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A CONVERGENCE ANALYSIS OF A NEWTON-LIKE METHOD WITHOUT INVERSES

Ioannis K. Argyros

Department of Mathematical Sciences
Cameron University
Lawton, OK 73505, USA
e-mail: iargyros@cameron.edu

Abstract: A convergence analysis of a Newton-like method without inverses on a Banach space is provided under p-Hölder continuity conditions to approximate a locally unique solution of a nonlinear equation in a Banach space. The order of the method is $1+p$.

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1. Introduction

In this study we are concerned with the problem of approximating a locally unique solution x^* of the nonlinear equation

$$F(x) = 0, \tag{1.1}$$

where F is a Fréchet-differentiable operator defined on a convex subset D of a Banach space X with values in a Banach space Y.

The most popular mehtod for generating a sequence approximating x^* is given by the Newton-like iteration in the form

$$x_{n+1} = x_n - G(x_n)F(x_n) \quad (n \geq 0) \ \ (x_0 \in D), \tag{1.2}$$

where $G(x)(x \in D) \in L(Y,X)$ is "close" to the inverse of the Fréchet-derivative

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$F'(x)$ of operator F (to be precised later, see (2.2)). The advantage of this method over a Newton-like method of the form

$$y_{n+1} = y_n - A(y_n)^{-1}F(y_n) \quad (n \geq 0) \quad (y_0 \in D) \tag{1.3}$$

is that in (1.3) the inverse $A(y_n)^{-1} \in L(Y, X)$ should be computed at every step.

The local and semilocal convergence analysis of methods (1.2) and (1.3) has been studied extensively by several authors and under various conditions. A survey of such results can be found in [1]–[8], and the references there. Here we provide a convergence analysis of method (1.2) under p-Hölder continuity conditions and show that the order of convergence is $1 + p$, $p \in [0, 1]$. Special cases of our results reduce to corresponding ones already in the literature or improve upon them.

2. Convergence Analysis of Method (1.2)

We can show the main semilocal convergence theorem for method (1.2).

Theorem 2.1. *Assume*:

F is continuously p-Hölder differentiable on D:

$$\|F'(x) - F'(y)\| \leq a\|x - y\|^p, \quad \text{for some } p \in [0, 1] \text{ and all } x, y \in D; \tag{2.1}$$

For every $x \in D$ there exist nonnegative constants b and c, an operator $G(x) \in L(Y, X)$, a constant $q \geq p$ such that:

$$\|F'(x)G(x) - I\| \leq b\|F(x)\|^q \tag{2.2}$$

and

$$\|G(x)\| \leq c; \tag{2.3}$$

There exists $x_0 \in D$ such that

$$\alpha = d\|F(x_0)\| < 1, \tag{2.4}$$

where,

$$d^p = b\|F(x_0)\|^{q-p} + \frac{ac^{1+p}}{1+p}\,; \tag{2.5}$$

The following inclusion holds:

$$\overline{U}(x_0, r) = \{x \in X \mid \|x - x_0\| \leq r\} \subseteq D, \tag{2.6}$$

where,

$$r = \left(\frac{c}{d} + \beta\alpha^p\right)\alpha, \tag{2.7}$$

and

$$\beta = \frac{c}{d(1-\alpha^{1+p})}\,. \tag{2.8}$$

Then sequence $\{x_n\}$ $(n \geq 0)$ generated by Newton-like method (1.2) is well defined, remains in $\overline{U}(x_0, r)$ for all $n \geq 0$ and converges to a solution $x^* \in \overline{U}(x_0, r)$ of equation $F(x) = 0$. The solution x^* is unique in $\overline{U}(x_0, r_0)$ for $r_0 \geq r$ provided that

$$\frac{ca}{1+p} r_0^p + \frac{b\alpha^q}{d^q} \leq 1, \tag{2.9}$$

and

$$\overline{U}(x_0, r_0) \subseteq D. \tag{2.10}$$

Moreover the following estimates hold for all $n \geq 0$:

$$\|x_{n+1} - x_n\| \leq c\|F(x_n)\|, \tag{2.11}$$

$$\|F(x_n)\| \leq \frac{\alpha^{(1+p)^n}}{d}, \tag{2.12}$$

and

$$\|x_n - x^*\| \leq \beta\alpha^{(1+p)^n}. \tag{2.13}$$

Proof. We shall first show using induction on the integer n that estimates (2.11), (2.12) hold true and $x_n \in U(x_0, r)$ for all $n \geq 0$. It follows from (1.2), (2.4) and (2.7) that (2.11), (2.12) hold for $n = 0$ and $x_1 \in U(x_0, r)$, since

$$\|x_1 - x_0\| = \|G(x_0)F(x_0)\| \leq \|G(x_0)\|\,\|F(x_0)\| \leq c\|F(x_0)\| \leq r. \tag{2.14}$$

Let us assume condition (2.11), (2.12) and $x_n \in U(x_0, r)$ hold true for $n = 0, 1, \ldots, k$.

In view of (1.2) we obtain the approximation

$$\begin{aligned} F(x_k) &= -[F'(x_{k-1})G(x_{k-1}) - I]F(x_{k-1}) \\ &\quad + F(x_k) - F(x_{k-1}) - F'(x_{k-1})(x_k - x_{k-1}). \end{aligned} \tag{2.15}$$

Using (2.1), (2.2), (2.4), (2.5), (2.15) and the induction hypotheses we obtain

$$\begin{aligned} \|F(x_k)\| &\leq b\|F(x_{k-1})\|^{1+q} + \frac{a}{1+p}\|x_k - x_{k-1}\|^{1+p} \\ &\leq b\|F(x_{k-1})\|^{1+q} + \frac{a}{1+p}c^{1+p}\|F(x_{k-1})\|^{1+p} \end{aligned}$$

$$
\begin{aligned}
&\le \left[b\|F(x_{k-1})\|^{q-p} + \frac{ac^{1+p}}{1+p}\right]\|F(x_{k-1})\|^{1+p}\\
&\le d^p\|F(x_{k-1})\|^{1+p},
\end{aligned}
$$

or

$$
\begin{aligned}
d\|F(x_k)\| \le [d\|F(x_{k-1})\|]^{(1+p)^1} \le [d\|F(x_{k-2})\|]^{(1+p)^2}&\\
\le [d\|F(x_0)\|]^{(1+p)^k},& \quad (2.16)
\end{aligned}
$$

which shows (2.12). We shall show $x_{k+1} \in \overline{U}(x_0, r)$. Indeed, we can have in turn

$$
\begin{aligned}
\|x_{k+1} - x_0\| &\le \|x_{k+1} - x_k\| + \|x_k - x_{k-1}\| + \cdots + \|x_1 - x_0\|\\
&\le c[\|F(x_k)\| + \|F(x_{k-1})\| + \cdots + \|F(x_0)\|]\\
&\le \frac{c}{d}\left[\alpha^{(1+p)^k} + \cdots + \alpha^{(1+p)^1}\right] + c\|F(x_0)\|\\
&\le \frac{c}{d}\left[\alpha^{(1+p)k} + \cdots + \alpha^{1+p}\right] + c\|F(x_0)\|\\
&\le \frac{c}{d}\left[\frac{1-\alpha^{(1+p)(k+1)}}{1-\alpha^{1+p}} - 1\right] + c\|F(x_0)\|\\
&\le \frac{c\alpha^{1+p}}{d(1-\alpha^{1+p})} + c\|F(x_0)\|\\
&= \beta\alpha^{1+p} + c\|F(x_0)\| = r, \quad (2.17)
\end{aligned}
$$

which implies $x_{k+1} \in \overline{U}(x_0, r)$. It now follows from (1.2) and (2.3) that

$$
\|x_{k+1} - x_k\| \le c\|F(x_k)\|, \quad (2.18)
$$

which shows (2.11).

Let $m \ge 0$. Then we can have by (2.17) and (2.18)

$$
\begin{aligned}
\|x_{k+m} - x_k\| &\le \|x_{k+m} - x_{k+m-1}\| + \|x_{k+m-1} - x_{k+m-2}\|\\
&\quad + \cdots + \|x_{k+1} - x_k\|\\
&\le c\left[\|F(x_{k+m-1})\| + \|F(x_{k+m-2})\| + \cdots + \|F(x_k)\|\right]\\
&\le \frac{c}{d}\left[\alpha^{(1+p)^{k+m-1}} + \cdots + \alpha^{(1+p)^k}\right]\\
&\le \frac{c}{d}\left[\alpha^{(1+p)(m-1)} + \cdots + 1\right]\alpha^{(1+p)^k}\\
&\le \frac{c}{d}\,\frac{1-\alpha^{m(1+p)}}{1-\alpha^{1+p}}\alpha^{(1+p)^k} \longrightarrow_{m\to\infty} \beta\alpha^{(1+p)^k}. \quad (2.19)
\end{aligned}
$$

It follows from (2.19) that sequence $\{x_n\}$ is Cauchy on a Banach space X and as such it converges to some $x^* \in \overline{U}(x_0, r)$ (since $\overline{U}(x_0, r)$ is a closed set). By letting $k \to \infty$ in (2.16) we obtain $F(x^*) = 0$.

Finally to show uniqueness, let $y^* \in \overline{U}(x_0, r_0)$ be a solution of equaiton $F(x) = 0$.

In view of (1.2) we obtain the approximation

$$\begin{aligned} x_{k+1} - y^* &= -G(x_k)[F(x_k) - F(y^*) - F'(x_k)(x_k - y^*)] \\ &\quad - [G(x_k)F'(x_k) - I](x_k - y^*). \end{aligned} \tag{2.20}$$

Using (2.1), (2.2), (2.3), (2.9), and (2.10) we get

$$\begin{aligned} \|x_{k+1} - y^*\| &\leq \frac{ca}{1+p}\|x_k - y^*\|^{1+p} + b\|F(x_k)\|^q \|x_k - y^*\| \\ &\leq \left[\frac{ca}{1+p}\|x_k - y^*\|^p + b\|F(x_k)\|^q\right] \|x_k - y^*\| \\ &< \left[\frac{ca}{1+p}\|x_0 - y^*\|^p + b\|F(x_0)\|^q\right] \|x_k - y^*\| \\ &\leq \left[\frac{ca}{1+p}r_0^p + b\|F(x_0)\|^q\right] \|x_k - y^*\| \\ &\leq \|x_k - y^*\|. \end{aligned} \tag{2.21}$$

That is we showed for all k

$$\|x_{k+1} - y^*\| < \|x_k - y^*\|, \tag{2.22}$$

from which we get $\lim_{k\to\infty} x_k = y^*$. But we know $\lim_{k\to\infty} x_k = x^*$.

Hence, we deduce

$$x^* = y^*.$$

That completes the proof of Theorem 2.1. □

Remark 2.2. If

$$p = q = 1, \quad a = 2M, \quad b = M \quad \text{and} \quad c = M \tag{2.23}$$

our Theorem 2.1 essentially reduces to Theorem 2.3 in [6, p. 277]. Otherwise (if $p = q = 1$) it constitutes an improvement over it. Note also that the uniqueness of the solution x^* was not studied in [6].

We state the local convergence result for method (1.2).

Theorem 2.3. *Under hypotheses (2.1)–(2.4) further assume there exists a simple solution x^* of equation $F(x) = 0$ and*

$$\overline{U}(x^*, r) \subseteq D, \tag{2.24}$$

where r is given by (2.7) and satisfies (2.9) as equality.

Then sequence $\{x_n\}$ $(n \geq 0)$ generated by Newton-like method (1.2) is well defined, remains in $\overline{U}(x^*, r)$ for all $n \geq 0$ and converges to x^*.

Proof. Using (2.20) and y^* replaced by x^* as in (2.21) we arrive at

$$\|x_{k+1} - x^*\| < \|x_k - x^*\| \leq r, \tag{2.25}$$

which imply $x_k \in \overline{U}(x^*, r)$ for all k and $\lim_{k\to\infty} x_k = x^*$.

That completes the proof of Theorem 2.3. □

Note that the local convergence of method (1.2) was not studied in [6]. Applications of our results can also be found in [4], [6].

Remark 2.4. (a) The results obtained here hold in a weaker Newton–Mysovskikh-type setting where (2.1)–(2.3) are replaced by

$$\|G(x)[F'(y + t(x - y)) - F'(y)](x - y)\| \leq a_1 t^p \|x - y\|^{1+p}, \tag{2.26}$$

and

$$\|G(x)[F'(x)G(x) - I]\| \leq b_1 \|F(x)\|^q. \tag{2.27}$$

Note that

$$a_1 \leq ca, \tag{2.28}$$

and

$$b_1 \leq cb \tag{2.29}$$

hold in general. However in this case we only arrive at $G(x^*)F(x^*) = 0$, which does not necessarily imply $F(x^*) = 0$ unless if we have additionally assumed that

$$G(x^*)F(x^*) = 0 \Rightarrow F(x^*) = 0, \tag{2.30}$$

or

$$G(x)F(x) = 0 \Rightarrow F(x) = 0 \quad (x \in D). \tag{2.31}$$

(b) In case $G(x) = A(x)^{-1} \in L(Y, X)$ the results obtained here hold for the iteration

$$x_{n+1} = x_n - A(x_n)^{-1}F(x_n) \quad (n \geq 0). \tag{2.32}$$

Moreover if conditions (2.26) and (2.27) hold then additional conditions (2.30) or (2.31) are not needed since in those cases $G(x^*)F(x^*) = 0$ imply $F(x^*) = 0$.

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