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TRANSLATIONS IN SIMPLY TRANSITIVE AFFINE ACTIONS OF FREE 2-STEP NILPOTENT LIE GROUPS

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Abstract: In this paper we study simply transitive affine actions of free 2-step nilpotent Lie groups. We show that for the free 2-step nilpotent Lie group on 3 generators there is always a non-trivial subgroup, which will act as a group of pure translations. We prove that for any odd number of generators ≥ 5, one can find a simply transitive action without translations. Finally, we also show that for an even number of generators, there does not exist a similar kind of action.

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1. Introduction

In 1977, J. Milnor [13] conjectured that any simply connected and connected solvable Lie group does admit a simply transitive affine action. However, in 1992, Y. Benoist [2] found a counter example, even in the nilpotent case (see also Burde et al [3], [4]). Since then, one has been interested in the question of which (nilpotent) Lie groups do admit such an action and in case such an action exists, one wants to classify all of them.

With respect to the classification of all simply transitive affine actions, there

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was another conjecture, due to L. Auslander [1], whose truth implied a possible classification of all simply transitive affine actions of a given nilpotent Lie group in an iterative way, using cohomological techniques. In fact, his conjecture stated that any nilpotent simply transitive group of affine motions must contain a one-parameter group of pure translations in its centre. Unfortunately, also this conjecture was disproved. It was D. Fried [8] who was able to find the first counter example to this conjecture. In fact, he showed that the 3-step nilpotent 4-dimensional connected and simply connected nilpotent Lie group admits a simply transitive affine action where the identity element is the only element acting as a pure translation. On the other hand, in the same paper he also showed that Auslander's conjecture does hold in the case of abelian Lie groups. So in principle, one can classify all simply transitive affine actions of abelian Lie groups, by induction on the dimension. This was carried out in detail up to dimension 5 (Dekimpe et al [7]).

In the case of Lie groups for which Auslander's conjecture does not hold, one has to classify the simply transitive actions without translations separately (and still use the iteration process in the other cases).

It was H. Kim [11] who showed that, up to equivalence, there are exactly two simply transitive affine actions without translations of the 4-dimensional 3-step nilpotent connected and simply connected nilpotent Lie group (in the same paper a complete classification of all simply transitive affine actions of nilpotent Lie groups of dimensions ≤ 4 was obtained).

This example of D. Fried (and H. Kim) of a nilpotent Lie group admitting a simply transitive affine action without translations stayed isolated until 2001, when Medina et al [12] extended the above example to a whole collection of examples of n-dimensional Lie groups which are $(n-1)$-step nilpotent (such Lie groups are said to be filiform), where n is even and $n \geq 4$. They also proved that for all odd dimensional filiform Lie groups the Auslander conjecture does hold.

In Dekimpe et al [6] we have been studying the Lie groups of nilpotency class 2 and obtained that for all 2-step nilpotent Lie groups with a one-dimensional commutator subgroup the Auslander conjecture holds. This result cannot be extended to groups with a higher dimensional commutator subgroup. (We found an example of a 5-dimensional nilpotent Lie group with a 2-dimensional commutator subgroup for which the Auslander conjecture does not hold). Thereafter (Dekimpe et al [5]), we classified all simply transitive affine actions without nontrivial translations of the 5-dimensional nilpotent Lie groups. One of the results obtained in this paper is that the free 3-step nilpotent Lie group on 2 generators (by which we mean that its Lie algebra is free 3-step nilpotent on 2 generators) does not admit such actions, so the Auslander conjecture is valid for this Lie group. Together with the fact that the Auslander conjecture is also valid for abelian Lie groups (free 1-step nilpotent) and the 3-dimensional Heisenberg Lie group (free 2-step nilpotent on 2 generators), one is lead naturally to the idea that the Auslander conjecture could hold for all free nilpotent Lie groups.

We tackle this question in this paper for the free 2-step nilpotent case. In fact, we are able to construct counter examples on any free 2-step nilpotent Lie group with an odd number of generators $k \geq 5$. On the other hand we also prove that the conjecture of Auslander does hold for the free 2-step nilpotent Lie group on 3 generators. Unfortunately, we have not been able to treat completely the case of a free 2-step nilpotent Lie group on an even number of generators. Nevertheless, we can show that a possible counter example in this situation, must be of a considerable different genre than the one we constructed in the case of an odd number of generators.

Although our results concern simply transitive affine actions of Lie groups, we shall prove them at the Lie algebra level. There is a very nice and well known correspondence between such actions of a nilpotent Lie group and the complete left symmetric structures on the corresponding Lie algebra. Simply transitive affine actions without nontrivial translations will correspond to complete left symmetric structures with trivial centre. This will be explained in the next section. For details, we refer the reader to Fried et al [9], [10], Kim [11] and Segal [14].

2. Simply Transitive Actions and Left Symmetric Structures

In studying simply transitive affine actions, it turns out to be comfortable to translate everything to the Lie algebra level. In this section we recall how this can be done. As usual we consider the group $\mathrm{Aff}(\mathbb{R}^n)$ of invertible affine motions, as a subgroup of $\mathrm{GL}_{n+1}(\mathbb{R})$. Given a simply transitive affine action

$$\rho : G \to \mathrm{Aff}(\mathbb{R}^n) : \ g \mapsto \begin{pmatrix} \rho_\ell(g) & \rho_t(g) \\ 0 & 1 \end{pmatrix}$$

of a nilpotent Lie group G with Lie algebra $\mathfrak{g}$, we can consider the differential $d\rho$ of ρ which is a Lie algebra morphism of the form

$$d\rho : \mathfrak{g} \to \mathfrak{aff}(\mathbb{R}^n) : \ g \mapsto \begin{pmatrix} d\rho_\ell(g) & d\rho_t(g) \\ 0 & 0 \end{pmatrix}.$$

We call $d\rho_\ell$ the linear part and $d\rho_t$ the translational part. It is well known that for a nilpotent Lie group G, $d\rho_\ell(\mathfrak{g})$ consists of nilpotent matrices and $d\rho_t : \mathfrak{g} \to \mathbb{R}^n$ is a linear isomorphism. Conversely, for any Lie algebra morphism $\varphi : \mathfrak{g} \to \mathfrak{aff}(\mathbb{R}^n)$, for which the linear part $\varphi_\ell(\mathfrak{g})$ consists of nilpotent matrices and the translational part $\varphi_t : \mathfrak{g} \to \mathbb{R}^n$ is bijective, there exists a simply transitive affine action $\rho : G \to \mathrm{Aff}(\mathbb{R}^n)$, such that $d\rho = \varphi$. Such a φ is called a complete affine structure on $\mathfrak{g}$. As a conclusion, we have that there is a one-to-one correspondence between simply transitive affine actions ρ of a nilpotent Lie group G and complete affine structures $d\rho$ on its Lie algebra $\mathfrak{g}$.

As a second step we can pass from affine structures to the so called left symmetric structures on a Lie algebra.

Definition 2.1. Let $\mathfrak{g}$ be a Lie algebra. A left symmetric structure on $\mathfrak{g}$ consists of a bilinear product $\bullet : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, satisfying:

1. $\forall X, Y \in \mathfrak{g} :\ [X, Y] = X{\bullet}Y - Y{\bullet}X$, and
2. $\forall X, Y, Z \in \mathfrak{g} :\ [X, Y]{\bullet}Z = X{\bullet}(Y{\bullet}Z) - Y{\bullet}(X{\bullet}Z)$.

Moreover, a left symmetric structure is said to be complete if and only if $\forall Y \in \mathfrak{g}$, the map $t_Y : \mathfrak{g} \to \mathfrak{g} :\ X \mapsto X + X{\bullet}Y$ is bijective.

Given a complete affine structure $\varphi : \mathfrak{g} \to \mathfrak{aff}(\mathbb{R}^n)$ on a Lie algebra $\mathfrak{g}$, we can define a left symmetric structure on $\mathfrak{g}$, which turns out also to be complete, by $\forall X, Y \in \mathfrak{g} : X{\bullet}Y = \varphi_t^{-1}(\varphi_\ell(X)\varphi_t(Y))$.

Conversely, any left symmetric structure $\bullet$ on a nilpotent Lie algebra gives rise to an affine representation as follows; after choosing a basis, we identify $\mathfrak{g}$ with $\mathbb{R}^n$ via its coordinate map $\mathrm{co} : \mathfrak{g} \to \mathbb{R}^n :\ X \mapsto \mathrm{co}(X)$ sending each element onto its coordinate with respect to the chosen basis. The affine representation $\varphi : \mathfrak{g} \to \mathfrak{aff}(\mathbb{R}^n)$ is defined by taking

$$\varphi_t = \mathrm{co} \text{ and } \forall X \in \mathfrak{g},\ \forall a \in \mathbb{R}^n : \varphi_\ell(X)a = \mathrm{co}(X{\bullet}\mathrm{co}^{-1}(a)).$$

It turns out that this affine structure is complete if and only if $\bullet$ is a complete left symmetric structure. This explains the one-to-one correspondence between complete affine structures and complete left symmetric structures on a nilpotent Lie algebra.

As indicated in the introduction of this paper, our interest lies in the simply transitive affine actions of a nilpotent Lie group, for which the identity element is the only element acting as a pure translation.

Going through the steps mentioned above, such actions are in one-to-one correspondence with the complete left symmetric structures on the corresponding Lie algebra $\mathfrak{g}$ for which the set $T(\mathfrak{g}) = 0$, where

$$T(\mathfrak{g}) = \{X \in \mathfrak{g} \parallel X{\bullet}Y = 0,\ \forall Y \in \mathfrak{g}\}.$$

In order to be able to present our results, we have to recall some basic facts and notions concerning complete left symmetric structures $\bullet$ on a nilpotent Lie algebra $\mathfrak{g}$.

For any left symmetric structure $\bullet$, we can define the left multiplication map $\lambda : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) : \ X \mapsto \lambda_X$, with $\lambda_X : \mathfrak{g} \to \mathfrak{g} : \ Y \mapsto \lambda_X(Y) = X \bullet Y$ and the right multiplication map $\rho : \rho : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) : \ X \mapsto \rho_X$, with $\rho_X : \mathfrak{g} \to \mathfrak{g} : \ Y \mapsto \rho_X(Y) = Y \bullet X$.

Definition 2.1 implies that λ is a Lie algebra morphism. On the other hand, in general ρ will not be a Lie algebra morphism. H. Kim [11] showed that, in case $\bullet$ is a complete left symmetric structure on $\mathfrak{g}$, λ_X and ρ_X will be nilpotent linear maps for any $X \in \mathfrak{g}$. Also the converse is true, if λ_X is nilpotent for any $X \in \mathfrak{g}$, then the left symmetric structure is complete (Kim [11], Segal [14]).

Associated to any complete left symmetric structure on a nilpotent Lie algebra $\mathfrak{g}$, there are two decreasing sequences of Lie subalgebras of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{g}^1 \supseteq \mathfrak{g}^2 = \mathfrak{g} \cdot \mathfrak{g} \supseteq \mathfrak{g}^3 = \mathfrak{g} \cdot \mathfrak{g}^2 \supseteq \cdots \supseteq \mathfrak{g}^{i+1} = \mathfrak{g} \cdot \mathfrak{g}^i \supseteq \cdots \quad \text{and}$$
$$\mathfrak{g} = \mathfrak{g}_1 \supseteq \mathfrak{g}_2 = \mathfrak{g} \cdot \mathfrak{g} \supseteq \mathfrak{g}_3 = \mathfrak{g}_2 \cdot \mathfrak{g} \supseteq \cdots \supseteq \mathfrak{g}_{i+1} = \mathfrak{g}_i \cdot \mathfrak{g} \supseteq \cdots .$$

For the first of these sequences, we have that, for sufficiently large n, $\mathfrak{g}^n = 0$. This follows from the fact that $\lambda : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is a Lie algebra homomorphism whose image consists of nilpotent endomorphisms. The second sequence need not tend to zero, but nevertheless stabilizes after a finite number of steps at the space $\mathfrak{g}_\infty = \bigcap_{n=1}^{\infty} \mathfrak{g}_n$.

We will denote the centre of $\mathfrak{g}$ as a Lie algebra, by

$$Z(\mathfrak{g}) = \{X \in \mathfrak{g} \parallel [X,Y] = 0, \forall Y \in \mathfrak{g}\} \; = \; \{X \in \mathfrak{g} \parallel X \bullet Y = Y \bullet X, \ \forall Y \in \mathfrak{g}\},$$

while the centre of $\mathfrak{g}$, considered as a left symmetric algebra equals

$$C(\mathfrak{g}) = \{X \in \mathfrak{g} \parallel X \bullet Y = 0 = Y \bullet X, \ \forall Y \in \mathfrak{g}\}.$$

This centre $C(\mathfrak{g})$ is important in the context of this paper, because for a complete left symmetric structure $\bullet$ on a nilpotent Lie algebra $\mathfrak{g}$, one has that

$$T(\mathfrak{g}) = 0 \Leftrightarrow C(\mathfrak{g}) = 0.$$

This is easily seen to follow from the fact that $T(\mathfrak{g})$ is a (Lie algebra) ideal of $\mathfrak{g}$, and hence it has a non-empty intersection with $Z(\mathfrak{g})$, in case $T(\mathfrak{g}) \neq 0$.
Analogously, the centralizer of an element $Y \in \mathfrak{g}$ is defined by

$$C_{\mathfrak{g}}(Y) = \{X \in \mathfrak{g} \parallel [X,Y] = 0\}.$$

In the next proposition and lemma we recall some results obtained by Kim [11].

Proposition 2.2. *Let $\bullet$ be a complete left symmetric structure on a nilpotent Lie algebra $\mathfrak{g}$. Then:*

1. *each $\mathfrak{g}_i$ is a two-sided ideal of $\mathfrak{g}$ (thus $\mathfrak{g}\bullet\mathfrak{g}_i \subseteq \mathfrak{g}_i$ and $\mathfrak{g}_i\bullet\mathfrak{g} \subseteq \mathfrak{g}_i$).*
2. *$\mathfrak{g}_\infty\bullet\mathfrak{g} = \mathfrak{g}_\infty$.*
3. *$\mathfrak{g}_\infty$ is a proper ideal of $\mathfrak{g}$.*
4. *if $\mathfrak{g}_\infty \neq 0$, then $\dim(\mathfrak{g}_\infty) \geq 3$.*
5. *if $T(\mathfrak{g}) = 0$, then $\mathfrak{g}_\infty \neq 0$.*

Lemma 2.3. *Let $\bullet$ be a complete left symmetric structure on a nilpotent Lie algebra $\mathfrak{g}$, then there exists a basis $e_1, \ldots, e_r, e_{r+1}, \ldots, e_n$ of $\mathfrak{g}$ such that $\mathfrak{g}_\infty = \langle e_1, e_2, \ldots, e_r \rangle$, and with respect to this basis, the matrices of $\lambda(\mathfrak{g})$ are of the form*

$$\begin{pmatrix} A_r & * \\ 0 & B_{n-r} \end{pmatrix},$$

and those of $\rho(\mathfrak{g})$ are of the form

$$\begin{pmatrix} C_r & * \\ 0 & D_{n-r} \end{pmatrix},$$

where A_r ($r \times r$ matrix), B_{n-r} and D_{n-r} ($(n-r) \times (n-r)$ matrices) are simultaneously strict upper triangular.

Let V be a vector space, equipped with a filtration of subspaces

$$0 = V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n = V,$$

where the dimension of $V_i/V_{i-1} = k_i$ ($1 \leq i \leq n$). We say that a basis $v_1, v_2, \ldots, v_k$ (with $k = k_1 + k_2 + \cdots + k_n$) is compatible with the given filtration, if V_i is spanned by $v_1, v_2, \ldots, v_{k_1+k_2+\cdots+k_i}$, for each i.

To obtain the basis $e_1, e_2, \ldots, e_n$ referred to in Lemma 2.3, one chooses the subbasis $e_1, \ldots, e_r$ for $\mathfrak{g}_\infty$ to be compatible with the filtration of $\mathfrak{g}_\infty$ induced by left multiplication:

$$0 \subset \ldots \subset \mathfrak{g}\bullet(\mathfrak{g}\bullet\mathfrak{g}_\infty) \subset \mathfrak{g}\bullet\mathfrak{g}_\infty \subset \mathfrak{g}_\infty,$$

and the rest of the vectors $e_{r+1}, \ldots, e_n$ in such a way that their natural projections $\overline{e_{r+1}}, \ldots, \overline{e_n}$ in $\mathfrak{g}/\mathfrak{g}_\infty$ form a basis which is compatible with the filtration of $\mathfrak{g}/\mathfrak{g}_\infty$ induced by right multiplication:

$$0 \subset \ldots (\mathfrak{g}\bullet\mathfrak{g})\bullet\mathfrak{g}/\mathfrak{g}_\infty \subset \mathfrak{g}\bullet\mathfrak{g}/\mathfrak{g}_\infty \subset \mathfrak{g}/\mathfrak{g}_\infty.$$

Although there might exist other bases for which Lemma 2.3 holds, we will in this paper always assume that $e_1, e_2, \ldots, e_n$ are chosen in the way mentioned above and we will call such a basis an *adapted basis*. So, for an adapted basis Lemma 2.3 is valid.

Remark 2.4. Note that for an adapted basis, $X{\bullet}e_1 = 0$, for all $X \in \mathfrak{g}$.

As an immediate consequence of the definition of an adapted basis, we find the following lemma, the proof of which is left to the reader.

Lemma 2.5. *Let $\bullet$ be a complete left symmetric structure on a nilpotent Lie algebra $\mathfrak{g}$, and assume that $e_1, e_2, \ldots, e_n$ is an adapted basis for $\mathfrak{g}$, with $\mathfrak{g}_\infty = \langle e_1, \ldots, e_r \rangle$. Then, for any $k > 0$ we have that $\langle e_1, e_2, \ldots, e_{r+k} \rangle$ is an ideal of $\mathfrak{g}$ (as a left symmetric algebra).*

The following lemma presents results obtained in Dekimpe et al [6] and [5].

Lemma 2.6. *Let $\bullet$ be a complete left symmetric structure on a nilpotent Lie algebra $\mathfrak{g}$, then:*

1. $[\mathfrak{g}, \mathfrak{g}] \not\subseteq \mathfrak{g}{\bullet}\mathfrak{g}_\infty$.

2. If $T(\mathfrak{g}) = 0$, then $\mathfrak{g}_\infty \not\subseteq Z(\mathfrak{g})$. In fact, if $e_1, \ldots, e_n$ is an adapted basis, then $e_1 \notin Z(\mathfrak{g})$.

To finish this section, we prove a few more small results, which we will need later on. Let $\mathfrak{g}$ be a nilpotent Lie algebra, and define

$$T_1 = \{t \in C_{\mathfrak{g}}(e_1) \parallel t{\bullet}A = 0, \forall A \in C_{\mathfrak{g}}(e_1)\}.$$

Lemma 2.7. *Let $\bullet$ be a complete left symmetric structure on $\mathfrak{g}$, with $C_{\mathfrak{g}}(e_1)$ abelian and a right ideal of $\mathfrak{g}$, then T_1 is a right ideal of $\mathfrak{g}$, containing e_1.*

Proof. By definition of e_1, we know that $e_1{\bullet}C_{\mathfrak{g}}(e_1) = [e_1, C_{\mathfrak{g}}(e_1)] = 0$, so $e_1 \in T_1$. Let t be an element of T_1, $A \in C_{\mathfrak{g}}(e_1)$ and $X \in \mathfrak{g}$, then

$$(t{\bullet}X){\bullet}A = A{\bullet}(t{\bullet}X) = [A, t]{\bullet}X + t{\bullet}(A{\bullet}X) = 0,$$

where we used the fact that $C_{\mathfrak{g}}(e_1)$ is abelian and a right ideal of $\mathfrak{g}$. Hence $T_1{\bullet}\mathfrak{g} \subseteq T_1$ or T_1 is a right ideal of $\mathfrak{g}$. □

Corollary 2.8. *Let $\bullet$ be a complete left symmetric structure on $\mathfrak{g}$ with $C_{\mathfrak{g}}(e_1) \subseteq \langle e_1, Z(\mathfrak{g}) \rangle$ and a right ideal of $\mathfrak{g}$, then T_1 is an ideal of $\mathfrak{g}$.*

Proof. Let t be an element of T_1, then $t = \alpha e_1 + z$ with $z \in Z(\mathfrak{g}) \cap T_1$, $\alpha \in \mathbb{R}$. Then for any $X \in \mathfrak{g}$ we have

$$X{\bullet}t = X{\bullet}(\alpha e_1 + z) = z{\bullet}X \in T_1{\bullet}\mathfrak{g} \subseteq T_1.$$

This follows from the fact that T_1 is a right ideal of $\mathfrak{g}$ (Lemma 2.7). So we have $\mathfrak{g}{\bullet}T_1 \subseteq T_1$. □

Lemma 2.9. *Let $\bullet$ be a complete left symmetric structure on $\mathfrak{g}$, then* $C_{\mathfrak{g}}(e_1)\bullet C_{\mathfrak{g}}(e_1) \subseteq C_{\mathfrak{g}}(e_1)$.

Proof. Take $A \in C_{\mathfrak{g}}(e_1)$ and $B \in C_{\mathfrak{g}}(e_1)$, then $e_1\bullet(A\bullet B) = [e_1, A]\bullet B + A\bullet(e_1\bullet B) = 0$, thus $A\bullet B \in C_{\mathfrak{g}}(e_1)$. □

3. Free Two-Step Nilpotent Lie Algebras

From now onwards, we will restrict our attention to free 2-step nilpotent Lie algebras. The free 2-step nilpotent Lie algebra generated by k elements will be denoted by $\mathfrak{g}_k$.

As $\mathfrak{g}_k$ is free 2-step nilpotent on k generators, $\mathfrak{g}_k/[\mathfrak{g}_k, \mathfrak{g}_k]$ is a k-dimensional vector space. For any choice of elements $x_1, x_2, \ldots, x_k \in \mathfrak{g}_k$, such that their images under the natural projection $p : \mathfrak{g}_k \to \mathfrak{g}_k/[\mathfrak{g}_k, \mathfrak{g}_k]$ form a basis of $\mathfrak{g}_k/[\mathfrak{g}_k, \mathfrak{g}_k]$, we have that $x_1, x_2, \ldots, x_k \in \mathfrak{g}_k$ generate $\mathfrak{g}_k$ and the elements $y_{i,j} = [x_i, x_j]$ $(1 \leq i < j \leq k)$ form a basis of $[\mathfrak{g}_k, \mathfrak{g}_k] = Z(\mathfrak{g}_k)$.

For these Lie algebras $\mathfrak{g}_k$ we want to know if they admit complete left symmetric structures with $T(\mathfrak{g}_k) = 0$. Recall that in this case, Lemma 2.6 implies that $e_1 \notin Z(\mathfrak{g}_k)$ and hence, we can fix a set of generators $x_1, x_2, \ldots, x_k$ of $\mathfrak{g}_k$, with $x_1 = e_1$. Having fixed such a set of generators, we obtain a vector space decomposition of $\mathfrak{g}_k$ of the form

$$\mathfrak{g}_k = \langle e_1\rangle \oplus \langle x_2, \ldots, x_k\rangle \oplus Z(\mathfrak{g}_k).$$

In the sequel, we will use the notation $W = \langle x_2, \ldots, x_k\rangle$. Note that W depends on the chosen set of generators and can in fact be any subspace complementary to $\langle e_1\rangle \oplus Z(\mathfrak{g}_k)$.

Note that $C_{\mathfrak{g}_k}(e_1) = \langle e_1\rangle \oplus Z(\mathfrak{g}_k) = \langle e_1\rangle \oplus [\mathfrak{g}_k, \mathfrak{g}_k]$ is abelian.

4. Free Two-Step Nilpotent Lie Groups not Admitting Simply Transitive Affine Actions without Translations

The free two-step nilpotent Lie algebra on 2 generators $\mathfrak{g}_2$ is isomorphic to the 3-dimensional Heisenberg Lie algebra, so for this Lie algebra it was already known that for any complete left symmetric structure on $\mathfrak{g}_2$, $T(\mathfrak{g}_2) \neq 0$ (Kim [11], Dekimpe et al [6]).

In this section we will prove that also the free two-step nilpotent Lie algebra on 3 generators $\mathfrak{g}_3$ does not admit complete left symmetric structures with

$T(\mathfrak{g}_3) = 0$. In the course of proving this we will also show that in the case k is even, then for any complete left symmetric structure with $\mathfrak{g}_k \bullet \mathfrak{g}_k$ abelian, $T(\mathfrak{g}_k) \neq 0$. This is in big contrast with the situation in which k is odd as we will see in the following section.

Lemma 4.10. *Let $\bullet$ be a complete left symmetric structure on $\mathfrak{g}_k$, with $T(\mathfrak{g}_k) = 0$ and $\mathfrak{g}_k \bullet \mathfrak{g}_k$ abelian then:*

1. $C_{\mathfrak{g}_k}(e_1) = \mathfrak{g}_k \bullet \mathfrak{g}_k$ is an ideal of $\mathfrak{g}_k$.

2. for $t \in T_1 \cap Z(\mathfrak{g}_k)$ and $X \in \mathfrak{g}_k$, if $t \bullet X \in Z(\mathfrak{g}_k)$, then $t \bullet X = 0$.

Proof. 1. By definition of e_1, we know that $e_1 \in \mathfrak{g}_{k,\infty} \subseteq \mathfrak{g}_k \bullet \mathfrak{g}_k$. From the fact that $\mathfrak{g}_k \bullet \mathfrak{g}_k$ is abelian, it follows that $\mathfrak{g}_k \bullet \mathfrak{g}_k \subseteq C_{\mathfrak{g}_k}(e_1)$. On the other hand, we have $C_{\mathfrak{g}_k}(e_1) = \langle e_1, [\mathfrak{g}_k, \mathfrak{g}_k] \rangle \subseteq \mathfrak{g}_k \bullet \mathfrak{g}_k$. So, $C_{\mathfrak{g}_k}(e_1) = \mathfrak{g}_k \bullet \mathfrak{g}_k$ and this proves the first statement.

2. Assume $t \bullet X \in Z(\mathfrak{g}_k)$, then $\forall Y \in \mathfrak{g}_k$ we have

$$(t \bullet X) \bullet Y = Y \bullet (t \bullet X) = [Y, t] \bullet X + t \bullet (Y \bullet X) = 0,$$

where we used 1. Therefore we find that $t \bullet X$ belongs to $T(\mathfrak{g}_k)$. This is only possible if $t \bullet X = 0$. □

Corollary 4.11. *For any complete left symmetric structure on $\mathfrak{g}_k$, with $T(\mathfrak{g}_k) = 0$ and $\mathfrak{g}_k \bullet \mathfrak{g}_k$ abelian, we have that T_1 is an ideal of $\mathfrak{g}_k$.*

Proof. This follows immediately from Lemma 4.10 and Corollary 2.8. □

Corollary 4.12. *Let $\bullet$ be a complete left symmetric structure on $\mathfrak{g}_k$, with $T(\mathfrak{g}_k) = 0$ and $\mathfrak{g}_k \bullet \mathfrak{g}_k$ abelian. Let $t, t' \in T_1 \cap Z(\mathfrak{g}_k)$ and $X, X' \in \mathfrak{g}_k$, then:*

1. $t \bullet X \sim t \bullet X'$,

2. $t \bullet X \sim t' \bullet X$.

(with $A \sim B$, we mean that the vectors A, B are linearly depended)

Proof. 1. We can assume that $t \bullet X \neq 0 \neq t \bullet X'$. By Lemma 4.10 we know that $t \bullet X, t \bullet X' \in C_{\mathfrak{g}_k}(e_1)$ and $t \bullet X, t \bullet X' \notin Z(\mathfrak{g}_k)$. Therefore there exist $\alpha, \beta \in \mathbb{R}_0$, $z, z' \in Z(\mathfrak{g}_k)$, such that $t \bullet X = \alpha e_1 + z$ and $t \bullet X' = \beta e_1 + z'$. This implies that $t \bullet (\beta X - \alpha X')$ belongs to the centre $Z(\mathfrak{g}_k)$. By Lemma 4.10 we find that $t \bullet (\beta X - \alpha X') = 0$. Hence $t \bullet X \sim t \bullet X'$.

2. The second claim is proved in an analogous way. □

We keep focusing on a complete left symmetric structure on $\mathfrak{g}_k$ with $T(\mathfrak{g}_k) = 0$ and $\mathfrak{g}_k \bullet \mathfrak{g}_k$ abelian, and take any element $w_2 \in W$. We define $t_2 = [e_1, w_2]$. This commutator is not zero and belongs to $T_1 \cap Z(\mathfrak{g}_k)$. This follows from the fact that $e_1 \in T_1$ and T_1 is an ideal of $\mathfrak{g}_k$. That means $t_2{\bullet}(\mathfrak{g}_k{\bullet}\mathfrak{g}_k) = 0$. The following computation shows that also the product $t_2{\bullet}w_2 = 0$:

$$t_2{\bullet}w_2 = [e_1, w_2]{\bullet}w_2 = e_1{\bullet}(w_2{\bullet}w_2) - w_2{\bullet}(e_1{\bullet}w_2)$$
$$= [e_1, w_2{\bullet}w_2] - t_2{\bullet}w_2 \Rightarrow 2t_2{\bullet}w_2 = 0\,,$$

where we used that $\mathfrak{g}_k{\bullet}\mathfrak{g}_k = C_{\mathfrak{g}_k}(e_1)$. This implies that, since $T(\mathfrak{g}_k) = 0$, there must exist an element $w \in W, w \notin \langle w_2 \rangle$ such that $t_2{\bullet}w \neq 0$. From Lemma 4.10 it follows that $t_2{\bullet}w$ doesn't belong to $Z(\mathfrak{g}_k)$. So, we can assume that $t_2{\bullet}w = e_1 + z_2$ for some $z_2 \in Z(\mathfrak{g}_k)$. And finally, by Corollary 4.12 we can find a basis $w_2, w_3, \ldots, w_{k-1}, w_k = w$ for W such that $t_2{\bullet}w_i = 0$, $\forall i \in \{2, \ldots, k-1\}$ and $t_2{\bullet}w_k = e_1 + z_2$ (we already knew $t_2{\bullet}(\mathfrak{g}_k{\bullet}\mathfrak{g}_k) = 0$.) Let us now define $t_i = [e_1, w_i], \quad 2 \leq i \leq k$.

For any i, t_i has following properties:

— $t_i \in T_1 \cap Z(\mathfrak{g}_k)$,

— $t_i{\bullet}w_i = 0$,

— $t_i{\bullet}w_j = -t_j{\bullet}w_i, \quad \forall j \in \{2, \ldots, k\}, j \neq i$.

The last two proporties follow from this computation:

$$t_i{\bullet}w_j = [e_1, w_i]{\bullet}w_j = [e_1, w_i{\bullet}w_j] - w_i{\bullet}[e_1, w_j] = -t_j{\bullet}w_i.$$

Theorem 4.13. *Let $\mathfrak{g}_k$ be the free two-step nilpotent Lie algebra on k generators, with k even. Then for any complete left symmetric structure on $\mathfrak{g}_k$, with $\mathfrak{g}_k{\bullet}\mathfrak{g}_k$ abelian, we have that $T(\mathfrak{g}_k) \neq 0$.*

Proof. Assume that $\bullet$ is a complete left symmetric structure on $\mathfrak{g}_k$, with $\mathfrak{g}_k{\bullet}\mathfrak{g}_k$ abelian and for which $T(\mathfrak{g}_k) = 0$. Above we have constructed a basis $w_2, w_3, \ldots, w_k$ for W and elements $t_i = [e_1, w_i] \in T_1 \cap Z(\mathfrak{g}_k)$, $(2 \leq i \leq k)$ satisfying

$$t_i{\bullet}w_j = -t_j{\bullet}w_i,\ \forall i \in \{2, 3, \ldots, k\}. \tag{1}$$

Note that $t_i{\bullet}(\mathfrak{g}_k{\bullet}\mathfrak{g}_k) = 0$.

We shall now prove that we can also assume that our basis satisfies the following assertion for any $j \in \{2, \ldots, \frac{k}{2} + 1\}$ and $i \in \{2, \ldots, k\}$:

$$t_j{\bullet}w_i = \alpha_j(e_1 + z_j) \text{ with } \begin{array}{lll} \alpha_j = 0 & \text{if} & j \neq k+2-i \\ \alpha_j \neq 0 & \text{if} & j = k+2-i \end{array} \quad \text{and} \quad z_j \in Z(\mathfrak{g}_k).$$

We will proceed by induction on j to adjust the original basis of W to a basis satisfying also these extra conditions.

For the element t_2, we already know that $t_2{\bullet}\langle w_2, w_3, \ldots, w_{k-1} \rangle = 0$ and $t_2{\bullet}w_k = e_1 + z_2$, so $\alpha_2 = 1$.

Let us now assume that we have a basis $w_2, w_3, \ldots, w_k$ satisfying (1) and such that the assertion is valid till $j-1$. We will now adjust the basis so that the statement also holds for j. We know that $t_j \bullet w_i = -t_i \bullet w_j$. For $i < j$, we can use the induction hypothesis. Hence, because $j \leq \frac{k}{2} + 1$ and $i < j$ we find $t_i \bullet w_j = 0$ and $t_j \bullet w_i = 0$. When $i = j$ we know that $t_i \bullet w_i = 0$. Finally, we take a look at the case $i > j$. As $T(\mathfrak{g}_k) = 0$ and $t_j \bullet \langle \mathfrak{g}_k \bullet \mathfrak{g}_k, w_2, \ldots, w_j \rangle = 0$, there exists at least one element w_l with $j < l \leq k$ such that $t_j \bullet w_l \neq 0$. Define m as the minimum of integers l for which that holds. Let us now distinguish two cases.

Case 1. $m > k + 2 - j$. For any product $t_j \bullet w_l$, with $l \geq m$ there are two possibilities; or the product is zero, or the product does not belong to the centre $Z(\mathfrak{g}_k)$. For the second case, we have $t_j \bullet w_l = \alpha_{j,l} e_1 + z$, for some $z \in Z(\mathfrak{g}_k)$ and $\alpha_{j,l} \neq 0$. By the induction hypothesis we know that $t_{k+2-l} \bullet w_l = \alpha_{k+2-l}(e_1 + z_{k+2-l})$ with $\alpha_{k+2-l} \neq 0$ (remark $k+2-l < j$) So, $t_j \bullet w_l \sim t_{k+2-l} \bullet w_l$ (Corollary 4.12). Thus for any $l \geq m$ we have $t_j \bullet w_l = \alpha_{j,l}(e_1 + z_{k+2-l})$ (where some of the $\alpha_{j,l}$ can be 0). Define

$$t = t_j - \sum_{l=m}^{k} \frac{\alpha_{j,l}}{\alpha_{k+2-l}} t_{k+2-l}.$$

It is easy to see that t is not zero and $t \in T_1$. So, $t \bullet (\mathfrak{g}_k \bullet \mathfrak{g}_k) = 0$. By definition of m, it follows that $t \bullet \langle w_2, \ldots, w_{m-1} \rangle = 0$. By construction of t, we have that $t \bullet \langle w_m, \ldots, w_k \rangle = 0$. Thus, $t \in T(\mathfrak{g}_k)$, which is a contradiction. It follows that this case cannot happen.

Case 2. $m \leq k + 2 - j$. By Lemma 4.10, it follows that we can write $t_j \bullet w_m = \alpha_j(e_1 + z_j)$ with $\alpha_j \neq 0$ and $z_j \in Z(\mathfrak{g}_k)$. For any element $l > m$ with $t_j \bullet w_l \neq 0$, we know by Corollary 4.12 that $t_j \bullet w_m \sim t_j \bullet w_l$. Hence we can take a change of basis of W (where we replace w_l by adding a suitable multiple of w_m to w_l) such that $t_j \bullet w_l = 0$ (remark that t_l is now defined as $[e_j, w_l]$, with w_l the new basisvector). Note this operation does not have any effect on the value of $t_i \bullet w_l$ for all $i < j$, because $t_i \bullet w_m = 0$.

So, the only product of the form $t_j \bullet w_l$ that is not zero, we get for $l = m$. Now, we take another change of basis such that w_m becomes w_{k+2-j} and w_{k+2-j} becomes w_m. Hence we find $t_j \bullet w_{k+2-j} = \alpha_j(e_1 + z_j)$ with $\alpha_j \neq 0$ and $z_j \in Z(\mathfrak{g}_k)$. Again, this last operation does not have any effect on the values of $t_i \bullet w_r$, for $i < j$, since all products $t_i \bullet w_m = 0$ and $t_i \bullet w_{k+2-j} = 0$. This finishes the proof concerning the assertion on the possible choice of a basis for W.

For $j = \frac{k}{2} + 1$ this leads to a contradiction, namely we know that $t_j \bullet w_j = 0$ on the one hand and on the other hand, by the assertion we also have $t_j \bullet w_{k+2-j} = t_j \bullet w_{\frac{k}{2}+1} = t_j \bullet w_j = \alpha_j(e_1 + z_j)$ with $\alpha_j \neq 0$. This proves the theorem. □

We now investigate the case $k = 3$. $\mathfrak{g}_3$ is a 6-dimensional two-step nilpotent Lie algebra, with 3 generators and a 3-dimensional centre. We assume that $\bullet$ is a complete left symmetric structure on $\mathfrak{g}_3$ with $T(\mathfrak{g}_3) = 0$ and we take e_1 as one of the generators of $\mathfrak{g}_3$ as before.

If $\mathfrak{g}_3 \bullet \mathfrak{g}_3$ is not abelian, then $C_{\mathfrak{g}_3}(e_1) \subsetneq \mathfrak{g}_3 \bullet \mathfrak{g}_3 \subsetneq \mathfrak{g}_3$. It follows that $\mathfrak{g}_3 \bullet \mathfrak{g}_3$ is 5-dimensional. By the construction of the adapted basis, we know that $\mathfrak{g}_3 \bullet \mathfrak{g}_3 = \langle e_1, e_2, e_3, e_4, e_5 \rangle$. Let j be the smallest integer such that $e_j \notin C_{\mathfrak{g}_3}(e_1)$, then $j \geq 2$. We choose the second generator of $\mathfrak{g}_3$ to be $w_2 = e_j$. So, we have that $\mathfrak{g}_3 \bullet \mathfrak{g}_3 = \langle e_1, w_2, Z(\mathfrak{g}_3) \rangle$. Note that we can adjust the other basis vectors e_i $(2 \leq i \leq 5,\ i \neq j)$ to obtain that for these values of i, $e_i \in Z(\mathfrak{g}_3)$.

Fix a third generator w_3 of $\mathfrak{g}_3$ by choosing any vector outside $\mathfrak{g}_3 \bullet \mathfrak{g}_3$ (e.g. we can take $w_3 = e_6$), and define $t_i = [e_1, w_i]$ for $i = 2$ and $i = 3$ and $t_{2,3} = [w_2, w_3]$. Hence $\mathfrak{g}_3 \bullet \mathfrak{g}_3 = \langle e_1, t_2, t_3, t_{2,3}, w_2 \rangle$.

Lemma 4.14. *Let $\bullet$ be a complete left symmetric structure on $\mathfrak{g}_3$, with $T(\mathfrak{g}_3) = 0$ and $\mathfrak{g}_3 \bullet \mathfrak{g}_3$ not abelian, then $\mathfrak{g}_{3,\infty}$ is abelian.*

Proof. Assume $\mathfrak{g}_{3,\infty}$ is not abelian, then $\mathfrak{g}_{3,\infty} = \langle e_1, w_2, t_2, t_3, t_{2,3} \rangle = \mathfrak{g}_3 \bullet \mathfrak{g}_3 = \langle e_1, e_2, e_3, e_4, e_5 \rangle$, where the adapted basis is chosen as explained just above the statement of this lemma, i.e. $\exists j \geq 2$ with $e_j = w_2$ and $e_i \in Z(\mathfrak{g}_3)$ for $2 \leq j \leq 5, i \neq j$.

j cannot be 2, otherwise we would have that $t_2 = e_1 \bullet e_2 \in \langle e_1 \rangle$, which is impossible since $e_1 \notin Z(\mathfrak{g}_3)$.

Now we want to prove that $\langle e_1, w_2, t_2, t_3, t_{2,3} \rangle \bullet \mathfrak{g}_3 \subseteq C_{\mathfrak{g}_3}(e_1)$. This will lead to a contradiction. By construction of the adapted basis we find $e_1 \bullet \mathfrak{g}_3 = [e_1, \mathfrak{g}_3] \subseteq C_{\mathfrak{g}_3}(e_1)$ and for any $r \geq 1$, ${\mathfrak{g}_3}^r(w_2) = \underbrace{\mathfrak{g}_3 \bullet (\mathfrak{g}_3 \bullet (\dots (\mathfrak{g}_3}_{r \text{ times}} \bullet w_2))) \in C_{\mathfrak{g}_3}(e_1)$. Because $w_2 \bullet \mathfrak{g}_3 \subseteq \mathfrak{g}_3 \bullet w_2 + [w_2, \mathfrak{g}_3]$, we also have $w_2 \bullet \mathfrak{g}_3 \subseteq C_{\mathfrak{g}_3}(e_1)$. For $i, l \in \{2, 3\}$ we obtain

$$t_i \bullet w_l = [e_1, w_i] \bullet w_l = e_1 \bullet (w_i \bullet w_l) - w_i \bullet (e_1 \bullet w_l) \Rightarrow t_i \bullet w_l + w_i \bullet t_l = [e_1, w_i \bullet w_l],$$

and hence $t_i \bullet w_l + t_l \bullet w_i \in C_{\mathfrak{g}_3}(e_1)$. This, together with Lemma 2.9 (implying that $t_i \bullet C_{\mathfrak{g}_3}(e_1) \subseteq C_{\mathfrak{g}_3}(e_1)$) and the fact that $\mathfrak{g}_3 \bullet w_2 \in C_{\mathfrak{g}_3}(e_1)$, we see that $t_2 \bullet \mathfrak{g}_3 \subseteq C_{\mathfrak{g}_3}(e_1)$ and $t_3 \bullet \mathfrak{g}_3 \subseteq C_{\mathfrak{g}_3}(e_1)$. Finally we take a look at the product $t_{2,3} \bullet \mathfrak{g}_3$. We already know that $t_{2,3} \bullet \langle C_{\mathfrak{g}_3}(e_1), w_2 \rangle \subseteq C_{\mathfrak{g}_3}(e_1)$. Let us now compute the product $t_{2,3} \bullet w_3$:

$$\begin{aligned} t_{2,3} \bullet w_3 = [w_2, w_3] \bullet w_3 &= w_2 \bullet (w_3 \bullet w_3) - w_3 \bullet (w_2 \bullet w_3) \\ &= w_2 \bullet (w_3 \bullet w_3) - w_3 \bullet (w_3 \bullet w_2 + [w_2 \bullet w_3]) = c - t_{2,3} \bullet w_3, \end{aligned}$$

with $c \in C_{\mathfrak{g}_3}(e_1)$. So, $t_{2,3}{\bullet}w_3 \in C_{\mathfrak{g}_3}(e_1)$ and $t_{2,3}{\bullet}\mathfrak{g}_3 \subseteq C_{\mathfrak{g}_3}(e_1)$. Hence we have proven that $\mathfrak{g}_{3,\infty}{\bullet}\mathfrak{g}_3 = \langle e_1, w_2, t_2, t_3, t_{2,3}\rangle{\bullet}\mathfrak{g}_3 \subseteq C_{\mathfrak{g}_3}(e_1) \subsetneq \mathfrak{g}_{3,\infty}$. This contradicts the definition of $\mathfrak{g}_{3,\infty}$. □

Lemma 4.15. *Let • be a complete left symmetric structure on $\mathfrak{g}_3$, with $T(\mathfrak{g}_3) = 0$, then $C_{\mathfrak{g}_3}(e_1) = \langle e_1, e_2, e_3, e_4\rangle$ and is an ideal of $\mathfrak{g}_3$.*

Proof. The case $\mathfrak{g}_3{\bullet}\mathfrak{g}_3$ abelian is already proven by Lemma 4.10.

If $\mathfrak{g}_3{\bullet}\mathfrak{g}_3$ is not abelian, $\mathfrak{g}_3{\bullet}\mathfrak{g}_3 = \langle e_1, t_2, t_3, t_{2,3}, w_2\rangle$. Because $\mathfrak{g}_{3,\infty} \subseteq C_{\mathfrak{g}_3}(e_1)$ (Lemma 4.14) there are two possibilities, or $\mathfrak{g}_{3,\infty} = C_{\mathfrak{g}_3}(e_1)$ or $\mathfrak{g}_{3,\infty} \subsetneq C_{\mathfrak{g}_3}(e_1)$. When $\mathfrak{g}_{3,\infty} = C_{\mathfrak{g}_3}(e_1)$ there is nothing to prove. When $\mathfrak{g}_{3,\infty} \subsetneq C_{\mathfrak{g}_3}(e_1)$, by construction of the adapted basis, we know that $\mathfrak{g}_{3,\infty} = \langle e_1, t_2, t_3\rangle = \langle e_1, e_2, e_3\rangle$. On the other hand, because $w_2 \in \mathfrak{g}_3{\bullet}\mathfrak{g}_3$, we have $t_{2,3} = [w_2, w_3] \in (\mathfrak{g}_3{\bullet}\mathfrak{g}_3){\bullet}\mathfrak{g}_3 - \mathfrak{g}_3{\bullet}(\mathfrak{g}_3{\bullet}\mathfrak{g}_3)$. By $\mathfrak{g}_{3,\infty} \subsetneq \mathfrak{g}_3{\bullet}\mathfrak{g}_3$, $(\mathfrak{g}_3{\bullet}\mathfrak{g}_3){\bullet}\mathfrak{g}_3 \subsetneq \mathfrak{g}_3{\bullet}\mathfrak{g}_3$ and $(\mathfrak{g}_3{\bullet}\mathfrak{g}_3){\bullet}\mathfrak{g}_3 \subseteq \langle e_1, \dots, e_4\rangle$. Also $\mathfrak{g}_3{\bullet}(\mathfrak{g}_3{\bullet}\mathfrak{g}_3) \subseteq \mathfrak{g}_3{\bullet}\langle e_1, \dots, e_5\rangle \subseteq \langle e_1, \dots, e_4\rangle$. Hence $t_{2,3} \in \langle e_1, \dots, e_4\rangle$ and therefore $C_{\mathfrak{g}_3}(e_1) = \langle e_1, \dots, e_4\rangle$. It follows from Lemma 2.5 that $C_{\mathfrak{g}_3}(e_1)$ is an ideal of $\mathfrak{g}$. □

Corollary 4.16. *For any complete left symmetric structure on $\mathfrak{g}_3$, with $T(\mathfrak{g}_3) = 0$, we have that T_1 is an ideal of $\mathfrak{g}_k$.*

Proof. Immediate consequence of Lemma 4.15 and Corollary 2.8. □

Theorem 4.17. *Let $\mathfrak{g}_3$ be the free two-step nilpotent Lie algebra on 3 generators. Then for any complete left symmetric structure on $\mathfrak{g}_3$, we have that $T(\mathfrak{g}_3) \neq 0$.*

Proof. Assume that $T(\mathfrak{g}_3) = 0$. As before, we write $\mathfrak{g}_3$ as a direct sum of vector spaces $\mathfrak{g}_3 = \langle e_1\rangle \oplus \langle w_2, w_3\rangle \oplus Z(\mathfrak{g}_3)$. In case $\mathfrak{g}_3{\bullet}\mathfrak{g}_3$ is not abelian, we choose w_2 as explained just before Lemma 4.14. Let us distinguish two cases (where in both cases we keep the notation $t_2 = [e_1, w_2]$, $t_3 = [e_1, w_3]$ and $t_{2,3} = [w_2, w_3]$):

Case 1. $\mathfrak{g}_3{\bullet}\mathfrak{g}_3$ is abelian. We know that in this case $\mathfrak{g}_3{\bullet}\mathfrak{g}_3 = C_{\mathfrak{g}_3}(e_1) = \langle e_1, t_2, t_3, t_{2,3}\rangle$ (Lemma 4.10) and by definition of this basis, it follows (as explained before Theorem 4.13) that $t_2{\bullet}w_2 = 0, t_2{\bullet}w_3 = e_1 + z = -t_3{\bullet}w_2$ (for some $z \in Z(\mathfrak{g}_3)$) and $t_3{\bullet}w_3 = 0$.

Define $T_2 = \langle e_1, t_2, t_3\rangle$, so $T_2 \subseteq T_1$ (Corollary 4.11).

We want to show that $t_{2,3}{\bullet}t_{2,3} = 0$.

Assume that $t_{2,3}{\bullet}t_{2,3} \neq 0$, then $t_{2,3} \notin T_1$, which implies that $T_2 = T_1$ is an ideal of $\mathfrak{g}_3$. We claim that in this case $t_2{\bullet}w_3 = e_1$ (i.e. the z-factor is 0). Indeed, as $T_1 = T_2 = \langle e_1, t_2, t_3\rangle$ is an ideal, we find must have that $t_2{\bullet}w_3 = e_1+\alpha t_2+\beta t_3$, for some $\alpha, \beta \in \mathbb{R}$. Let us compute:

$$\begin{aligned}
\lambda_{w_3}(t_2{\bullet}w_3) &= \alpha t_2{\bullet}w_3 = \alpha e_1+\alpha^2 t_2+\alpha\beta t_3\,,\\
\lambda^2_{w_3}(t_2{\bullet}w_3) &= \alpha^2 t_2{\bullet}w_3\,,\\
&\vdots\\
\lambda^n_{w_3}(t_2{\bullet}w_3) &= \alpha^n t_2{\bullet}w_3.
\end{aligned}$$

As $\lambda^n_{w_3} = 0$ for n big enough, we must have that $\alpha = 0$. In a similar way one shows that $\beta = 0$, and hence $t_2{\bullet}w_3 = e_1$.

The following computation

$$\begin{aligned}
\mathfrak{g}_3{\bullet}T_2 = \mathfrak{g}_3{\bullet}\langle e_1, t_2, t_3\rangle = \mathfrak{g}_3{\bullet}\langle t_2, t_3\rangle = \langle t_2, t_3\rangle{\bullet}\mathfrak{g}_3\\
= \langle t_2, t_3\rangle{\bullet}\langle C_{\mathfrak{g}_3}(e_1), w_2, w_3\rangle = \langle t_2, t_3\rangle{\bullet}\langle w_2, w_3\rangle
\end{aligned}$$

shows that

$$\mathfrak{g}_3{\bullet}T_2 \subseteq \langle e_1\rangle \tag{2}$$

and because $t_{2,3} \in Z(\mathfrak{g}_3)$, we have that

$$t_{2,3}{\bullet}T_2 = T_2{\bullet}t_{2,3} \subseteq T_1{\bullet}C_{\mathfrak{g}_3}(e_1) = 0. \tag{3}$$

Let us define $w_2{\bullet}w_2 = t' + \alpha t_{2,3}$, $w_2{\bullet}w_3 = t'' + \beta t_{2,3}$, $w_3{\bullet}w_3 = t''' + \gamma t_{2,3}$ with $t', t'', t''' \in T_2$ and $\alpha, \beta, \gamma \in \mathbb{R}$. From $[w_3, w_2]{\bullet}w_2 = w_3{\bullet}(w_2{\bullet}w_2) - w_2{\bullet}(w_3{\bullet}w_2)$ and $[w_3, w_2]{\bullet}w_3 = w_3{\bullet}(w_2{\bullet}w_3) - w_2{\bullet}(w_3{\bullet}w_3)$, it follows that

$$f_1 = (\beta - 2)t_{2,3}{\bullet}w_2 - \alpha t_{2,3}{\bullet}w_3 \in \langle e_1\rangle\,, \tag{4}$$

$$f_2 = (\beta + 1)t_{2,3}{\bullet}w_3 - \gamma t_{2,3}{\bullet}w_2 \in \langle e_1\rangle\,, \tag{5}$$

where we used (2). When we take the products $w_2{\bullet}f_1, w_3{\bullet}f_1$ and $w_2{\bullet}f_2$, we get

$$\begin{aligned}
\alpha t_{2,3}{\bullet}t_{2,3} &= 0\,,\\
((\beta - 1)(\beta - 2) - \alpha\gamma)t_{2,3}{\bullet}t_{2,3} &= 0\,,\\
(\beta(\beta + 1) - \alpha\gamma)t_{2,3}{\bullet}t_{2,3} &= 0
\end{aligned}$$

(we used (3) to find these equations). The only solution is $t_{2,3}{\bullet}t_{2,3} = 0$.

The fact that $t_{2,3}{\bullet}t_{2,3} = 0$ implies that $t_{2,3} \in T_1$. By Corollary 4.12 we have that $t_{2,3}{\bullet}w_2 = \alpha t_3{\bullet}w_2$ and $t_{2,3}{\bullet}w_3 = \beta t_2{\bullet}w_3$ for some $\alpha, \beta \in \mathbb{R}$. Therefore $(t_{2,3} - \alpha t_3 - \beta t_2){\bullet}\mathfrak{g}_3 = 0$ which contradicts the fact that $T(\mathfrak{g}_3) = 0$.

Case 2. $\mathfrak{g}_3\bullet\mathfrak{g}_3$ is not abelian. In this case we have $\mathfrak{g}_3\bullet\mathfrak{g}_3 = \langle e_1, t_2, t_3, t_{2,3}, w_2\rangle$. By Lemma 4.15 we get $\mathfrak{g}_3\bullet C_{\mathfrak{g}_3}(e_1) \subseteq C_{\mathfrak{g}_3}(e_1)$ and $\mathfrak{g}_3\bullet w_2 \subseteq \mathfrak{g}_3\bullet\langle e_1, \dots, e_5\rangle \subseteq \langle e_1, \dots, e_4\rangle = C_{\mathfrak{g}_3}(e_1)$. Also $\langle C_{\mathfrak{g}_3}(e_1), w_2\rangle\bullet w_3 \subseteq C_{\mathfrak{g}_3}(e_1)$. Hence $w_3\bullet w_3 \notin C_{\mathfrak{g}_3}(e_1)$. So there exist elements $\alpha \in \mathbb{R}_0$ and $c \in C_{\mathfrak{g}_3}(e_1)$ such that $w_3\bullet w_3 = \alpha w_2 + c$. As a consequence we have $t_3\bullet w_3 = \frac{\alpha}{2}t_2$.

Because $t_3 \in T_1$ (Corollary 4.16), we have $t_3\bullet(w_3\bullet w_3) = \alpha t_3\bullet w_2$. On the other hand, we get $t_3\bullet(w_3\bullet w_3) = [t_3, w_3]\bullet w_3 + w_3\bullet(t_3\bullet w_3) = \frac{\alpha}{2}t_2\bullet w_3$. So, $t_3\bullet w_2 = \frac{1}{2}t_2\bullet w_3$.

Finally, we compute $t_3\bullet w_2 = [e_1, w_3]\bullet w_2 = e_1\bullet(w_3\bullet w_2) - w_3\bullet(e_1\bullet w_2) = -t_2\bullet w_3$. This is a contradiction. □

5. Free Two-Step Nilpotent Lie Groups Admitting Simply Transitive Affine Actions without Translations

Let $\bullet$ be a complete left symmetric structure on $\mathfrak{g}_k$, a free two-step nilpotent Lie algebra on k generators. For k even, we have proven in previous section that if $\mathfrak{g}_k\bullet\mathfrak{g}_k$ is abelian, then $T(\mathfrak{g}_k) \neq 0$. It is not possible to generalize this result to all k. In this section we give an example of a complete left symmetric structure on $\mathfrak{g}_k$, for $k \geq 5$ and odd, for which $\mathfrak{g}_k\bullet\mathfrak{g}_k$ is abelian and $T(\mathfrak{g}_k) = 0$.

We are looking for a complete left symmetric structure on $\mathfrak{g}_k$ such that $T(\mathfrak{g}_k) = 0$ and $\mathfrak{g}_k\bullet\mathfrak{g}_k$ is abelian (k is odd). In this case, we know already:

— there is a vector space decomposition $\mathfrak{g}_k = \langle e_1\rangle \oplus W \oplus Z(\mathfrak{g}_k)$ with $W = \langle w_2, \dots, w_k\rangle$,

— we can define elements $t_i = [e_1, w_i] \in T_1 \cap Z(\mathfrak{g}_k), 2 \leq i \leq k$ such that:

* $t_2\bullet\langle C_{\mathfrak{g}}(e_1), w_2, \dots, w_{k-1}\rangle = 0$ and $t_2\bullet w_k \neq 0$,
* $t_i\bullet w_i = 0$,
* $t_i\bullet w_j = -t_j\bullet w_i$, $\forall j \in \{2, \dots, k\}, j \neq i$.

Let us define $z_{a,b} = [w_a, w_b]$ with $a, b \in \{2, \dots, k\}$. So we have $z_{b,a} = -z_{a,b}, z_{a,a} = 0$ and the centre $Z(\mathfrak{g}_k)$ is as vector space generated by the elements t_i and $z_{a,b}$ with $2 \leq i \leq k$, $2 \leq a < b \leq k$.

With the above properties in mind, we now define the product $\bullet$ on any pair of elements e_1, w_i, t_j, $z_{a,b}$ $(2 \leq i, j \leq k, 2 \leq a < b \leq k)$ (which form a basis for $\mathfrak{g}_k$ as a vector space).

— By definition of e_1, we have

$$\langle e_1, w_i, t_j, z_{a,b}\rangle\bullet e_1 = 0, \quad e_1\bullet w_i = t_i, \quad e_1\bullet t_j = 0, \quad e_1\bullet z_{a,b} = 0.$$

— By Corollary 4.11 we know $t_i \bullet C_{\mathfrak{g}}(e_1) = 0 = C_{\mathfrak{g}}(e_1) \bullet t_i$. When we take a look at the strategy of the proof of Theorem 4.13, it is natural to define

$$\begin{array}{ccccccc} t_i \bullet w_j & = & 0 & = & w_j \bullet t_i, & & j \neq k+2-i, \\ t_i \bullet w_{k+2-i} & = & (-1)^{i+1} e_1 & = & w_{k+2-i} \bullet t_i. & & \end{array}$$

— Yet, we know $z_{a,b} \bullet \langle e_1, t_2, \dots, t_k \rangle = 0 = \langle e_1, t_2, \dots, t_k \rangle \bullet z_{a,b}$ and we define
$z_{a,b} \bullet z_{c,d} = \beta_{abcd} e_1$, with $\beta_{abcd} = 0$ if a, b, c and d are not 4 different integers and $\beta_{abcd} = 1$ when $a < c < b < d$
For the other cases, we use following rules: if m, n, p, q are 4 different integers between 2 and k then $\beta_{mnpq} = -\beta_{nmpq} = -\beta_{mnqp} = -\beta_{mpnq}$

Finally, if $c = a$ or $c = b$ then $z_{a,b} \bullet w_c = 0 = w_c \bullet z_{a,b}$, else, $z_{a,b} \bullet w_c = -\frac{1}{2} \sum_{d=2}^{k} (-1)^d \; \beta_{abcd} \; t_{k+2-d} = w_c \bullet z_{a,b}$.

— We still need to define the products $w_i \bullet w_j$:

$$w_i \bullet w_i = 0, \quad w_i \bullet w_j = \frac{1}{2} z_{i,j} \quad \text{if } i \neq j.$$

Yet, we have to check whether $\bullet$ is indeed a complete left symmetric structure . By Definition 2.1 we must check two conditions.

Firstly, we have to consider any two elements $A, B \in \mathfrak{g}_k$, and check if $A \bullet B - B \bullet A = [A, B]$. As an example, take $A = w_i$ and $B = w_j$, $i \neq j$, then

$$w_i \bullet w_j - w_j \bullet w_i = \frac{1}{2} z_{i,j} - \frac{1}{2} z_{j,i} = \frac{1}{2} z_{i,j} + \frac{1}{2} z_{i,j} = z_{i,j} = [w_i, w_j].$$

All other cases are analogously.

Secondly, we need to pick 3 elements $A, B, C \in \mathfrak{g}_k$ and check if $[A, B] \bullet C = A \bullet (B \bullet C) - B \bullet (A \bullet C)$. Most of the cases are trivial, but let us, as an example, compute the case $A = w_i, B = w_j$, $2 \leq i, j \leq k, i \neq j$ and $C = z_{a,b}$, $2 \leq a < b \leq k$. Hence we must check if $z_{i,j} \bullet z_{a,b} = w_i \bullet (w_j \bullet z_{a,b}) - w_j \bullet (w_i \bullet z_{a,b})$.

Case 1. $j = a$ or $j = b$. In this case we know $z_{i,j} \bullet z_{a,b} = 0$, and also $w_j \bullet z_{a,b} = 0$ and hence $w_i \bullet (w_j \bullet z_{a,b}) = 0$. Note that $w_i \bullet z_{a,b} \in \langle t_{k+2-d},\ 2 \leq d \leq k, d \neq a, b, i \rangle$ and $w_j \bullet t_{k+2-d} \neq 0$ implies that $d = j$. As $j = a$ or $j = b$, it follows that $w_j \bullet (w_i \bullet z_{a,b}) = 0$. Hence $w_i \bullet (w_j \bullet z_{a,b}) - w_j \bullet (w_i \bullet z_{a,b}) = 0$.

Case 2. $i = a$ or $i = b$. Analogously as previous case.

Case 3. i, j, a, b are 4 different integers.

By definition of the product, we have that $z_{i,j} \bullet z_{a,b} = \beta_{ijab} e_1$. As in the previous cases, we obtain that $w_j \bullet z_{a,b} \in \langle t_{k+2-d},\ 2 \leq d \leq k, d \neq a, b, j \rangle$. But because a, b, i and j are all different, $w_j \bullet z_{a,b}$ has a t_{k+2-i} term, with coefficient $-\frac{1}{2}(-1)^i \beta_{abji}$. So,

$$w_i \bullet (w_j \bullet z_{a,b}) = -\frac{1}{2}(-1)^i \beta_{abji} w_i \bullet t_{k+2-i} = -\frac{1}{2}(-1)^i (-1)^{k+2-i+1} \beta_{abji} e_1$$
$$= \frac{1}{2}\beta_{ijab} e_1.$$

Analogously we find $w_j \bullet (w_i \bullet z_{a,b}) = -\frac{1}{2}\beta_{ijab} e_1$. Hence

$$w_i \bullet (w_j \bullet z_{a,b}) - w_j \bullet (w_i \bullet z_{a,b}) = \beta_{ijab} e_1.$$

This completes the proof that $z_{i,j} \bullet z_{a,b} = w_i \bullet (w_j \bullet z_{a,b}) - w_j \bullet (w_i \bullet z_{a,b})$.

All the other cases can be found in an analogously way.

Therefore the product $\bullet$ we defined is really a left symmetric structure. Because

$$\begin{aligned}
\mathfrak{g}_k \bullet \mathfrak{g}_k &= \langle e_1, t_i, z_{a,b},\ i, a, b \in \{2, \ldots, k\},\ a < b \rangle = C_{\mathfrak{g}}(e_1)\,, \\
\mathfrak{g}_k \bullet (\mathfrak{g}_k \bullet \mathfrak{g}_k) &= \langle e_1, t_i,\ 2 \le i \le k \rangle\,, \\
\mathfrak{g}_k \bullet (\mathfrak{g}_k \bullet (\mathfrak{g}_k \bullet \mathfrak{g}_k)) &= \langle e_1 \rangle\,, \\
\mathfrak{g}_k \bullet (\mathfrak{g}_k \bullet (\mathfrak{g}_k \bullet (\mathfrak{g}_k \bullet \mathfrak{g}_k))) &= 0\,,
\end{aligned}$$

we have that λ_X is nilpotent $\forall X \in \mathfrak{g}_k$. So $\bullet$ is a left symmetric structure which is complete. Note that here we have $\mathfrak{g}_k \bullet \mathfrak{g}_k$ abelian.

Finally we prove that $T(\mathfrak{g}_k) = 0$.

Firstly, we want to show that for any nonzero element $h \in W \oplus Z(\mathfrak{g}_k)$, there exists an element $h' \in W \oplus Z(\mathfrak{g}_k)$ such that $h \bullet h' \notin W \oplus Z(\mathfrak{g}_k)$, what means that $h \bullet h' = \rho e_1 + h''$ with $\rho \in \mathbb{R}_0$ and $h'' \in W \oplus Z(\mathfrak{g})$. To prove this, we create the following table, where each entry is the coefficient of the e_1-term of the product $A \bullet B$:

$\bullet$	t_2	$\dots$	t_k	$z_{2,3}$	$\dots$	$z_{k-1,k}$	w_2	$\dots$	w_k
t_2	0	$\dots$	0	0	$\dots$	0			
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$		M_1	
t_k	0	$\dots$	0	0	$\dots$	0			
$z_{2,3}$	0	$\dots$	0				0	$\dots$	0
$\vdots$	$\vdots$	$\ddots$	$\vdots$		M_2		$\vdots$	$\ddots$	$\vdots$
$z_{k-1,k}$	0	$\dots$	0				0	$\dots$	0
w_2				0	$\dots$	0	0	$\dots$	0
$\vdots$		M_3		$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
w_k				0	$\dots$	0	0	$\dots$	0

Now, it is enough to show that the determinant of the matrix M formed by these entries is not zero.

The only nonzero entries of the $(k-1)\times(k-1)$ matrix M_1, are $(M_1)_{i,j} = (-1)^i$ with $i+j=k$, so $\det(M_1)=1$. M_3 is the transposed matrix of M_1, hence $\det(M_3)=\det(M_1)=1$. Let us now take a better look to the matrix M_2. This matrix comes from the the products $z_{a,b}\bullet z_{c,d}$. The only possibilities for the entries are $0, 1$ and -1.

Define α_i as the following $(k-2i-1)\times(k-2i-1)$ matrix and β as:

$$\alpha_i = \begin{pmatrix} 0 & 1 & \dots & 1 \\ -1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ -1 & \dots & -1 & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence, $\det(\alpha_i)=1$ (recall that k is odd) and $\det(\beta)=-1$.

The matrix M_2 now has the following structure:

$\bullet$	$z_{2,3}$	$z_{2,4}$ … $z_{2,k}$	$z_{3,4}$ … $z_{3,k}$	$z_{4,5}$	$z_{4,6}$ … $z_{4,k}$	$z_{5,6}$ … $z_{5,k}$	… …	$z_{k-5,k-4}$	$z_{k-5,k-3}$ … $z_{k-5,k}$	$z_{k-4,k-3}$ … $z_{k-4,k}$	$z_{k-3,k-2}$ … $z_{k-1,k}$
$z_{2,3}$	0	0 … 0	0 … 0	−1	−1 … −1	−1 … −1	… …	−1	−1 … −1	−1 … −1	−1 … −1
$z_{2,4}$ ⋮ $z_{2,k}$	0 ⋮ 0	0 … 0 ⋮ ⋱ ⋮ 0 … 0	α_1	γ_1							
$z_{3,4}$ ⋮ $z_{3,k}$	0 ⋮ 0	${}^t\alpha_1$	0 … 0 ⋮ ⋱ ⋮ 0 … 0	γ'_1							
$z_{4,5}$	−1	${}^t\gamma_1$	${}^t\gamma'_1$	0	0 … 0	0 … 0	… …	−1	−1 … −1	−1 … −1	−1 … −1
$z_{4,6}$ ⋮ $z_{4,k}$	−1 ⋮ −1			0 ⋮ 0	0 … 0 ⋮ ⋱ ⋮ 0 … 0	α_2	γ_2				
$z_{5,6}$ ⋮ $z_{5,k}$	−1 ⋮ −1			0 ⋮ 0	${}^t\alpha_2$	0 … 0 ⋮ ⋱ ⋮ 0 … 0	γ'_2				
⋮ ⋮	⋮ ⋮			⋮ ⋮			⋮ ⋮				
$z_{k-5,k-4}$	−1			−1	${}^t\gamma_2$	${}^t\gamma'_2$	… …	0	0 … 0	0 … 0	−1 … −1
$z_{k-5,k-3}$ ⋮ $z_{k-5,k}$	−1 ⋮ −1			−1 ⋮ −1				0 ⋮ 0	0 … 0 ⋮ ⋱ ⋮ 0 … 0	α_{k-1}	γ_{k-1}
$z_{k-4,k-3}$ ⋮ $z_{k-4,k}$	−1 ⋮ −1			−1 ⋮ −1				0 ⋮ 0	${}^t\alpha_{k-1}$	0 … 0 ⋮ ⋱ ⋮ 0 … 0	γ'_{k-1}
$z_{k-3,k-2}$ ⋮ $z_{k-1,k}$	−1 ⋮ −1			−1 ⋮ −1				−1 ⋮ −1	${}^t\gamma_{k-1}$	${}^t\gamma'_{k-1}$	β

When we take a look at matrix α_i, we see that ${}^t\alpha_i = -\alpha_i$. Now we prove that $\gamma'_i = \gamma_i$, $1 \leq i \leq k-1$. Therefore we need to show that $z_{2i,j}\bullet z_{a,b} = z_{2i+1,j}\bullet z_{a,b}$

with $2i+2 \leq j \leq k$ and $2i+2 \leq a < b \leq k$.

Case 1. $z_{2i,j} \bullet z_{a,b} = 0$. This is the case when $a = j$ or $b = j$. Then it follows immediately that $z_{2i+1,j} \bullet z_{a,b} = 0$.

Case 2. $z_{2i,j} \bullet z_{a,b} = 1$. This is only possible if $2i < a < j < b$. We know that $a \geq 2i+2$, hence $2i+1 < a < j < b$. Thus also $z_{2i+1,j} \bullet z_{a,b} = 1$.

Case 3. $z_{2i,j} \bullet z_{a,b} = -1$. This holds in the following cases: $2i < j < a < b$ and $2i < a < b < j$. Because $a \geq 2i+2$ and $j \geq 2i+2$, we have respectively $2i+1 < j < a < b$ and $2i+1 < a < b < j$. So for these cases we find $z_{2i+1,j} \bullet z_{a,b} = -1$.

This proves that for all possible cases we have $z_{2i,j} \bullet z_{a,b} = z_{2i+1,j} \bullet z_{a,b}$ or $\gamma_i' = \gamma_i$, $1 \leq i \leq k-1$.

Using this knowledge on ${}^t\alpha_i$ and the γ_i', it is now an interesting exercise in linear algebra to show that $\det(M_2) = -\frac{k-3}{2}$. Because $k \geq 5$, we have that $\det(M_2) \neq 0$. Therefore we can conclude that $\det(M) \neq 0$.

Finally we need to prove that for any nonzero element $g \in \mathfrak{g}_k$, there exists an element $g' \in \mathfrak{g}_k$ such that $g \bullet g' \neq 0$. Let us first look to the case $g \in \langle e_1 \rangle$. Then $g = \alpha e_1$ (with $\alpha \neq 0$) and $g \bullet w_2 = \alpha e_1 \bullet w_2 = \alpha[e_1, w_2] = \alpha t_2$. In the case $g \notin \langle e_1 \rangle$, we have $g = \alpha e_1 + h$ with $h \in W \oplus Z(\mathfrak{g}_k)$ and $h \neq 0$. Then we know that there exists an element $h' \in W \oplus Z(\mathfrak{g}_k)$ such that $h \bullet h' = \rho e_1 + h''$ with $\rho \in \mathbb{R}_0$ and $h'' \in W \oplus Z(\mathfrak{g}_k)$. Hence

$$g \bullet h' = (\alpha e_1 + h) \bullet h' = \alpha[e_1, h'] + h \bullet h' = \rho e_1 + h''',$$

with $\rho \neq 0$ and $h''' \in W \oplus Z(\mathfrak{g}_k)$. This proves that $\bullet$ is a complete left symmetric structure with $T(\mathfrak{g}_k) = 0$.

As an illustration of this construction, we take a better look to the case $k = 5$. The free 2-step nilpotent Lie algebra $\mathfrak{g}_5$ on 5 generators is 15-dimensional with a 10-dimensional centre $Z(\mathfrak{g}_5) = [\mathfrak{g}_5, \mathfrak{g}_5]$.

Every element $g \in \mathfrak{g}_5$ can be written as a linear combination of the elements $e_1, w_2, \ldots, w_5, t_2, \ldots, t_5, z_{2,3}, z_{2,4}, z_{2,5}, z_{3,4}, z_{3,5}, z_{4,5}$:

$$g = \lambda_1 e_1 + \sum_{i=2}^{5} \lambda_i w_i + \sum_{i=2}^{5} \rho_i t_i + \sum_{a=2}^{4} \sum_{b=a+1}^{5} \nu_{a,b} z_{a,b}.$$

The construction above gives us a complete left symmetric structure on $\mathfrak{g}_5$ with $T(\mathfrak{g}_5) = 0$. The associated affine structure on $\mathfrak{g}_5$ is given by $\varphi : \mathfrak{g}_5 \mapsto \mathfrak{aff}(\mathbb{R}^{15})$:

$$\varphi(g)=\left(\begin{array}{ccccccccccccccc|c}
0 & -\lambda_5 & \lambda_4 & -\lambda_3 & \lambda_2 & -\nu_{4,5} & \nu_{3,5} & -\nu_{3,4} & -\nu_{2,5} & \nu_{2,4} & -\nu_{2,3} & \rho_5 & -\rho_4 & \rho_3 & -\rho_2 & \lambda_1\\
0 & 0 & 0 & 0 & 0 & \frac{-\lambda_4}{2} & \frac{\lambda_3}{2} & 0 & \frac{-\lambda_2}{2} & 0 & 0 & \lambda_1-\frac{\nu_{3,4}}{2} & \frac{\nu_{2,4}}{2} & \frac{-\nu_{2,3}}{2} & 0 & \rho_2\\
0 & 0 & 0 & 0 & 0 & \frac{-\lambda_5}{2} & 0 & \frac{\lambda_3}{2} & 0 & \frac{-\lambda_2}{2} & 0 & \frac{-\nu_{3,5}}{2} & \lambda_1+\frac{\nu_{2,5}}{2} & 0 & \frac{-\nu_{2,3}}{2} & \rho_3\\
0 & 0 & 0 & 0 & 0 & 0 & \frac{-\lambda_5}{2} & \frac{\lambda_4}{2} & 0 & 0 & \frac{-\lambda_2}{2} & \frac{-\nu_{4,5}}{2} & 0 & \lambda_1+\frac{\nu_{2,5}}{2} & \frac{-\nu_{2,4}}{2} & \rho_4\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-\lambda_5}{2} & \frac{\lambda_4}{2} & \frac{-\lambda_3}{2} & 0 & \frac{-\nu_{4,5}}{2} & \frac{\nu_{3,5}}{2} & \lambda_1-\frac{\nu_{3,4}}{2} & \rho_5\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-\lambda_3}{2} & \frac{\lambda_2}{2} & 0 & 0 & \nu_{2,3}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-\lambda_4}{2} & 0 & \frac{\lambda_2}{2} & 0 & \nu_{2,4}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-\lambda_5}{2} & 0 & 0 & \frac{\lambda_2}{2} & \nu_{2,5}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-\lambda_4}{2} & \frac{\lambda_3}{2} & 0 & \nu_{3,4}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-\lambda_5}{2} & 0 & \frac{\lambda_3}{2} & \nu_{3,5}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-\lambda_5}{2} & \frac{\lambda_4}{2} & \nu_{4,5}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_2\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_3\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_4\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_5\\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right).$$

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