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ON FACTORIZATION OF THE GENERALIZED FIBONACCI NUMBERS

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Abstract: Several authors gave various factorizations of the Fibonacci and Lucas numbers. In this paper some results on factorizations of the generalized Fibonacci numbers W_n which satisfy the recurrence of the second order $W_{n+2} = pW_{n+1} - qW_n$ are derived. Proofs are made with the help of connections between determinants of tridiagonal matrices and the numbers W_n using the Chebyshev polynomials.

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1. Introduction

Connections between recurrence relations for the Fibonacci numbers and Chebyshev polynomials are discussed in [6]. Zeros of Chebyshev polynomials give the possibility to factorize the Fibonacci numbers. A rather different approach to their factorization applied other authors. They found several factorizations of the Fibonacci or Lucas numbers and some specific linear subsequences of them. In [1] Cahill et al studied certain families of tridiagonal matrices and their correspondence to these sequences. In [2] the authors derived the following complex

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factorizations:

$$F_n = \prod_{k=1}^{n-1}\left(1 - 2i\cos\frac{k\pi}{n}\right),\ n \geq 2\,,$$

and

$$L_n = \prod_{k=1}^{n}\left(1 - 2i\cos\frac{(2k-1)\pi}{2n}\right),\ n \geq 1\,.$$

They proved them by considering how the realized numbers can be connected to Chebyshev polynomials by determinants of sequences of suitable tridiagonal matrices. In the case of the Fibonacci numbers they used the $n \times n$ tridiagonal matrix $M(n)$ with entries $m_{k,k} = 1$, $1 \leq k \leq n$, and $m_{k-1,k} = m_{k,k-1} = i$, $2 \leq k \leq n$. Thus,

$$M(n) = \begin{pmatrix} 1 & i & & & \\ i & 1 & i & & \\ & i & 1 & \ddots & \\ & & \ddots & \ddots & i \\ & & & i & 1 \end{pmatrix}.$$

For the Lucas numbers they used the $n \times n$ tridiagonal matrix $S(n)$ with entries $s_{1,1} = \frac{1}{2}$, $s_{k,k} = 1$, $2 \leq k \leq n$, and $s_{k-1,k} = s_{k,k-1} = i$, $2 \leq k \leq n$. That is,

$$S(n) = \begin{pmatrix} \frac{1}{2} & i & & & \\ i & 1 & i & & \\ & i & 1 & \ddots & \\ & & \ddots & \ddots & i \\ & & & i & 1 \end{pmatrix}.$$

In [3] Cahill and Narayan extended the previous results to construct families of tridiagonal matrices whose determinants generate an arbitrary linear subsequence F_{an+b} or L_{an+b} for any positive integers a, n and nonnegative integer b. They chose a specific linear subsequence of the Fibonacci numbers and used it to derive the factorization

$$F_{2mn} = F_{2m}\prod_{k=1}^{n-1}\left(L_{2m} - 2\cos\frac{k\pi}{n}\right),$$

which was a generalization of the factorization

$$F_{2n} = \prod_{k=1}^{n-1}\left(3 - 2\cos\frac{k\pi}{n}\right)$$

presented in [2].

In this paper we derive factorizations of the generalized Fibonacci numbers, especially the Fibonacci–type and the Lucas–type numbers.

2. Preliminary Results

Lemma 1. (see [2], Lemma 1) *Let* $\{H(n), n = 1, 2, \dots\}$ *be a sequence of tridiagonal matrices of the form:*

$$H(n) = \begin{pmatrix} h_{1,1} & h_{1,2} & & & & \\ h_{2,1} & h_{2,2} & h_{2,3} & & & \\ & h_{3,2} & h_{3,3} & h_{3,4} & & \\ & & h_{4,3} & h_{4,4} & \ddots & \\ & & & \ddots & \ddots & h_{n-1,n} \\ & & & & h_{n,n-1} & h_{n,n} \end{pmatrix}.$$

Then the successive determinants of $H(n)$ *are given by the recursive formula*

$$\begin{aligned} |H(1)| &= h_{1,1}\,, \\ |H(2)| &= h_{1,1}h_{2,2} - h_{1,2}h_{2,1}\,, \\ |H(n)| &= h_{n,n}|H(n-1)| - h_{n-1,n}h_{n,n-1}|H(n-2)|\,. \end{aligned} \tag{1}$$

Some connections between determinants of tridiagonal matrices and the Fibonacci and the Lucas numbers are known. It is easy to compute by Lemma 1 that $|M(n)| = F_{n+1}$ and $|S(n)| = \frac{1}{2}L_n$, where $M(n)$ and $S(n)$ are above the mentioned matrices.

In the notation of Horadam [4], let us denote

$$W_n = W_n(a, b; p, q)\,,$$

so that

$$W_n = p\,W_{n-1} - q\,W_{n-2}\,, \quad n \geq 2\,, \tag{2}$$

where $W_0 = a, W_1 = b$.

The n-th terms of the Fibonacci and Lucas sequences are then

$$F_n = W_n(0, 1; 1, -1)\,, \quad L_n = W_n(2, 1; 1, -1)\,.$$

More generally, we name the Fibonacci–type sequence $U_n = W_n(0, 1; p, q)$ and the Lucas–type sequence $V_n = W_n(2, p; p, q)$. The Binet formulas for U_n and V_n are

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}\,, \quad V_n = \alpha^n + \beta^n\,,$$

where $\alpha = \frac{p+\sqrt{p^2-4q}}{2}$ and $\beta = \frac{p-\sqrt{p^2-4q}}{2}$ are the roots (mutually distinct) of the quadratic equation $x^2 - px + q = 0$.

The sequence of Chebyshev polynomials of the first kind is the sequence $\{T_n(x)\}_{n=0}^{\infty}$, where $x \in \mathbb{C}$, defined by the recurrence relation

$$T_{n+2}(x) = 2x\,T_{n+1}(x) - T_n(x)\,,$$

with $T_0(x) = 1$ and $T_1(x) = x$. These polynomials $T_n(x)$ can be generated by determinants of successive $n \times n$ matrices, $n \geq 1$, of the form

$$T(n,x) = \left(\begin{array}{cccccc} x & 1 & & & & \\ 1 & 2x & 1 & & & \\ & 1 & 2x & 1 & & \\ & & 1 & 2x & \ddots & \\ & & & \ddots & \ddots & 1 \\ & & & & 1 & 2x \end{array} \right). \tag{3}$$

The sequence of Chebyshev polynomials of the second kind is the sequence $\{S_n(x)\}_{n=0}^{\infty}$, where $x \in \mathbb{C}$, defined by the same recurrence relation

$$S_{n+2}(x) = 2x\,S_{n+1}(x) - S_n(x)\,,$$

with $S_0(x) = 1$ and $S_1(x) = 2x$. Now, the polynomials $S_n(x)$ can be generated by determinants of successive $n \times n$ matrices, $n \geq 1$, of the form

$$S(n,x) = \left(\begin{array}{cccccc} 2x & 1 & & & & \\ 1 & 2x & 1 & & & \\ & 1 & 2x & 1 & & \\ & & 1 & 2x & \ddots & \\ & & & \ddots & \ddots & 1 \\ & & & & 1 & 2x \end{array} \right). \tag{4}$$

There is a connection between the sequence of the Fibonacci numbers and the sequences of Chebyshev polynomials $S_n(x)$ and $T_n(x)$. For example Rivlin in [6] derived the following relations

$$S_n\left(\frac{i}{2}\right) = i^n F_{n+1}\,, \quad T_n\left(\frac{i}{2}\right) = \frac{i^n}{2}(2F_{n+1} - F_n) = \frac{i^n}{2}\frac{F_{2n}}{F_n}$$

and

$$F_{2n} = S_{n-1}\left(\frac{3}{2}\right)\,.$$

Morgado in [5] gave some analogical relations.

3. The Main Result

Theorem 2. *The factorizations of the Fibonacci–type numbers and the Lucas–type numbers are given as follows:*

$$(i) \qquad U_n = \prod_{k=1}^{n-1} \left(p - 2\sqrt{q} \cos \frac{k\pi}{n} \right), \quad n \geq 2, \tag{5}$$

$$(ii) \qquad V_n = \prod_{k=1}^{n} \left(p - 2\sqrt{q} \cos \frac{(2k-1)\pi}{2n} \right), \quad n \geq 1. \tag{6}$$

4. Proof of Main Result

Proof of Theorem 2. (i) In order to derive (5), we introduce the sequence of $n \times n$ tridiagonal matrices $\{U(n),\, n = 1, 2, \dots\}$, where

$$U(n) = \begin{pmatrix} 1 & 0 & & & \\ 0 & p & \sqrt{q} & & \\ & \sqrt{q} & p & \ddots & \\ & & \ddots & \ddots & \sqrt{q} \\ & & & \sqrt{q} & p \end{pmatrix}.$$

With respect to Lemma 1 we can write $|U(n)| = p\,|U(n-1)| - q\,|U(n-2)|$ for $n > 2$ and it is easy to see that $|U(1)| = 1$, $|U(2)| = p$. Thus, we have $|U(n)| = U_n$ for any positive integer n.

The determinant of a matrix can be expressed by taking the product of its eigenvalues. Therefore, we will compute the spectrum of the matrix $U(n)$ in order to find an alternative formulation for $|U(n)|$. It is obvious that one of the eigenvalues of $U(n)$ equals 1. The other eigenvalues are the ones of the $(n-1) \times (n-1)$ tridiagonal matrix $R(n-1)$ given for $n > 1$ in the form

$$R(n-1) = \begin{pmatrix} p & \sqrt{q} & & & \\ \sqrt{q} & p & \sqrt{q} & & \\ & \sqrt{q} & p & \ddots & \\ & & \ddots & \ddots & \sqrt{q} \\ & & & \sqrt{q} & p \end{pmatrix}.$$

Let us denote

$$G(n-1) = \begin{pmatrix} 0 & 1 & & & & \\ 1 & 0 & 1 & & & \\ & 1 & 0 & 1 & & \\ & & 1 & 0 & \ddots & \\ & & & \ddots & \ddots & 1 \\ & & & & 1 & 0 \end{pmatrix}$$

the $(n-1)\times(n-1)$ tridiagonal matrix. Note that $R(n-1) = pI + \sqrt{q}\,G(n-1)$, where I is the $(n-1)\times(n-1)$ identity matrix.

Let λ_k, $k = 1, 2, \ldots, n-1$, be the eigenvalues of $G(n-1)$ with associated eigenvectors $\mathbf{y}_k$. Then, for each k,

$$\begin{aligned} R(n-1)\,\mathbf{y}_k = (pI + \sqrt{q}\,G(n-1))\,\mathbf{y}_k &= pI\,\mathbf{y}_k + \sqrt{q}\,G(n-1)\,\mathbf{y}_k \\ &= p\,\mathbf{y}_k + \sqrt{q}\,\lambda_k\,\mathbf{y}_k = (p + \sqrt{q}\,\lambda_k)\,\mathbf{y}_k\,. \end{aligned}$$

Therefore $p + \sqrt{q}\,\lambda_k$, $k = 1, 2, \ldots, n-1$, are the eigenvalues of $R(n-1)$. Hence $|R(n-1)| = \prod_{k=1}^{n-1}(p + \sqrt{q}\,\lambda_k)$ for $n > 1$.

Now, we will use the Chebyshev polynomials of the second kind which are generated by the determinants of matrices $S(n,x)$ from (4).

Clearly $R(n-1) = \sqrt{q}\,S\left(n-1, \frac{p}{2\sqrt{q}}\right)$, then we have

$$|R(n-1)| = (\sqrt{q})^{n-1}\left|S\left(n-1, \frac{p}{2\sqrt{q}}\right)\right| = (\sqrt{q})^{n-1} S_{n-1}\left(\frac{p}{2\sqrt{q}}\right).$$

It is a well–known fact that defining $x = \cos\vartheta$, ϑ is complex, allows the Chebyshev polynomials of the second kind to be written as $S_n(x) = \frac{\sin(n+1)\vartheta}{\sin\vartheta}$.

The roots of $S_n(x)$ are given by the relation $\vartheta_k = \frac{k\pi}{n+1}$, $k = 1, 2, \ldots, n$, or $x_k = \cos\vartheta_k = \cos\frac{k\pi}{n+1}$. Applying the transformation $\lambda = -2x$, now we obtain the eigenvalues of $G(n-1)$ in the form

$$\lambda_k = -2\cos\frac{k\pi}{n}\,, \quad k = 1, 2, \ldots, n-1\,.$$

Combining the previous relations, we have

$$U_n = |U(n)| = \prod_{k=1}^{n-1}\left(p - 2\sqrt{q}\,\cos\frac{k\pi}{n}\right), \; n \geq 2\,.$$

(*ii*) In this case we introduce the sequence of tridiagonal $n \times n$ matrices $\{V(n),\, n = 1, 2, \dots\}$, where

$$V(n) = \begin{pmatrix} p & \sqrt{2q} & & & \\ \sqrt{2q} & p & \sqrt{q} & & \\ & \sqrt{q} & p & \ddots & \\ & & \ddots & \ddots & \sqrt{q} \\ & & & \sqrt{q} & p \end{pmatrix}.$$

Using Lemma 1 we have $|V(n)| = p\,|V(n-1)| - q\,|V(n-2)|$ for $n > 2$. Further $|V(1)| = p$, $|V(2)| = p^2 - 2q$. It means that $|V(n)| = V_n$ for any positive integer n.

Let $P(n)$ be the $n \times n$ tridiagonal matrix defined as $P(n) = V(n) - pI$. We see that

$$P(n)\,\mathbf{y}_k = V(n)\,\mathbf{y}_k - pI\mathbf{y}_k = \lambda_k \mathbf{y}_k - p\,\mathbf{y}_k = (\lambda_k - p)\,\mathbf{y}_k\,,$$

where λ_k are the eigenvalues of $V(n)$ with associated eigenvectors $\mathbf{y}_k$.

Then $\gamma_k = \lambda_k - p$ are the eigenvalues of $P(n)$. An eigenvalue γ of $P(n)$ is a root of the characteristic polynomial $|P(n) - \gamma I|$. Note that

$$\begin{aligned} &|P(n) - \gamma I| \\ &= \begin{vmatrix} -\gamma & \sqrt{2q} & & & \\ \sqrt{2q} & -\gamma & \sqrt{q} & & \\ & \sqrt{q} & -\gamma & \ddots & \\ & & \ddots & \ddots & \sqrt{q} \\ & & & \sqrt{q} & -\gamma \end{vmatrix} = 2(\sqrt{q})^n \begin{vmatrix} \frac{-\gamma}{2\sqrt{q}} & 1 & & \\ 1 & \frac{-\gamma}{\sqrt{q}} & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & \frac{-\gamma}{\sqrt{q}} \end{vmatrix}. \end{aligned}$$

We know that the Chebyshev polynomials $T_n(x)$ of the first kind can be generated by the determinants of matrices $T(n,x)$ from (3). Then we have

$$|P(n) - \gamma I| = 2\,(\sqrt{q})^n\, T\left(n, -\frac{\gamma}{2\sqrt{q}}\right) = 2\,(\sqrt{q})^n\, T_n\left(-\frac{\gamma}{2\sqrt{q}}\right).$$

Defining $x = \cos\vartheta$, $\vartheta \in \mathbb{C}$, allows the Chebyshev polynomials of the first kind to write as $T_n(x) = \cos n\vartheta$. We can see that the roots of this polynomial are given by

$$\vartheta_k = \frac{(2k-1)\pi}{2n},\ \ k = 1, 2, \dots, n, \quad \text{or } x_k = \cos\vartheta_k = \cos\frac{(2k-1)\pi}{2n}\,.$$

It means that $\gamma_k = -2\sqrt{q}\cos\frac{(2k-1)\pi}{2n}$ and $\lambda_k = p - 2\sqrt{q}\cos\frac{(2k-1)\pi}{2n}$, $k = 1, 2, \ldots, n$. Therefore

$$V_n = |V(n)| = \prod_{k=1}^{n}\left(p - 2\sqrt{q}\cos\frac{(2k-1)\pi}{2n}\right) .$$

This completes the proof of Theorem 2. □

5. Concluding Remarks

Special cases of the sequence $\{W_n\}$ which interest us in the number theory are above all the following ones. Their factorizations are derived from Theorem 2 (the first two factorizations are the same as the ones given in [2]):

The Fibonacci sequence $\{F_n\}$:

$$F_n = \prod_{k=1}^{n-1}\left(1 - 2i\cos\frac{k\pi}{n}\right) , \quad n \geq 2 ,$$

the Lucas sequence $\{L_n\}$:

$$L_n = \prod_{k=1}^{n}\left(1 - 2i\cos\frac{(2k-1)\pi}{2n}\right) , \quad n \geq 1 ,$$

the Pell sequence $\{P_n\} = \{W_n(0, 1; 2, -1)\}$:

$$P_n = \prod_{k=1}^{n-1}\left(2 - 2i\cos\frac{k\pi}{n}\right) , \quad n \geq 2 ,$$

the Pell–Lucas sequence $\{Q_n\} = \{W_n(2, p; 2, -1)\}$:

$$Q_n = \prod_{k=1}^{n}\left(2 - 2i\cos\frac{(2k-1)\pi}{2n}\right) , \quad n \geq 1 ,$$

the Fermat sequence $\{f_n\} = \{W_n(0, 1\,; 3, 2)\}$ (its terms are also known as the Mersenne numbers $M_n = 2^n - 1$):

$$f_n = \prod_{k=1}^{n-1}\left(3 - 2\sqrt{2}\cos\frac{k\pi}{n}\right) , \quad n \geq 2 ,$$

n	U_n	V_n
1	1	p
2	p	$(p-\sqrt{2q})(p+\sqrt{2q})$
3	$(p-\sqrt{q})(p+\sqrt{q})$	$p(p-\sqrt{3q})(p+\sqrt{3q})$
4	$p(p\pm\sqrt{2q})$	$\left(p\pm\sqrt{\delta\, q}\right)\left(p\pm\sqrt{\varepsilon\, q}\right)$
5	$(p\pm\sqrt{\varrho\, q})\left(p\pm\sqrt{(2-\alpha)q}\right)$	$p\left(p\pm\sqrt{\nu\, q}\right)\left(p\pm\sqrt{(3-\alpha)q}\right)$
6	$p(p\pm\sqrt{3q})(p\pm\sqrt{q})$	$\left(p\pm\sqrt{\eta\, q}\right)\left(p\pm\sqrt{2q}\right)\left(p\pm\sqrt{\vartheta\, q}\right)$

Table 1: The factorizations of U_n and V_n for $1 \le n \le 6$

the Fermat–Lucas sequence $\{g_n\} = \{W_n(2,p;3,2)\}$:

$$g_n = \prod_{k=1}^{n}\left(3 - 2\sqrt{2}\cos\frac{(2k-1)\pi}{2n}\right), \quad n \ge 1\,.$$

There are the factorizations of the Fibonacci–type numbers U_n and the Lucas–type numbers V_n for $1 \le n \le 6$ in Table 1. We use the notation $\alpha = \frac{1+\sqrt{5}}{2}$, $\delta = 2+\sqrt{2}$, $\varepsilon = 2-\sqrt{2}$, $\eta = 2+\sqrt{3}$, $\vartheta = 2-\sqrt{3}$, $\varrho = 1+\alpha$, $\nu = 2+\alpha$.

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