Thus E is a rank 2 stable vector bundle on $\mathbf{P}^3$ with $c_1(E) = 0$, $c_2(E) = c_2$ and $\alpha = 1$. It is easy to find Y as above defined over $\mathbb{F}_{q^t}$, but not over $\mathbb{F}_{q^{t-1}}$.

(d) By [3], Remark 3.1.4, the integers c_1, c_2 and α covered in parts (a), (b) and (c) are the only numerical invariants allowed for rank 2 stable vector bundles on $\mathbf{P}^3(\bar{\mathbb{F}}_q)$.

(e) For the "furthermore" part, just use that for the vector bundles we constructed, the vector bundle uniquely Y and Y is not defined over a proper subfield of $\mathbb{F}_q$. □

Proof of Theorem 2. We will use the four constructions of stable rank two reflexive sheave.

(a) Fix integers c_2, r, d such that $c_2 \geq 4$ and $(c_2/2) + 1 \geq r \geq 1$ and $0 \leq d \leq r(r-1)$. Let $Y_1 \subset \mathbf{P}^3$ be a smooth degree c_2 plane curve $Y_2 \subset \mathbf{P}^3$ a smooth degree $(c_2/2) + 1$ curve complete intersection of two surfaces of degree r and $r - 1$ such that $\sharp((Y_1 \cap Y_2)_{red}) = d$ and the curve $Y := Y_1 \cup Y_2$ is nodal (with the convention $Y_2 = \emptyset$ if $r = 1$). We also assume that Y_1 and Y_2 are defined over $\mathbb{F}_q$ and that Y_1 is not defined over any proper subfield of $\mathbb{F}_q$. For the existence of such curves Y_1, Y_2, see below. The reflexive sheaf E is given by an extension

$$0 \to \mathcal{O}_{\mathbf{P}^3} \to E(r) \to \mathcal{I}_Y(2r-1) \to 0\,. \tag{2}$$

By construction E is stable, $c_1(E) = -1$, $c_2(E) = c_2$ and $c_3(E) = 2p_a(Y) - 2 + (5-2r)\cdot \deg(Y) = c_2^2 - 2(r-1)c_2 + 2d$. We will check the existence of Y only in the case $r \ge 2$, because the case $r = 1$ (i.e. $Y_2 = \emptyset$) is easier. We start with a smooth degree r surface $A \subset \mathbf{P}^3$ defined over $\mathbb{F}_q$, but not over a proper subfield of $\mathbb{F}_q$. Let $v : \mathbf{P}^3 \to \mathbf{P}^N$, $N := \binom{r+2}{3} - 1$, be the order $r-1$ Veronese embedding of $\mathbf{P}^3$. Hence v is defined over $\mathbb{F}_q$, $\dim(v(A)) = 2$ and $\deg(v(A)) = r(r-1)^2$. Since $q \ge (r^3 - 2r^2 + r - 1)r^2(r-1)^4$, we may apply [1], Theorem 1.1, and get the existence of a hyperplane M of $\mathbf{P}^N$ intersecting transversally $v(A)$, i.e. the existence of a degree r surface $B \subset \mathbf{P}^3$ defined over $\mathbb{F}_q$ such that $Y_2 := A \cap B$ is a smooth complete intersection and it is defined over $\mathbb{F}_q$. The same assumption on q gives the existence of a plane W defined over $\mathbb{F}_q$ intersecting transversally Y_2 and defined over $\mathbb{F}_q$. However, we do not know if for large d we may find a smooth Y_1 such that $Y_1 \cup Y_2$ is nodal and $Y_1 \cap Y_2$ contains d points whose union is defined over $\mathbb{F}_q$. Hence (unless, say, $d \le r$) we will use a different construction. We will find Y_1, Y_2 as above, except that Y_1 and Y_2 are reducible and nodal. We take as Y_2 a nodal intersection of A with $r-1$ transversal planes A_i, $1 \le i \le r-1$. The existence of A_1 is given by [1], Theorem 1.1, under weaker assumption on q. Then we apply again the same result inductively $r-2$ times to a pair $(A, A \cap (A_1 \cup \cdot \cup A_{i-1}))$. Then we find Y_2 as a nodal union of lines. Since any set $A \cap D$, D a line defined over $\mathbb{F}_q$ and transversal to A, is defined over $\mathbb{F}_q$, in this way we need to find at most $r-1$ points of $Y_1 \cap Y_2$, each of them defined over $\mathbb{F}_q$. We use that by Hasse-Weil inequality any degree r smooth plane curve defined over $\mathbb{F}_q$ points has at least $q + 1 - (r-1)(r-2)\sqrt{q}$ $\mathbb{F}_q$-points.

(b) Fix an odd integer $c_2 \ge 5$ and an integer m such that $2 \le m \le ((c_2 - 1)/2) + 1$. Let Y be the union of m mutually disjoint rational curves of degree ≥ 2, each of them defined over $\mathbb{F}_q$ and such that the scheme Y is not defined over any proper subfield of $\mathbb{F}_q$. It is easy to check the existence of such curve Y. The reflexive sheaf E is given by an extension

$$0 \to \mathcal{O}_{\mathbf{P}^3} \to E(2) \to \mathcal{I}_Y(3) \to 0\,. \tag{3}$$

By construction E is stable and $c_1(E) = -1$, $c_2(E) = c_2$ and $c_3(E) = 2p_a(Y) - 2 + \deg(Y) = 2p_a(Y) + c_2 = c_2 - 2m + 2$.

(c) Fix integers c_2, r, d such that $c_2 \geq 6$, $(c_2+1)/2 > r \geq 1$ and $0 \leq d \leq r^2$. Let Y_1 be a smooth plane curve of degree c_2 and Y_2a smooth complete intersection of two degree r surfaces such that $Y := Y_1 \cup Y_2$ is nodal and $\sharp(Y_1 \cap Y_2) = d$. We also assume that Y_1 and Y_2 are defined over $\mathbb{F}_q$ and that Y_1 is not defined over any proper subfield of $\mathbb{F}_q$. For the existence of such curves Y_1, Y_2, see below. The reflexive sheaf E is given by an extension

$$0 \to \mathcal{O}_{\mathbf{P}^3} \to E(r) \to \mathcal{I}_Y(2r) \to 0\,. \tag{4}$$

By construction E is stable and $c_1(E) = 0$, $c_2(E) = c_2$, and $c_3(E) = c_2^2 - (2r-1)c_2 + 2d$. As in part (b) an application of [1], Th. 1.1, gives the existence of a nodal $Y_1 \cup Y_2$ defined over $\mathbb{F}_q$.

(d) Fix an even integer $c_2 \geq 6$ and an integer m such that $c_2 + 1 \geq m \geq (c_2/2)+1$. Let Y be the union of m mutually disjoint rational curves such that $\deg(Y) = c_2 + 1$. We also assume that Y is $\mathbb{F}_q$ and that Y_1 is not defined over any proper subfield of $\mathbb{F}_q$. It is easy to check the existence of such a curve Y, see below. The reflexive sheaf E is given by an extension

$$0 \to \mathcal{O}_{\mathbf{P}^3} \to E(1) \to \mathcal{I}_Y(2) \to 0 \tag{5}$$

By construction E is stable, $c_1(E) = 0$, $c_2(E) = c_2$, and $c_3(E) = 2c_2 - 2m + 2$.

(e) It was cheked in [6], proof of Theorem B at p. 321, that the cases listed in part (a), (b), (c) and (d) (called there respectively Constructions 1, 2, 3 and 4) cover the range of Chern classes given in the statement of Theorem 2. □

Proof of Theorem 3. All example are constructed in [7] as extension of an example constructed in the proof of Theorem 2 or with a similar example (Construction 2.10 and Construction 2.11 of [7]). We may fix a non-zero extension defined over $\mathbb{F}_q$. Such a non-zero extension exists because it exists over $\bar{\mathbb{F}}_q$ and hence we may apply the last part of Remark 2. □

Acknowledgements

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

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