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FUNCTIONS CLOSE-TO-CONVEX AND QUASI-CONVEX WITH RESPECT TO OTHER POINTS

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Abstract: In [6], Sakaguchi introduced the class of functions starlike with respect to symmetric points. Then, El-Ashwah and Thomas in [2], introduced two more classes, namely starlike with respect to conjugate points and starlike with respect to symmetric conjugate points. Das and Singh in [1], considered the class of functions convex and close-to-convex with respect to symmetric points, respectively. This paper considers other classes, namely functions close-to-convex with respect to conjugate points and functions close-to-convex with respect to symmetric conjugate points. Next, the classes of functions quasi-convex with respect to symmetric points, with respect to conjugate points and with respect to symmetric conjugate points are introduced. We give some properties for functions belonging to these classes. The behaviour of certain integral operators are also considered.

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1. Introduction

Let $\mathcal{S}$ be the class of analytic functions f, univalent in the unit disc $\mathcal{D} = \{z : |z| < 1\}$, with

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

In [6], Sakaguchi introduced the class $S_s^\star$ of analytic functions f, normalised by (1) which are starlike with respect to symmetric points. Two other classes introduced by El-Ashwah and Thomas in [2] are functions starlike with respect to conjugate points, $S_c^\star$, and functions starlike with respect to symmetric conjugate points, $S_{sc}^\star$. As a generalization of the Sakaguchi functions, Das and Singh in [1], introduced the class C_s and K_s of analytic functions f, normalised by (1) which are convex and close-to-convex with respect to symmetric points respectively.

This paper concentrates on the class K_s and two other classes, namely functions close-to-convex with respect to conjugate points and functions close-to-convex with respect to symmetric conjugate points. Next, we look at the class of functions quasi-convex with respect to symmetric points and two other classes, namely functions quasi-convex with respect to conjugate points and functions quasi-convex with respect to symmetric conjugate points. Before we begin defining the notion of all the classes, consider first three classes of functions, C_s, C_c and C_{sc}.

Definition 1.1. Let f be analytic in $\mathcal{D}$ with $f(0) = f'(0) - 1 = 0$. Then:

(i) $f \in C_s$ is said to be convex with respect to symmetric points if and only if

$$\text{Re}\ \left\{ \frac{(zf'(z))'}{(f(z) - f(-z))'} \right\} > 0;$$

(ii) $f \in C_c$ is said to be convex with respect to conjugate points if and only if

$$\text{Re}\ \left\{ \frac{(zf'(z))'}{(f(z) + \overline{f(\bar{z})})'} \right\} > 0;$$

(iii) $f \in C_{sc}$ is said to be convex with respect to symmetric conjugate points if and only if

$$\text{Re}\ \left\{ \frac{(zf'(z))'}{(f(z) - \overline{f(-\bar{z})})'} \right\} > 0.$$

Note. Definitions in (i), (ii) and (iii) above are also respectively equivalent to the following:

(i) $f \in C_s$ if and only if there is a $h = zf'(z) \in S_s^\star$ such that

$$\text{Re}\left\{\frac{zh'(z)}{h(z)-h(-z)}\right\} > 0;$$

(ii) $f \in C_c$ if and only if there is a $h = zf'(z) \in S_c^\star$ such that

$$\text{Re}\left\{\frac{zh'(z)}{h(z)+\overline{h(\bar{z})}}\right\} > 0;$$

(iii) $f \in C_{sc}$ if and only if there is a $h = zf'(z) \in S_{sc}^\star$ such that

$$\text{Re}\left\{\frac{zh'(z)}{h(z)-\overline{h(-\bar{z})}}\right\} > 0.$$

2. Close-to-Convex with Respect to Other Points

Definition 2.1. Let f be analytic in $\mathcal{D}$ with $f(0) = f'(0) - 1 = 0$. Then:
(i) $f \in K_s$ is said to be close-to-convex with respect to symmetric points if there exists a $g \in C_s$ such that

$$\text{Re}\left\{\frac{f'(z)}{g'(z)+g'(-z)}\right\} > 0;$$

(ii) $f \in K_c$ is said to be close-to-convex with respect to conjugate points if there exists a $g \in C_c$ such that

$$\text{Re}\left\{\frac{f'(z)}{g'(z)+\overline{g'(\bar{z})}}\right\} > 0;$$

(iii) $f \in K_{sc}$ is said to be close-to-convex with respect to symmetric conjugate points if there exists a $g \in C_{sc}$ such that

$$\text{Re}\left\{\frac{f'(z)}{g'(z)+\overline{g'(-\bar{z})}}\right\} > 0.$$

Note. Definitions in (i), (ii) and (iii) above are also respectively equivalent to the following:

(i) $f \in K_s$, if there exists a $h = zg' \in S_s^\star$ such that

$$\mathrm{Re}\ \left\{\frac{zf'(z)}{h(z)-h(-z)}\right\} > 0;$$

(ii) $f \in K_c$, if there exists a $h = zg' \in S_c^\star$ such that

$$\mathrm{Re}\ \left\{\frac{zf'(z)}{h(z)+\overline{h(\bar{z})}}\right\} > 0;$$

(iii) $f \in K_{sc}$, if there exists a $h = zg' \in S_{sc}^\star$ such that

$$\mathrm{Re}\ \left\{\frac{zf'(z)}{h(z)-\overline{h(-\bar{z})}}\right\} > 0.$$

We list some preliminary lemmas, required for proving our results.

Lemma 2.1. (see [4]) *If M and N are analytic in $\mathcal{D}$ and $M(0) = 0 = N(0)$, N maps $\mathcal{D}$ onto a many sheeted region which is starlike with respect to origin and $(M'/N') \in \mathcal{P}$, then $(M/N) \in \mathcal{P}$, where $\mathcal{P}$ is the class of all analytic functions ϕ such that $\phi(0) = 1$, $\mathrm{Re}\,\{\phi(z)\} > 0$.*

Lemma 2.2. (see [1]) *If $g \in C_s$ then $\frac{1}{2}\,[g(z) - g(-z)] \in C$, the class of convex functions.*

Lemma 2.3. *If $g \in C_c$ then $\frac{1}{2}\left[g(z) + \overline{g(\bar{z})}\right] \in C$, the class of convex functions.*

Lemma 2.4. *If $g \in C_{sc}$ then $\frac{1}{2}\left[g(z) - \overline{g(-\bar{z})}\right] \in C$, the class of convex functions.*

Lemma 2.5. *Let $g \in C_s$. Then the function G defined by*

$$G(z) = \frac{a+1}{2z^a}\int_0^z\ t^{a-1}[g(t)-g(-t)]dt, \tag{2}$$

also belongs to C_s for $z \in \mathcal{D}$ and $a > 0$.

Proof. Since $g \in C_s$, (2) gives

$$\frac{2(zG'(z))'}{(G(z)-G(-z))'} = \frac{z^{a+1}[g(z)-g(-z)]' - az^a[g(z)-g(-z)]}{z^a[g(z)-g(-z)] - a\int_0^z\ t^{a-1}[g(t)-g(-t)]dt}$$
$$+\ \frac{a^2\int_0^z\ t^{a-1}[g(t)-g(-t)]dt}{z^a[g(z)-g(-z)] - a\int_0^z\ t^{a-1}[g(t)-g(-t)]dt} = \frac{M(z)}{N(z)}, \quad \text{say.}$$

Note that $M(0) = N(0) = 0$ and for $g \in C_s$,

$$\begin{aligned} \operatorname{Re}\left(1+\frac{zN''(z)}{N'(z)}\right) &= a+\operatorname{Re}\left(1+\frac{z[g(z)-g(-z)]''}{(g(z)-g(-z))'}\right) \\ &> a+0=a. \end{aligned}$$

Thus $N(z)$ is starlike if and only if $a > 0$. Furthermore, since

$$\operatorname{Re}\frac{M'(z)}{N'(z)} = \operatorname{Re}\left(\frac{[z[g(z)-g(-z)]']'}{[g(z)-g(-z)]'}\right) > 0.$$

Lemma 2.1 shows that $G \in C_s$. □

Lemma 2.6. *Let $g \in C_c$. Then the function G defined by*

$$G(z) = \frac{a+1}{2z^a}\int_0^z t^{a-1}\left[g(t)+\overline{g(\bar{t})}\right]dt, \tag{2}$$

also belongs to C_c for $z \in \mathcal{D}$ and $a > 0$.

Proof. Let $g \in C_c$. Halim in [3] proved that (2.6) gives

$$\overline{\int_0^{\bar{z}} t^{a-1}\left[g(t)+\overline{g(\bar{t})}\right]dt} = \int_0^z t^{a-1}\left[g(t)+\overline{g(\bar{t})}\right]dt. \tag{2}$$

Thus

$$\begin{aligned} \frac{2(zG'(z))'}{\left(G(z)+\overline{G(\bar{z})}\right)'} &= \frac{z^{a+1}\left[g(z)+\overline{g(\bar{z})}\right]' - az^a\left[g(z)+\overline{g(\bar{z})}\right]}{z^a\left[g(z)+\overline{g(\bar{z})}\right] - a\int_0^z t^{a-1}\left[g(t)+\overline{g(\bar{t})}\right]dt} \\ &+ \frac{a^2\int_0^z t^{a-1}\left[g(t)+\overline{g(\bar{t})}\right]dt}{z^a\left[g(z)+\overline{g(\bar{z})}\right] - a\int_0^z t^{a-1}\left[g(t)+\overline{g(\bar{t})}\right]dt} = \frac{M(z)}{N(z)}, \end{aligned}$$

where $M(0) = N(0) = 0$ and $N \in S^\star$ for $a > 0$. By using Lemma 2.1, it follows that $G \in C_c$. □

Lemma 2.7. *Let $g \in C_{sc}$. Then the function G defined by*

$$G(z) = \frac{a+1}{2z^a}\int_0^z t^{a-1}\left[g(t)-\overline{g(-\bar{t})}\right]dt, \tag{2}$$

also belongs to C_{sc} for $z \in \mathcal{D}$ and $a > 0$.

Proof. Let $g \in C_{sc}$. Again, Halim in [3] proved that (2.7) gives

$$\overline{G(-\bar{z})} = \frac{-(a+1)}{2z^a} \int_0^z t^{a-1} \left[g(t) - \overline{g(-\bar{t})} \right] dt.$$

Thus

$$\frac{2(zG'(z))'}{\left(G(z) - \overline{G(-\bar{z})}\right)'} = \frac{z^{a+1}\left[g(z) - \overline{g(-\bar{z})}\right]' - az^a\left[g(z) - \overline{g(-\bar{z})}\right]}{z^a\left[g(z) - \overline{g(-\bar{z})}\right] - a\int_0^z t^{a-1}\left[g(t) - \overline{g(-\bar{t})}\right] dt}$$
$$+ \frac{a^2\int_0^z t^{a-1}\left[g(t) - \overline{g(-\bar{t})}\right] dt}{z^a\left[g(z) - \overline{g(-\bar{z})}\right] - a\int_0^z t^{a-1}\left[g(t) - \overline{g(-\bar{t})}\right] dt} = \frac{M(z)}{N(z)},$$

where $M(0) = N(0) = 0$ and $N \in S^\star$ for $a > 0$. Hence, Lemma 2.1 shows that $G \in C_{sc}$. □

In [1], Das and Singh showed that $K_s \subset K$, where K is the class of close-to-convex functions. Using Lemma 2.3 and Lemma 2.4 respectively, one can trivially show that $K_c \subset K$ and $K_{sc} \subset K$ and thus forms subclasses of univalent functions.

Next, we need the following lemma.

Lemma 2.8. *If* $f \in K_s$ *then* $\mathrm{Re}\ \left(\frac{(f(z)-f(-z))'}{(g(z)-g(-z))'}\right) > 0.$

Proof. Let $f \in K_s$. Then, there is a function $g \in C_s$ such that

$$\mathrm{Re}\ \left(\frac{f'(z)}{g'(z)+g'(-z)}\right) > 0, \quad z \in \mathcal{D}. \tag{2}$$

Put $-z$ for z in (2), we have

$$\mathrm{Re}\ \left(\frac{f'(-z)}{g'(-z)+g'(z)}\right) > 0. \tag{2}$$

We combine (2) and (2), and obtain:

$$\mathrm{Re}\ \left(\frac{f'(z)}{g'(z)+g'(-z)} + \frac{f'(-z)}{g'(-z)+g'(z)}\right) = \mathrm{Re}\ \left(\frac{(f(z)-f(-z))'}{(g(z)-g(-z))'}\right) > 0.$$

This implies the result. □

Theorem 2.1. *Let $f \in K_s$. Then the function F defined by*

$$F(z) = \frac{a+1}{2z^a} \int_0^z t^{a-1}[f(t) - f(-t)]dt, \tag{2}$$

also belongs to K_s for $z \in \mathcal{D}$ and $a > 0$.

Proof. Since $f \in K_s$, there is a function $g \in C_s$. Lemma 2.5 states that $G \in C_s$. From the representation of F,

$$\begin{aligned} \frac{2F'(z)}{(G(z) - G(-z))'} &= \frac{z^a[f(z) - f(-z)] - a\int_0^z t^{a-1}[f(t) - f(-t)]dt}{z^a[g(z) - g(-z)] - a\int_0^z t^{a-1}[g(t) - g(-t)]dt} \\ &= \frac{M(z)}{N(z)}, \quad \text{say.} \end{aligned}$$

Note that $M(0) = N(0) = 0$ and for $g \in C_s$,

$$\begin{aligned} \text{Re}\left(1 + \frac{zN''(z)}{N'(z)}\right) &= a + \text{Re}\left(1 + \frac{z[g(z) - g(-z)]''}{(g(z) - g(-z))'}\right) \\ &> a + 0 = a. \end{aligned}$$

Thus $N(z)$ is starlike as $a > 0$. Furthermore, since by Lemma 2.8

$$\text{Re}\,\frac{M'(z)}{N'(z)} = \text{Re}\left(\frac{[f(z) - f(-z)]'}{[g(z) - g(-z)]'}\right) > 0.$$

Lemma 2.1 shows that $F \in K_s$. □

Lemma 2.9. *If $f \in K_c$ then* Re $\left(\frac{\left(f(z)+\overline{f(\bar{z})}\right)'}{\left(g(z)+\overline{g(\bar{z})}\right)'}\right) > 0$.

Proof. Let $f \in K_c$. Then, there is a function $g \in C_c$ such that

$$\text{Re}\left\{\frac{f'(z)}{\left(g(z) + \overline{g(\bar{z})}\right)'}\right\} > 0, \quad z \in \mathcal{D}. \tag{-5}$$

Put $\bar{z}$ for z in (2) and taking conjugate, we have

$$\text{Re}\left\{(\frac{\overline{f'(\bar{z})}}{\left(g(z) + \overline{g(\bar{z})}\right)'}\right\} > 0. \tag{-5}$$

We combine (2) and (2), to give

$$\begin{aligned} \operatorname{Re}\left\{\frac{f'(z)}{\left(g(z)+\overline{g(\bar{z})}\right)'}+\frac{\overline{f'(\bar{z})}}{\left(g(z)+\overline{g(\bar{z})}\right)'}\right\} &= \operatorname{Re}\left\{\frac{\left(f(z)+\overline{f(\bar{z})}\right)'}{\left(g(z)+\overline{g(\bar{z})}\right)'}\right\} \\ &> 0. \end{aligned}$$

This implies the result. □

Theorem 2.2. *Let $f \in K_c$. Then the function F defined by*

$$F(z)=\frac{a+1}{2z^a}\int_0^z t^{a-1}\left[f(t)+\overline{f(\bar{t})}\right]dt, \tag{-8}$$

also belongs to K_c for $z \in \mathcal{D}$ and $a > 0$.

Proof. Since $f \in K_c$, there is a function $g \in C_c$. Lemma 2.6 states that $G \in C_c$. (2) gives

$$\begin{aligned} \frac{2F'(z)}{\left(G(z)+\overline{G(\bar{z})}\right)'} = &= \frac{z^a[f(z)+\overline{f(\bar{z})}]-a\int_0^z t^{a-1}[f(t)+\overline{f(\bar{t})}]dt}{z^a[g(z)+\overline{g(\bar{z})}]-a\int_0^z t^{a-1}[g(t)+\overline{g(\bar{t})}]dt} \\ &= \frac{M(z)}{N(z)}, \end{aligned}$$

where $M(0)=N(0)=0$ and $N \in S^\star$ for $a > 0$. By Lemma 2.9, it can be shown that $\operatorname{Re}\frac{M'(z)}{N'(z)} > 0$ and hence using Lemma 2.1, it follows that $F \in K_c$. □

Lemma 2.10. *If $f \in K_{sc}$ then* $\operatorname{Re}\left(\frac{\left(f(z)-\overline{f(-\bar{z})}\right)'}{\left(g(z)-\overline{g(-\bar{z})}\right)'}\right) > 0.$

Using similar method and Lemma 2.10, analogous results for functions f belonging to K_{sc} are obtained.

Theorem 2.3. *Let $f \in K_{sc}$. Then the function F defined by*

$$F(z)=\frac{a+1}{2z^a}\int_0^z t^{a-1}[f(t)-\overline{f(-\bar{t})}]dt, \tag{-11}$$

also belongs to K_{sc} for $z \in \mathcal{D}$ and $a > 0$.

3. Quasi-Convex with Respect to Other Points

Now, we look at the class of functions which are quasi-convex with respect to symmetric points, quasi-convex with respect to conjugate points and quasi-convex with respect to symmetric conjugate points. All 3 classes are contained in $K^\star$, the class of quasi-convex functions introduced by Noor in [5].

Definition 3.1. Let f be analytic in $\mathcal{D}$ with $f(0) = f'(0) - 1 = 0$. Then:

(i) $f \in K_s^\star$ is said to be quasi-convex with respect to symmetric points if there exists a $g \in C_s$ such that

$$\text{Re} \left\{ \frac{(zf'(z))'}{(g(z) - g(-z))'} \right\} > 0;$$

(ii) $f \in K_c^\star$ is said to be quasi-convex with respect to conjugate points if there exists a $g \in C_c$ such that

$$\text{Re} \left\{ \frac{(zf'(z))'}{\left(g(z) + \overline{g(\bar{z})}\right)'} \right\} > 0;$$

(iii) $f \in K_{sc}^\star$ is said to be quasi-convex with respect to symmetric conjugate points if there exists a $g \in C_{sc}$ such that

$$\text{Re} \left\{ \frac{(zf'(z))'}{\left(g(z) - \overline{g(-\bar{z})}\right)'} \right\} > 0.$$

Note. Definitions in (i), (ii) and (iii) above are also respectively equivalent to the following:

(i) $f \in K_s^\star$, if there exists a $h = zg' \in S_s^\star$ such that

$$\text{Re} \left\{ \frac{z(zf'(z))'}{h(z) - h(-z)} \right\} > 0;$$

(ii) $f \in K_c^\star$, if there exists a $h = zg' \in S_c^\star$ such that

$$\text{Re} \left\{ \frac{z(zf'(z))'}{h(z) + \overline{h(\bar{z})}} \right\} > 0;$$

(iii) $f \in K_{sc}^\star$, if there exists a $h = zg' \in S_{sc}^\star$ such that

$$\text{Re} \left\{ \frac{z(zf'(z))'}{h(z) - \overline{h(-\bar{z})}} \right\} > 0.$$

Using Lemma 2.2, Lemma 2.3 and Lemma 2.4, one can trivially show respectively that $K_s^\star \subset K^\star$, $K_c^\star \subset K^\star$ and $K_{sc}^\star \subset K^\star$ and thus forms subclasses of univalent functions.

The following lemmas can be trivially shown using the same approaches as in previous lemmas.

Lemma 3.1. *If* $f \in K_s^\star$ *then* $\text{Re}\ \left\{\frac{(z(f(z)-f(-z))')'}{(g(z)-g(-z))'}\right\} > 0.$

Lemma 3.2. *If* $f \in K_c^\star$ *then* $\text{Re}\ \left\{\frac{\left(z\left(f(z)+\overline{f(\bar{z})}\right)'\right)'}{\left(g(z)+\overline{g(\bar{z})}\right)'}\right\} > 0.$

Lemma 3.3. *If* $f \in K_{sc}^\star$ *then* $\text{Re}\ \left\{\frac{\left(z\left(f(z)-\overline{f(-\bar{z})}\right)'\right)'}{\left(g(z)-\overline{g(-\bar{z})}\right)'}\right\} > 0.$

Theorem 3.1. *Let* $f \in K_s^\star$. *Then the function* F *as defined by (2.1) also belongs to* $K_s^\star$ *for* $z \in \mathcal{D}$ *and* $a > 0$.

Proof. Since $f \in K_s^\star$, there is a function $g \in C_s$. Lemma 2.5 states that $G \in C_s$. From the representation of F,

$$\begin{aligned}\frac{2(zF'(z))'}{(G(z)-G(-z))'} &= \frac{z^{a+1}[f(z)-f(-z)]' - az^a[f(z)-f(-z)]}{z^a[g(z)-g(-z)] - a\int_0^z t^{a-1}[g(t)-g(-t)]dt} \\ &\quad + \frac{a^2\int_0^z t^{a-1}[f(t)-f(-t)]dt}{z^a[g(z)-g(-z)] - a\int_0^z t^{a-1}[g(t)-g(-t)]dt} \\ &= \frac{M(z)}{N(z)}, \quad \text{say.}\end{aligned}$$

Note that $M(0) = N(0) = 0$ and for $g \in C_s$,

$$\begin{aligned}\text{Re}\ \left(1 + \frac{zN''(z)}{N'(z)}\right) &= a + \text{Re}\ \left(1 + \frac{z[g(z)-g(-z)]''}{(g(z)-g(-z))'}\right) \\ &> a + 0 = a.\end{aligned}$$

Thus $N(z)$ is starlike as $a > 0$. Furthermore, since by Lemma 3.1

$$\text{Re}\ \frac{M'(z)}{N'(z)} = \text{Re}\ \left(\frac{[z(f(z)-f(-z))']'}{[g(z)-g(-z)]'}\right) > 0.$$

Lemma 2.1 shows that $F \in K_s^\star$. □

Theorem 3.2. *Let* $f \in K_c^\star$. *Then the function* F *as defined by (2.2) also belongs to* $K_c^\star$ *for* $z \in \mathcal{D}$ *and* $a > 0$.

Proof. Since $f \in K_c^\star$, there is a function $g \in C_c$. Lemma 2.6 states that $G \in C_c$. (2) gives

$$\begin{aligned}\frac{2(zF'(z))'}{\left(G(z)+\overline{G(\bar z)}\right)'} &= \frac{z^{a+1}\left(f(z)+\overline{f(\bar z)}\right)' - az^a\left(f(z)+\overline{f(\bar z)}\right)}{z^a\left(g(z)+\overline{g(\bar z)}\right) - a\int_0^z t^{a-1}\left(g(t)+\overline{g(\bar t)}\right)dt} \\ &+ \frac{a^2\int_0^z t^{a-1}\left(f(t)+\overline{f(\bar t)}\right)dt}{z^a\left(g(z)+\overline{g(\bar z)}\right) - a\int_0^z t^{a-1}\left(g(t)+\overline{g(\bar t)}\right)dt} \\ &= \frac{M(z)}{N(z)},\end{aligned}$$

where $M(0) = N(0) = 0$ and $N \in S^\star$ for $a > 0$. Using Lemma 3.2 followed by Lemma 2.1, gives the result. □

Now, we give similar result for functions in $K_{sc}^\star$.

Theorem 3.3. *Let $f \in K_{sc}^\star$. Then the function F as defined by (2.3) also belongs to $K_{sc}^\star$ for $z \in \mathcal{D}$ and $a > 0$.*

Proof. Since $f \in K_{sc}^\star$, there is a function $g \in C_{sc}$. Lemma 2.7 states that $G \in C_{sc}$. Thus

$$\begin{aligned}\frac{2(zF'(z))'}{\left(G(z)-\overline{G(-\bar z)}\right)'} &= \frac{z^{a+1}\left(f(z)-\overline{f(-\bar z)}\right)' - az^a\left(f(z)-\overline{f(-\bar z)}\right)}{z^a\left(g(z)-\overline{g(-\bar z)}\right) - a\int_0^z t^{a-1}\left(g(t)-\overline{g(-\bar t)}\right)dt} \\ &+ \frac{a^2\int_0^z t^{a-1}\left(f(t)-\overline{f(-\bar t)}\right)dt}{z^a\left(g(z)-\overline{g(-\bar z)}\right) - a\int_0^z t^{a-1}\left(g(t)-\overline{g(-\bar t)}\right)dt} \\ &= \frac{M(z)}{N(z)},\end{aligned}$$

where $M(0) = N(0) = 0$ and $N \in S^\star$ for $a > 0$. As before, using Lemma 3.3 followed by Lemma 2.1, $F \in K_{sc}^\star$ can be easily be obtained. □

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