International Journal of Pure and Applied Mathematics
Volume 30 No. 1 2006, 93-102

DOUBLE PERIODIC SOLUTIONS OF THE (1 + 1)-DIMENSIONAL DISPERSIVE LONG WAVE EQUATIONS

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Abstract: By the aid of symbolic computation system *Maple*, a new extended F-expansion is presented by means of new solutions of the corresponding auxiliary equation and a general ansatz. The validity and reliability of the method is tested by its application to the (1+1)-dimensional dispersive long wave equations. The solitary wave solutions and triangular periodic solutions can be obtained at their limit condition.

AMS Subject Classification: 35C05, 35L05, 35Q51
Key Words: double periodic solution, (1+1)-dimensional dispersive long wave equations

1. Introduction

Nonlinear partial differential equations (NPDEs) are related to nonlinear science such as physics, mechanics, biology, etc. To further explain some physical phenomena, searching for exact solutions of NLEEs is very important. Much work has been focused on the various extensions and application of the known algebraic methods to construct the exact solutions of NPDEs[1]-[9].

Very recently, a unified method called the F-expansion, [1], [2] method has been developed to obtain double periodic solutions of some NPDEs. Compared with other direct ansatz methods, the F-expansion method gives more general

Received: June 6, 2006

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solutions without much extra effort and repetitive calculations. For example, in other direct ansatz methods, Jacobi elliptic function solutions in terms of $sn(\xi), cn(\xi), dn(\xi)$ may be obtained by applying *sn*-expansion, *cn*- expansion, *dn*-expansion. Thus many similar repetitive calculations have to be made. It allows one to find both Jacobi elliptic function solutions, triangular function solutions and soliton solutions using the same and unique procedure.

In this paper, by introducing many new solutions in terms of rational formal Jacobi elliptic functions of the auxiliary equation and a general ansatz, we propose a new extended F-expansion method so that it can be used to obtain more new exact double periodic solutions for the (1+1)-dimensional dispersive long wave equations.

2. Double Periodic Solutions of the (1+1)-Dimensional Dispersive Long Wave Equations

The (1+1)-dimensional dispersive long wave equations reads:

$$v_t + vv_x + w_x = 0, \tag{1a}$$

$$w_t + (wv)_x + \frac{1}{3}v_{xxx} = 0, \tag{1b}$$

where w is the elevation of the water wave, v is the surface velocity of water along x-direction.

The equations (1) can be traced back to the works of Broer [1], Kaup [4], Martinez [9], Kupershmidt [5], etc. A good understanding of the solutions of equations (1) is very helpful for coastal and civil engineers to apply the nonlinear water model in harbor and coastal design. In order to seek travelling wave solutions of equations (1), we make the following transformation:

$$w = W(\xi), \quad v = V(\xi), \quad \xi = x + \lambda t, \tag{2}$$

where λ is a constant to be determined later. Then equations (1) become

$$\lambda V' + VV' + W' = 0, \tag{3a}$$

$$\lambda W' + (WV)' + \frac{1}{3}V''' = 0. \tag{3b}$$

By balancing W' with VV' in equation (3a) and balancing V''' with $(WV)'$ in equation (3b), we get $n_1 = 2, n_2 = 1$, so we can suppose that equations (3) have the following formal solutions:

$$W = K(F) = a_{10} + a_{11}F + a_{12}F^2 + \frac{b_{11}}{F} + \frac{b_{12}}{F^2} + c_{11}\frac{F'}{F} + c_{12}\frac{F'^2}{F^2}, \tag{4a}$$

$$V = P(F) = a_{20} + a_{21}F + \frac{b_{21}}{F} + c_{21}\frac{F'}{F}, \tag{4b}$$

where $a_{10}, a_{11}, a_{12}, b_{11}, b_{12}, c_{11}, c_{12}, a_{20}, a_{21}, b_{21}$ and c_{21} are constants to be determined later. And the new variable $F = F(\xi)$ satisfying

$$F' - \sqrt{a + bF^2 + cF^4} = \frac{dF}{d\xi} - \sqrt{a + bF^2 + cF^4} = 0 \tag{5}$$

By the aid of *Maple*, substituting equations (4) into equations (3) along with equation (5), collecting the coefficients of $F^i(\sqrt{a + bF^2 + cF^4})^j$ $(j = 0, 1; i = 0, 1, 2...)$, and setting them to be zero, we can deduce the following set of equations with respect to $a_{10}, a_{11}, a_{12}, b_{11}, b_{12}, c_{11}, c_{12}, a_{20}$, a_{21}, b_{21}, c_{21} and λ.

$b_{21}c_{21}B = 0,$
$2\,a_{21}c_{21}C = 0,$
$a_{21}c_{21}B = 0,$
$2\,b_{21}c_{21}A = 0,$
$a_{20}a_{21} + a_{11} + \lambda\,a_{21} = 0,$
$b_{11} + \lambda\,b_{21} + a_{20}b_{21} = 0,$
$3\,a_{12}c_{21}A - 3\,Cb_{12}c_{21} = 0,$
$3\,a_{11}c_{21}B + 3\,c_{11}a_{21}B = 0,$
$6\,Ca_{11}c_{21} + 6\,Cc_{11}a_{21} = 0,$
$6\,b_{11}c_{21}A + 6\,c_{11}b_{21}A = 0,$
$3\,c_{11}b_{21}B + 3\,b_{11}c_{21}B = 0,$
$a_{20}c_{21}C + \lambda\,c_{21}C + c_{11}C = 0,$
$c_{11}A - a_{20}c_{21}A + \lambda\,c_{21}A = 0,$
$9\,b_{12}b_{21} + 6\,b_{21}A + 9\,{c_{12}}^2b_{21}A = 0,$
$6\,Ca_{21} + 9\,a_{12}a_{21} + 9\,C{c_{12}}^2a_{21} = 0,$
$2\,{c_{12}}^2A + {c_{21}}^2A + 2\,b_{12} - {b_{21}}^2 = 0,$
${a_{21}}^2 + 2\,{c_{12}}^2C + {c_{21}}^2C + 2\,a_{12} = 0,$
$6\,C^2c_{21} + 9\,Ca_{12}c_{21} + 9\,C^2{c_{12}}^2c_{21} = 0,$
$9\,b_{12}c_{21}A + 6\,c_{21}A^2 + 9\,{c_{12}}^2c_{21}A^2 = 0,$
$6\,\lambda\,a_{12} + 6\,C\lambda\,{c_{12}}^2 + 6\,Cc_{11}c_{21} + 6\,C{c_{12}}^2a_{20} + 6\,a_{11}a_{21} + 6\,a_{12}a_{20} = 0,$
$6\,\lambda\,{c_{12}}^2A + 6\,c_{11}c_{21}A + 6\,b_{12}a_{20} - 6\,\lambda\,b_{12} - 6\,b_{11}b_{21} - 6\,{c_{12}}^2a_{20}A = 0,$
$3\,{c_{12}}^2a_{21}B + 3\,C{c_{12}}^2b_{21} + a_{21}B + 3\,a_{11}a_{20} + 3\,\lambda\,a_{11} + 3\,a_{10}a_{21} + 3\,a_{12}b_{21} = 0,$
$3\,C\lambda\,c_{11} + 6\,a_{12}c_{21}B + 9\,C{c_{12}}^2c_{21}B + 3\,Ca_{10}c_{21} + 3\,Cc_{11}a_{20} + 4\,Cc_{21}B = 0,$
$3\,b_{12}a_{21} + 3\,{c_{12}}^2b_{21}B + 3\,\lambda\,b_{11} - 3\,b_{11}a_{20} + 3\,{c_{12}}^2a_{21}A + 3\,a_{10}b_{21} - b_{21}B = 0,$
$3\,a_{10}c_{21}A + 6\,b_{12}c_{21}B + 3\,c_{11}a_{20}A + 4\,c_{21}BA + 9\,{c_{12}}^2c_{21}AB + 3\,\lambda\,c_{11}A = 0.$

By use of the *Maple* we get the following results:

Case 1.

$$a_{21} = c_{11} = c_{21} = a_{11} = b_{11} = 0, \quad b_{21} = \frac{2}{3}\sqrt{3a}, \quad b_{12} = -{c_{12}}^2a - \frac{2}{3}a,$$

$$a_{10} = -\frac{1}{3}b - {c_{12}}^2 b, \quad a_{12} = -{c_{12}}^2 c, \quad \lambda = -a_{20}, \quad a_{20} = a_{20},$$
$$c_{12} = c_{12}, \quad d_{21} = d_{21}, \quad d_{11} = d_{11}, \quad d_{12} = d_{12}\,.$$

Case 2.

$$b_{12} = -{c_{12}}^2 a - \frac{2}{3}a, \quad \lambda = -a_{20}, \quad a_{20} = a_{20}, \quad c_{12} = c_{12}, \quad d_{21} = d_{21},$$
$$d_{11} = d_{11}, \quad a_{12} = -\frac{2}{3}c - {c_{12}}^2 c, \quad a_{11} = b_{11} = c_{11} = c_{21} = 0,$$
$$a_{21} = -\frac{2}{3}\sqrt{3c}, \quad a_{10} = -\frac{2}{3}\sqrt{ca} - \frac{1}{3}b - {c_{12}}^2 b, \quad b_{21} = \frac{2}{3}\sqrt{3a}, \quad d_{12} = d_{12}\,.$$

For simplicity, we set

$$Q = -\frac{2\sqrt{ac} + b + 3bc_{12}^2}{3}, \qquad M = -\frac{2c + 3cc_{12}^2}{3}, \qquad N = -\frac{2a + 3ac_{12}^2}{3}.$$

From (2), (4) and Case 1, Case 2, the double periodic solutions to the equations (1) expressed by various Jacobi elliptic functions can be obtained by using Appendix A.

Family 1. If $a = \frac{2-k^2+2k_1}{B^2}, b = 2\,k^2 - 4, c = B^2\left(k^2 - 2k_1 + 2\right)$, then

$$w_1 = Q + \frac{M\left(dn^2(\xi) + k_1\right)^2}{B^2\left(dn^2(\xi) - k_1\right)^2} + \frac{NB^2\left(dn^2(\xi) - k_1\right)^2}{\left(dn^2(\xi) + k_1\right)^2} + \frac{16{c_{12}}^4 k^4 k_1^2\, dn^2(\xi)\, cn^2(\xi)\, sn^2(\xi)}{\left(dn^4(\xi) - k_1^2\right)^2},$$

$$v_1 = a_{20} - \frac{2\sqrt{3c}\left(dn^2(\xi) + k_1\right)}{3B\left(dn^2(\xi) - k_1\right)} + \frac{2B\sqrt{3a}\left(dn^2(\xi) - k_1\right)}{3dn^2(\xi) + k_1}\,.$$

Family 2. If $a = \frac{k^2}{4(B^2-C^2)}, b = \frac{k^2}{2} - 1, c = \frac{k^2}{4(B^2-C^2)}$, we obtain the following rational formal double periodic solutions for equations (1):

$$w_2 = Q + \frac{M[\sqrt{B^2 - C^2}\,cn(\xi) - \sqrt{B^2k^2 + C^2 - B^2}\,sn(\xi)]^2}{(B^2 - C^2)\,(B\,dn(\xi) + C)^2} + \frac{N\left(B^2 - C^2\right)(B\,dn(\xi) + C)^2}{[\sqrt{B^2 - C^2}\,cn(\xi) - \sqrt{B^2k^2 + C^2 - B^2}\,sn(\xi)]^2}$$
$$+ \frac{{c_{12}}^4\left\{\sqrt{B^2 - C^2}\,sn(\xi)\,[B{k_1}^2 + C\,dn(\xi)] + \sqrt{B^2k^2 + C^2 - B^2}\,cn(\xi)\,[B + C\,dn(\xi)]\right\}^2}{[\sqrt{B^2 - C^2}\,cn(\xi) - \sqrt{B^2k^2 + C^2 - B^2}\,sn(\xi)]\,(B\,dn(\xi) + C)},$$

$$v_2 = a_{20} - \frac{2\sqrt{3}\sqrt{c}[\sqrt{B^2-C^2}\,cn\,(\xi) - \sqrt{B^2k^2+C^2-B^2}\,sn\,(\xi)]}{3\sqrt{B^2-C^2}\,(B\,dn\,(\xi)+C)} + \frac{2\sqrt{3}\sqrt{a}\sqrt{B^2-C^2}\,(B\,dn\,(\xi)+C)}{[3\sqrt{B^2-C^2}\,cn\,(\xi) - 3\sqrt{B^2k^2+C^2-B^2}\,sn\,(\xi)]}.$$

Family 3. If $a = \frac{1}{4(B^2-C^2)}, b = \frac{1}{2} - k^2, c = \frac{B^2-C^2}{4}$, we obtain the following rational formal double periodic solutions for equations (1):

$$w_3 = Q + \frac{M[\sqrt{B^2-C^2}\,dn\,(\xi) - \sqrt{B^2-B^2k^2+C^2k^2}\,sn\,(\xi)]^2}{(B^2-C^2)\,(B\,cn\,(\xi)+C)^2} + \frac{N\left(B^2-C^2\right)(B\,cn\,(\xi)+C)^2}{[\sqrt{B^2-C^2}\,dn\,(\xi) - \sqrt{B^2-B^2k^2+C^2k^2}\,sn\,(\xi)]^2} + \frac{{c_{12}}^4\left\{sn\,(\xi)\,[Ck^2\,cn\,(\xi) - 2\,Bk_1] + \sqrt{B^2-B^2k^2+k^2C^2}\,dn\,(\xi)\,(B+C\,cn\,(\xi))\right\}^2}{(B\,cn\,(\xi)+C)\,[\sqrt{B^2-B^2k^2+k^2C^2}\,sn\,(\xi) - dn\,(\xi)]},$$

$$v_3 = a_{20} - \frac{2\sqrt{3}\sqrt{c}[\sqrt{B^2-C^2}\,dn\,(\xi) - \sqrt{B^2-B^2k^2+C^2k^2}\,sn\,(\xi)]}{3\sqrt{B^2-C^2}\,(B\,cn\,(\xi)+C)} + \frac{2\sqrt{3}\sqrt{a}\sqrt{B^2-C^2}\,(B\,cn\,(\xi)+C)}{[3\sqrt{B^2-C^2}\,dn\,(\xi) - 3\sqrt{B^2-B^2k^2+C^2k^2}\,sn\,(\xi)]};$$

Family 4. If $a = \frac{k^4-2\,k^2+1}{4(B^2-C^2)}$, $b = \frac{k^2+1}{2}$, $c = \frac{B^2-C^2}{4}$, we obtain the following rational formal double periodic solutions for equations (1):

$$w_4 = Q + \frac{M[\sqrt{B^2-C^2}\,dn\,(\xi) - \sqrt{B^2-C^2k^2}\,cn\,(\xi)]^2}{(B^2-C^2)\,(B\,sn\,(\xi)+C)^2} + \frac{N\left(B^2-C^2\right)(B\,sn\,(\xi)+C)^2}{[\sqrt{B^2-C^2}\,dn\,(\xi) - \sqrt{B^2-C^2k^2}\,cn\,(\xi)]^2} + \frac{{c_{12}}^4\left\{cn\,(\xi)\,[Ck^2\,sn\,(\xi)+B] + \sqrt{B^2-k^2C^2}\,dn\,(\xi)\,[sn\,(\xi)\,C+B]\right\}^2}{(B\,sn\,(\xi)+C)\,[dn\,(\xi) + \sqrt{B^2-k^2C^2}\,cn\,(\xi)]},$$

$$v_4 = a_{20} - \frac{2\sqrt{3}\sqrt{c}[\sqrt{B^2-C^2}\,dn\,(\xi) - \sqrt{B^2-C^2k^2}\,cn\,(\xi)]}{3\sqrt{B^2-C^2}\,(B\,sn\,(\xi)+C)}$$

$$+\frac{2\sqrt{3}\sqrt{a}\sqrt{B^2-C^2}\,(B\,sn\,(\xi)+C)}{[3\sqrt{B^2-C^2}dn\,(\xi)-3\sqrt{B^2-C^2k^2}cn\,(\xi)]}.$$

Family 5. If $a=\frac{k^2+2k+1}{4A^2}, b=\frac{k^2-6k+1}{2}, c=\frac{A^2k^2+A^2+2kA^2}{4}$, we obtain the following rational formal double periodic solutions for equations (1):

$$w_5=Q+\frac{M\,(1+sn\,(\xi))^2\,(1-ksn\,(\xi))^2}{A^2cn^2\,(\xi)\,dn^2\,(\xi)}+\frac{NA^2cn^2\,(\xi)\,dn^2\,(\xi)}{(1+sn\,(\xi))^2\,(1-ksn\,(\xi))^2}$$
$$+\frac{{c_{12}}^4\left(k^2sn^2\,(\xi)-ksn^2\,(\xi)+k-1\right)^2}{cn^2\,(\xi)\,dn^2\,(\xi)},$$

$$v_5=a_{20}-\frac{2\sqrt{3c}\,(1+sn\,(\xi))\,(1-ksn\,(\xi))}{3Acn\,(\xi)\,dn\,(\xi)}+\frac{2\sqrt{3a}Acn\,(\xi)\,dn\,(\xi)}{3\,(1+sn\,(\xi))\,(1-ksn\,(\xi))}.$$

Family 6. If $a=4\,\frac{k}{A^2},\ b=-(k^2+1+6\,k),\ c=A^2k^2+A^2+2\,kA^2$, we obtain the following rational formal double periodic solutions for equations (1):

$$w_6=Q+\frac{M\left(1-ksn^2\,(\xi)\right)^2}{A^2cn^2\,(\xi)\,dn^2\,(\xi)}+\frac{NA^2cn^2\,(\xi)\,dn^2\,(\xi)}{(1-ksn^2\,(\xi))^2}$$
$$+\frac{{c_{12}}^4(k-1)^2\left(ksn^2\,(\xi)+1\right)^2}{(ksn^2\,(\xi)-1)^2\,cn^2\,(\xi)\,dn^2\,(\xi)},$$

$$v_6=a_{20}-\frac{2\sqrt{3c}\left(1-ksn^2\,(\xi)\right)}{3Acn\,(\xi)\,dn\,(\xi)}+\frac{2\sqrt{3a}Acn\,(\xi)\,dn\,(\xi)}{3\,(1-ksn^2\,(\xi))};$$

Family 7. If $a=\frac{k^4}{4}+k^2\,(k_1-1)+2-2k_1-k^2, b=\frac{k^2-2-6k_1}{2}, c=\frac{1}{4}$, we obtain the following rational formal double periodic solutions for equations (1):

$$w_7=Q+\frac{M\left((k_1-1)\,sn^2\,(\xi)+dn\,(\xi)+1\right)^2}{sn^2\,(\xi)\,cn^2\,(\xi)}+\frac{Nsn^2\,(\xi)\,cn^2\,(\xi)}{((k_1-1)\,sn^2\,(\xi)+dn\,(\xi)+1)^2}$$
$$+\frac{{c_{12}}^4[(sn\,(\xi))^2\,[(k_1+1)\,dn\,(\xi)+2]-(sn\,(\xi))^4\,k^2-1-dn\,(\xi)]^2}{sn^2\,(\xi)\,cn^2\,(\xi)\,[(k_1-1)\,(sn\,(\xi))^2+dn\,(\xi)+1]^2},$$

$$v_7=a_{20}-\frac{2\sqrt{3c}\left((k_1-1)\,sn^2\,(\xi)+dn\,(\xi)+1\right)}{3sn\,(\xi)\,cn\,(\xi)}$$
$$+\frac{2\sqrt{3a}sn\,(\xi)\,cn\,(\xi)}{3\,[(k_1-1)\,sn^2\,(\xi)+dn\,(\xi)+1]}.$$

Family 8. If $a = 8 - 4k_1k^2 + 8k_1 - 8\,k^2, b = 2 + 6k_1 - k^2, c = 1$, we obtain the following rational formal double periodic solutions for equations (1):

$$w_8 = Q + \frac{M\left(cn^2(\xi) - k_1 sn^2(\xi)\right)^2}{sn^2(\xi)\,cn^2(\xi)} + \frac{N sn^2(\xi)\,cn^2(\xi)}{\left(cn^2(\xi) - k_1 sn^2(\xi)\right)^2} + \frac{{c_{12}}^4 dn^2(\xi)\left[1 + (k_1 - 1)\,sn^2(\xi)\right]^2}{sn^2(\xi)\,cn^2(\xi)\left(k_1 - cn^2(\xi) + sn^2(\xi)\right)^2},$$

$$v_8 = a_{20} - \frac{2\sqrt{3c}\left(cn^2(\xi) - k_1 sn^2(\xi)\right)}{3sn(\xi)\,cn(\xi)} + \frac{2\sqrt{3a}sn(\xi)\,cn(\xi)}{3\left[cn^2(\xi) - k_1 sn^2(\xi)\right]}.$$

Family 9. If $a = \frac{4k^4\left(2\,k^2 - k_1k^2 + 2k_1 - 2\right)}{2 - k^2 - 2k_1}$, $b = \frac{k^4 + 8\,k^2 - 4\,k^2k_1 + 8k_1 - 8}{2 - k^2 - 2k_1}$, $c = \frac{1}{2 - k^2 - 2k_1}$, we obtain the following rational formal double periodic solutions for equations (1):

$$w_9 = Q + \frac{M\left(dn^2(\xi) - k_1\right)^2}{sn^2(\xi)\,cn^2(\xi)} + \frac{N sn^2(\xi)\,cn^2(\xi)}{\left(dn^2(\xi) - k_1\right)^2} + \frac{{c_{12}}^4 dn^2(\xi)\left[1 - k_1 - \left(2k_1 + k^2 - 2\right)sn^2(\xi)\right]^2}{sn^2(\xi)\,cn^2(\xi)\left(k_1 - dn^2(\xi)\right)^2},$$

$$v_9 = a_{20} - \frac{2\sqrt{3c}\left(dn^2(\xi) - k_1\right)}{sn(\xi)\,cn(\xi)} + \frac{2\sqrt{3a}sn(\xi)\,cn(\xi)}{3\left(dn^2(\xi) - k_1\right)}.$$

Here $\xi = x - a_{20}t$ in Family 1 – Family 9.

Remark 1. These solutions are obtained by substituting the Case 2 into equations (4), so they are only some double periodic solutions of equations (1). Other solutions are omitted.

Remark 2. All the above double periodic solutions of the (1+1)-dimensional dispersive long wave equations are quite new and have not been given in literature.

3. Conclusions

In this paper, we have presented the new extended F-expansion method which is more powerful than the method proposed recently by Wang [13] and Zhou [15]. We have applied our method to find abundant new double periodic solutions for the (1+1)-dimensional dispersive long wave equations. When the modulus $k \to$

1, these obtained solutions degenerate as solitary wave solutions of equations (1), and when the modulus $k \to 0$, these obtained solutions degenerate as triangular periodic solutions of equations (1). Our method can be applied to search for double periodic the solutions of the other NPDEs.

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Appendix A

The new double periodic solutions of the equation (5)

$$\frac{dF}{d\xi} - \sqrt{a + bF^2 + cF^4} = 0$$

are

Case 1. If $\begin{cases} a = \frac{2-k^2+2k_1}{B^2}, \\ b = 2\,k^2 - 4, \\ c = B^2\left(k^2 - 2k_1 + 2\right), \end{cases}$

then $F(\xi) = \frac{k_1 + dn^2(\xi,k)}{B\left(dn^2(\xi,k) - k_1\right)}$.

Case 2. If $\begin{cases} a = \frac{k^2}{4(B^2-C^2)}, \\ b = \frac{k^2}{2} - 1, \\ c = \frac{k^2}{4(B^2-C^2)}, \end{cases}$

then $F(\xi) = \frac{-\sqrt{\frac{B^2-B^2k^2-C^2}{B^2-C^2}}\,sn(\xi,k) + cn(\xi,k)}{B\,dn(\xi,k) + C}$.

Case 3. If $\begin{cases} a = \frac{1}{4(B^2-C^2)}, \\ b = \frac{1}{2} - k^2, \\ c = \frac{B^2-C^2}{4}, \end{cases}$

then $F(\xi) = \frac{-\sqrt{\frac{B^2+C^2k^2-k^2B^2}{B^2-C^2}}\,sn(\xi,k) + dn(\xi,k)}{B\,cn(\xi,k) + C}$.

Case 4. If $\begin{cases} a = \frac{k^4-2\,k^2+1}{4(B^2-C^2)}, \\ b = \frac{k^2+1}{2}, \\ c = \frac{B^2-C^2}{4}, \end{cases}$

then $F(\xi) = \frac{\sqrt{\frac{B^2-C^2k^2}{B^2-C^2}}\,cn(\xi,k) + dn(\xi,k)}{B\,sn(\xi,k) + C}$.

Case 5. If $\begin{cases} a = \frac{k^2+2\,k+1}{4A^2}, \\ b = \frac{k^2-6k+1}{2}, \\ c = \frac{A^2k^2+A^2+2kA^2}{4}, \end{cases}$

then $F(\xi) = \frac{(1+sn(\xi,k))(1-k\,sn(\xi,k))}{A\,cn(\xi,k)\,dn(\xi,k)}$.

Case 6. If $\begin{cases} a = 4\,\frac{k}{A^2}, \\ b = -(k^2 + 1 + 6\,k), \\ c = A^2k^2 + A^2 + 2\,kA^2, \end{cases}$

then $F(\xi) = \frac{1-k(sn(\xi,k))^2}{A\,cn(\xi,k)\,dn(\xi,k)}$.

Case 7. If $\begin{cases} a = \frac{k^4}{4} + k^2\left(k_1 - 1\right) + 2 - 2k_1 - k^2 \\ b = \frac{k^2 - 2 - 6k_1}{2}, \\ c = \frac{1}{4}, \end{cases}$

then $F(\xi) = \frac{(k_1-1)sn^2(\xi,k)+dn(\xi,k)+1}{sn(\xi,k)cn(\xi,k)}$.

Case 8. If $\begin{cases} a = 8 - 4k_1k^2 + 8k_1 - 8\,k^2, \\ b = 2 + 6k_1 - k^2, \\ c = 1, \end{cases}$

then $F(\xi) = \frac{cn^2(\xi,k)-k_1 sn^2(\xi,k)}{sn(\xi,k)cn(\xi,k)}$.

Case 9. If $\begin{cases} a = \frac{4k^4\left(2\,k^2 - k_1k^2 + 2k_1 - 2\right)}{2-k^2-2k_1}, \\ b = \frac{k^4+8\,k^2-4\,k^2k_1+8k_1-8}{2-k^2-2k_1}, \\ c = \frac{1}{2-k^2-2k_1}, \end{cases}$

then $F(\xi) = \frac{dn^2(\xi,k)-k_1}{sn(\xi,k)cn(\xi,k)}$.

Here $k_1 = \sqrt{1-k^2}$ and A, B, C are arbitrary constants. Jacobi elliptic functions $sn\left(\xi,k\right), cn\left(\xi,k\right),\ dn\left(\xi,k\right),\ k\ (0 < k < 1)$ is the modulus of the Jacobi elliptic function, are double periodic possess properties of triangular functions, namely

$$\mathrm{sn}^2\xi + \mathrm{cn}^2\xi = 1, \quad \mathrm{dn}^2\xi + k^2\mathrm{sn}^2\xi = 1, \quad \mathrm{sn}'\xi = \mathrm{cn}\xi\mathrm{dn}\xi,$$

$$\mathrm{cn}'\xi = -\mathrm{sn}\xi\mathrm{dn}\xi, \quad \mathrm{dn}'\xi = -k^2\mathrm{sn}\xi\mathrm{cn}\xi.$$

When $k \to 0$, the Jacobi elliptic functions degenerate to the triangle functions i.e., $\mathrm{sn}\xi \to \sin\xi$, $\mathrm{cn}\xi \to \cos\xi$, $\mathrm{dn}\xi \to 1$.

When $k \to 1$, the Jacobi elliptic functions degenerate to the hyperbolic functions i.e., $\mathrm{sn}\xi \to tanh\xi$, $\mathrm{cn}\xi \to sech\xi$, $\mathrm{dn}\xi \to sech\xi$.