International Journal of Pure and Applied Mathematics
Volume 30 No. 2 2006, 237-246

ON THE MATHEMATICAL MODEL OF THE MESSAGE SWITCHING SYSTEM

Stasys Steišūnas
Institute of Mathematics and Informatics
Akademijos 4, Vilnius, LT-08663, LITHUANIA
e-mail: stst@ktl.mii.lt

Abstract: The object of this research in the sphere of queueing theory is the theorem about the law of the iterated logarithm under the conditions of heavy traffic in multiphase queueing systems and its application to the model of the message switching system. First the law of the iterated logarithm is proved for values of important probabilistic characteristics of the queueing system investigated as well as sojourn time of a customer. Finally we present an application of the proved theorem for the mathematical model of the message switching system.

AMS Subject Classification: 60K25, 60G70, 60F17
Key Words: mathematical models of technical systems, queueing theory, multiphase queueing systems, heavy traffic, a law of the iterated logarithm, sojourn time of a customer

1. Introduction

At first, the law of the iterated logarithm is considered by investigating the sojourn time of a customer in multiphase queueing systems. The multiphase queueing system is a queueing system when a customer does not visit the same queueing node twice (see, for example, Karpelevich et al [13]). Therefore, such a system is a special case of the open Jackson network.

Interest in the field of multiphase queueing systems has been stimulated by the theoretical values of the results as well as by their possible applications in information and computing systems, communication networks, and automated technological processes (see, for example, Saati et al [25]). The methods of investigation of single-phase queueing systems are considered in Borovkov

Received: June 15, 2006

[2, 3], etc. The asymptotic analysis of models of queueing systems in heavy traffic is of special interest (see, for example, Kingman [14, 15], Iglehart et al [9, 10], etc.). The papers Kobyashi [17], Reiman [22] and others desribed the beginning of the investigation of diffusion approximation to queueing networks. Intermediate models – multiphase queueing systems – are considered rarer due to serious technical difficulties (see, for example, book Karpelevich et al [13]). The works on sojourn time for the multiphase queueing systems and open Jackson networks in heavy traffic are also sparse. In Szczotka et al [28], it is proved that the stationary distribution of random vectors of sojourn times in multiphase queueing systems is given as a functional of a Brownian motion. The papers Boxma [4], Boxma et al [5], Karpelevich et al [12] and Knessl et al [16] investigated the distribution of sojourn times in multiphase queueing systems with identical service times. Reiman [21] and Coffman et al [7] presented the proof of an expression for the stationary distribution of the diffusion approximation for sojourn times in open Jackson networks. The papers Reiman et al [23, 24] investigated the sojourn time distribution in networks of priority queues. In Zhang Han Qin et al [30] and Chen Hong et al [6], applying the method of strong approximations, sojourn time processes in open and multiclass feedforward queueing networks are investigated. The papers Reiman et al [23] and Harrison et al [8] deal with the simulation of sojourn time distribution in open queueing networks.

Let the sojourn time of a customer in the phases of a queueing system be unrestricted, the principle of service being "first come, first served". All the random variables studied are defined on one basic probability space $(\Omega, \mathfrak{F}, \mathbf{P})$.

We present some definitions in the theory of metric spaces (see, for example, Billinsley [1]).

Let $\mathbf{C}$ be a metric space consisting of real continuous functions in [0, 1] with a uniform metric

$$\rho(x, y) = \sup_{0 \leq t \leq 1} |x(t) - y(t)|, \quad x, y \in \mathbf{C}.$$

Let $\mathbf{D}$ be a space of all real-valued right-continuous functions in [0,1] having left limits and endowed with the Skorokhod topology induced by the metric d (under which $\mathbf{D}$ is complete and separable). Also, note that $d(x, y) \leq \rho(x, y)$ for $x, y \in D$.

In this paper, we will constantly use an analog of the theorem on converging together (see, for example, Iglehart [11]):

Theorem 1.1. *Let $\varepsilon > 0$ and $\boldsymbol{X}_n,\ \boldsymbol{Y}_n,\ \boldsymbol{X} \in \boldsymbol{D}$. If*

$$\boldsymbol{P}\left(\overline{\lim_{n\to\infty}}\, d(\boldsymbol{X}_n, \boldsymbol{X}) > \varepsilon\right) = 0 \ \ \textit{and} \ \ \boldsymbol{P}\left(\overline{\lim_{n\to\infty}}\, d(\boldsymbol{X}_n, \boldsymbol{Y}_n) > \varepsilon\right) = 0,$$

then

$$\boldsymbol{P}\left(\overline{\lim_{n\to\infty}}\, d(\mathbf{Y}_n, \mathbf{X}) > \varepsilon\right) = 0. \tag{1}$$

Finally, we present Theorem 6.1 from Whitt [29], p. 80.

Let $S = R$ and T have a closed left endpoint 0.

For any $x \in D$, let $x^{\uparrow}(t) = \sup_{0\le s\le t} x(s),\ t \in T$. The supremum function $\uparrow$ is easily seen to be continuous in each of Skorohod's topologies (see, for example, Whitt [29]).

Theorem 1.2. *For all $x, y \in D$,*

$$d(x^{\uparrow}, y^{\uparrow}) \le d(x, y)\,. \tag{2}$$

In this paper, the law of the iterated logarithm for the sojourn time of multiphase queueing systems in conditions of heavy traffic is proved. The main tool for the analysis of multiphase queueing systems in heavy traffic is a functional law of the iterated logarithm for partial sums of independent identically distributed random variables (the proof can be found in Strassen [27] and Iglehart [11]).

2. Statement of the Problem

We investigate here a k-phase queue (i.e., after a customer has been served in the j-th phase of the queue, he is routed to the $j+1$-st phase of the queue, and, after the service in the k-th phase of the queue, he leaves the queue). Let us denote by t_n the time of arrival of the n-th customer; by $S_n^{(j)}$– the service time of the n-th customer in the j-th phase; $z_n = t_{n+1} - t_n$; and by $\tau_{j,n+j}$ – departure of the n-th customer from the j-th phase of the queue, $j = 1, 2, \cdots, k$.

Let interarrival times (z_n) at the multiphase queueing system and service times $(S_n^{(j)})$ in each phase of the queue for $j = 1, 2, \cdots, k$ be mutually independent identically distributed random variables.

Next, denote by $W_n^{(j)}$ the waiting time of the n-th customer in the j-th phase of the queue; $T_{j,n} = \sum_{i=1}^{j}(W_n^{(i)} + S_n^{(i)})$ stands for the sojourn time of the n-th customer (time, which the n-th customer spent in the queueing system until the j-th phase), $j = 1, 2, \ldots, k$.

We form such a modified multiphase queueing system in which $W_n^{(j)} = 0,\ j = 1, 2, \ldots, k,\ n < k$. Limit distributions for a modified multiphase queueing system and the usual multiphase queueing system which, working in heavy traffic conditions, are coincidental (see, for example, Iglehart [9]). Thus, later we can investigate only the modified multiphase queueing system and assume that $n \geq k$.

When $j = 1, 2, \ldots, k$, let

$$\delta_{j,n} = \begin{cases} S^{(j)}_{n-(j-1)} - z_n, & \text{if } n \geq k\,, \\ 0, & \text{if } n < k. \end{cases}$$

Let us denote $S_{j,n} = \sum_{l=1}^{n-1} \delta_{j,l}$, $S_{0,n} \equiv 0$, $\hat{S}_{j,n} = S_{j-1,n} - S_{j,n}$, $x_{j,n} = \tau_{j,n} - t_n$, $x_{0,n} \equiv 0$, $\hat{x}_{j,n+1} = x_{j,n} - \delta_{j,n+1}$, $\hat{x}_{0,n} \equiv 0$, $y_{j,n} = \hat{x}_{j,n+(j-2)} - \hat{x}_{j-1,n+(j-2)}$, $\hat{y}_{j,n} = \hat{x}_{j,n} - \hat{x}_{j-1,n}$, $\hat{\delta}_{j,n} = \delta_{j,n+(j-2)} - \delta_{j-1,n+(j-2)}$, $\alpha_j = M\delta_{j,n}$, $\alpha_0 \equiv 0$, $Dz_n = \sigma_0^2$, $DS_n^{(j)} = \sigma_j^2$, $\tilde{\sigma}_j^2 = \sigma_0^2 + \sigma_j^2$, $S_n^{(0)} = z_n$, $j = 1, 2, \ldots, k$, $\hat{\delta}_n = \max\limits_{1 \leq j \leq k} \max\limits_{0 \leq l \leq 2n} |\delta_{j,l}|$, $[x]$ as the integer part of number x.

We assume that the following conditions are fulfilled:

there exists a constant $\gamma > 0$ such that

$$\sup_{n \geq 1} M|S_n^{(j)}|^{4+\gamma} < \infty, \quad j = 0, 1, 2, \ldots, k \tag{3}$$

and

$$\alpha_k > \alpha_{k-1} > \cdots > \alpha_1 > 0. \tag{4}$$

In this paper, we mostly use the equations presented in Minkevičius [18]:

$$\begin{aligned} \hat{x}_{j,n} &= \max_{0 \leq l \leq n} (\hat{x}_{j-1,l} - S_{j,l}) + S_{j,n}, \ \ \hat{x}_{0,n} \equiv 0,\ n \geq k, \\ x_{j,n} &= \max(x_{j-1,n-1} + \delta_{j,n};\ x_{j,n-1} + \delta_{j,n}),\ x_{0,n} \equiv 0, \end{aligned} \tag{5}$$

$$\begin{aligned} x_{j,n+1} &= \max_{0 \leq l_1 < l_2 < \ldots < l_j \leq n} \left(\sum_{m=l_1}^{l_2-1} \delta_{1,m} + \sum_{m=l_2}^{l_3-1} \delta_{2,m} + \cdots + \sum_{m=l_j}^{n} \delta_{j,m} \right), \\ x_{j,n} &= \max_{0 \leq l \leq n-1} (x_{j-1,l} - S_{j,l}) + S_{j,n-1},\ n \geq k,\ j = 1, 2, \ldots, k. \end{aligned}$$

3. Main Result

First we investigate the law of the iterated logarithm for the sojourn time of a customer in multiphase queues.

We prove such a theorem.

Theorem 3.1. *If conditions (3) and (4) are fulfilled, then*

$$\boldsymbol{P}\left(\overline{\lim_{n\to\infty}} \frac{T_{j,n} - \alpha_j \cdot n}{\tilde{\sigma}_j \cdot a(n)} = 1\right) = \boldsymbol{P}\left(\underline{\lim}_{n\to\infty} \frac{T_{j,n} - \alpha_j \cdot n}{\tilde{\sigma}_j \cdot a(n)} = -1\right) = 1,$$

$j = 1, 2, \ldots, k$ *and* $a(n) = \sqrt{2n \ln \ln n}$.

Proof. Denote random functions in D as follows

$$\bar{T}_j^n(t) = \frac{T_{j,[nt]} - \alpha_j \cdot [nt]}{a(n)}, \ \bar{X}_j^n(t) = \frac{\hat{x}_{j,[nt]} - \alpha_j \cdot [nt]}{(n)},$$
$$\bar{S}_j^n(t) = \frac{S_{j,[nt]} - \alpha_j \cdot [nt]}{a(n)}, \ \ j = 1, 2, \ldots, k \text{ and } \ 0 \le \mathrm{t} \le 1.$$

Using a triangle inequality and (2), we see that, for each fixed $\varepsilon > 0$,

$$\begin{aligned}
&\mathbf{P}\left(\overline{\lim_{n\to\infty}}\, d(\bar{T}_j^n, \bar{S}_j^n) > \varepsilon\right) \le \mathbf{P}\left(\overline{\lim_{n\to\infty}}\, d(\bar{T}_j^n, \bar{X}_j^n) > \frac{\varepsilon}{2}\right) \\
&+\mathbf{P}\left(\overline{\lim_{n\to\infty}}\, d(\bar{X}_j^n, \bar{S}_j^n) > \frac{\varepsilon}{2}\right) \le \mathbf{P}\left(\overline{\lim_{n\to\infty}}\, \rho(\bar{T}_j^n, \bar{X}_j^n) > \frac{\varepsilon}{2}\right) \\
&+\mathbf{P}\left(\overline{\lim_{n\to\infty}}\, \rho(\bar{X}_j^n, \bar{S}_j^n) > \frac{\varepsilon}{2}\right) = \mathbf{P}\left(\overline{\lim_{n\to\infty}} \frac{\sup_{0\le t\le 1} |T_{j,[nt]} - \hat{x}_{j,[nt]}|}{a(n)} > \frac{\varepsilon}{2}\right) \\
&+\mathbf{P}\left(\overline{\lim_{n\to\infty}} \frac{\sup_{0\le t\le 1} |\hat{x}_{j,[nt]} - S_{j,[nt]}|}{a(n)} > \frac{\varepsilon}{2}\right), \quad j = 1, 2, \ldots, k.
\end{aligned} \tag{6}$$

It is proved (see Minkevičius [20]) that, if conditions (4) are fulfilled, then, for each fixed $\varepsilon > 0$,

$$\mathbf{P}\left(\overline{\lim_{n\to\infty}} \frac{\sup_{0\le t\le 1} |T_{j,[nt]} - \hat{x}_{j,[nt]}|}{a(n)} > \varepsilon\right) = 0, \quad j = 1, 2, \ldots, k. \tag{7}$$

Also we obtain (see again Minkevičius [20]) that for each fixed $\varepsilon > 0$,

$$\mathbf{P}\left(\frac{\sup_{0\le t\le 1} |\hat{x}_{j,[nt]} - S_{j,[nt]}|}{a(n)} > \varepsilon\right) \le \sum_{i=1}^{k} \mathbf{P}\left(\frac{\max_{0\le l\le n} |\max_{0\le m\le l} \hat{S}_{i,m}|}{a(n)} > \frac{\varepsilon}{k}\right), \tag{8}$$

$j = 1, 2, \ldots, k$.

Note (see, for example, Iglehart et al [9]) that for each fixed $\varepsilon > 0$,

$$P\left(\overline{\lim_{n\to\infty}}\frac{\max\limits_{0\le l\le n}\hat{S}_{j,l}}{a(n)}>\varepsilon\right)=0,\quad j=1,2,\dots,k,\tag{9}$$

if conditions (4) are fulfilled.

Using the theorem on continuous mapping (see Billinsley [1]) for $|\cdot|$ and the maximum function, we also obtain that for each fixed $\varepsilon>0$,

$$P\left(\overline{\lim_{n\to\infty}}\frac{\max\limits_{0\le l\le n}|\max\limits_{0\le m\le l}\hat{S}_{j,m}|}{a(n)}>\varepsilon\right)=0,\quad j=1,2,\dots,k.\tag{10}$$

Thus, we have that, for each fixed $\varepsilon>0$,

$$\mathbf{P}\left(\overline{\lim_{n\to\infty}}\frac{\sup\limits_{0\le t\le 1}|\hat{x}_{j,[nt]}-S_{j,[nt]}|}{a(n)}>\varepsilon\right)=0,\quad j=1,2,\dots,k.\tag{11}$$

Besides, we get (see (7) and (11)) that, for each fixed $\varepsilon>0$,

$$\mathbf{P}\left(\overline{\lim_{n\to\infty}}\frac{\sup\limits_{0\le t\le 1}|T_{j,[nt]}-S_{j,[nt]}|}{a(n)}>\varepsilon\right)=0,\quad j=1,2,\dots,k.\tag{12}$$

Using the theorem on the law of the iterated logarithm for random functions $\bar{S}_{j,n}(t),\quad j=1,2,\dots,k$ (see, for example, Strassen [27]), we achieve that

$$\mathbf{P}\left(\overline{\lim_{n\to\infty}}\frac{S_{j,n}-\alpha_j\cdot n}{\tilde{\sigma}_j\cdot a(n)}=1\right)=1 \text{ and } \mathbf{P}\left(\underline{\lim_{n\to\infty}}\frac{S_{j,n}-\alpha_j\cdot n}{\tilde{\sigma}_j\cdot a(n)}=-1\right)=1,\tag{13}$$

$j=1,2,\dots,k$. Thus, applying (13) and (1) we obtain

$$\mathbf{P}\left(\overline{\lim_{n\to\infty}}\frac{T_{j,n}-\alpha_j\cdot n}{\tilde{\sigma}_j\cdot a(n)}=1\right)=1 \text{ and } \mathbf{P}\left(\underline{\lim_{n\to\infty}}\frac{T_{j,n}-\alpha_j\cdot n}{\tilde{\sigma}_j\cdot a(n)}=-1\right)=1,$$

$j=1,2,\dots,k$.

The proof of Theorem 3.1 is complete. □

4. On the Model of Switching Facility

In this part of the paper, we will present an application of the proved theorem – a mathematical model of message switching system.

As noted in the introduction, multiphase queueing systems are of special interest both in theory and in practical applications. Such systems consist of several service nodes, and each arriving customer is served at each of the consecutively located node (frequently called phases). A typical example is provided by queueing systems with identical service. Such systems are very important in applications, especially to message switching systems. In fact, in many comunication systems the transmission times of the customers do not vary in the delivery process.

So, we investigate a message switching system which consists of k phases and in which $S_n^j = S_n$, $j = 1,\ 2, \dots, k$ (the service process is identical in phases of the system).

Let

$$\delta_n = \begin{cases} S_{n-k} - z_n, & \text{if } n \geq k\,, \\ 0, & \text{if } n < k. \end{cases}$$

Also, let us note $\alpha = M\delta_n$, $Dz_n = \sigma_0^2$, $DS_n = \sigma^2$, $\tilde{\sigma}^2 = \sigma_0^2 + \sigma^2$, $\hat{T}_{j,n} = \sum_{i=1}^{j}(W_n^{(i)} + S_n)$.

We assume that the following conditions are fulfilled: there exists a constant $\gamma > 0$ such that

$$\sup_{n \geq 1} M|S_n|^{4+j} < \infty \tag{14}$$

and

$$\alpha > 0. \tag{15}$$

Similarly as in the proof of Theorem 3.1, we present the following theorem on the law of the iterated logarithm for the sojourn time of a data packet in message switching systems.

Theorem 4.1. *If conditions (14) and (15) are fulfilled, then*

$$\boldsymbol{P}\left(\overline{\lim_{n\to\infty}} \frac{\hat{T}_{j,n} - \alpha_j \cdot n}{\tilde{\sigma} \cdot a(n)} = 1\right) = \boldsymbol{P}\left(\underline{\lim_{n\to\infty}} \frac{\hat{T}_{j,n} - \alpha_j \cdot n}{\tilde{\sigma} \cdot a(n)} = -1\right) = 1,$$

$j = 1, 2, \dots, k.$

Corollary 4.2. *If conditions (14) and (15) are fulfilled, then for fixed $\varepsilon > 0$ there exists $n(\varepsilon)$ such that for every $n \geq n(\varepsilon)$,*

$$(1-\varepsilon)\cdot\tilde{\sigma}\cdot a(n) + \alpha\cdot n \leq \hat{T}_{j,n} \leq (1+\varepsilon)\cdot\tilde{\sigma}\cdot a(n) + \alpha\cdot n,\ \ j = 1, 2, \dots, k,$$

with probability one.

We see that the waiting time of data packet in the first phase gives a major contribution to the waiting time for the whole system (see, for example, Karpelevich et al [13], p. 71).

References

[1] P. Billinsley, *Convergence of Probability Measures*, Wiley, New York (1968).

[2] A. Borovkov, *Stochastic Processes in Queueing Theory*, Nauka, Moscow (1972), In Russian.

[3] A. Borovkov, *Asymptotic Methods in Theory of Queues*, Nauka, Moscow (1980), In Russian.

[4] O.J. Boxma, On a tandem queuing model with identical service times at both counters. I, II, *Advances in Applied Probability*, **11** (1979), 616-643, 644-659.

[5] O.J. Boxma, Q. Deng, Asymptotic behaviour of the tandem queuing system with identical service times at both queues, *Mathematical Methods of Operations Research*, **52** (2000), 307-323.

[6] Chen Hong, Shen Xinyang, Strong approximations for multiclass feedforward queueing networks, *Annals of Applied Probability*, **10** (2000), 828-876.

[7] E.G. Coffman, M.I. Reiman, Diffusion approximation for computer/communication systems, In: *Mathematical Computer Perfomance and Reliability*, North-Holland, Amsterdam (1984), 33-53.

[8] J. Harrison, V. Nguyen, Brownian models of multiclass queueing networks: Current status and open problems, *Queueing Systems*, **13** (1993), 5-40.

[9] D. Iglehart, W. Whitt, Multiple channel queues in heavy traffic. I, *Advances in Applied Probability*, **2** (1970), 150-177.

[10] D. Iglehart, W. Whitt, Multiple channel queues in heavy traffic. II: Sequences, networks and batches, *Advances in Applied Probability*, **2** (1970), 355-369.

[11] D. Iglehart, Multiple channel queues in heavy traffic. IV. Law of the iterated logarithm, *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, **17** (1971), 168-180.

[12] F.I. Karpelevich, A.I. Kreinin, Asymptotic analysis of switching systems for telegraph-type messages, *Problemy Peredachi Informatsii*, **26** (1990), 83-95, In Russian.

[13] F.I. Karpelevich, A.I. Kreinin, *Heavy Traffic Limits for Multiphase Queues*, American Mathematical Society, Providence (1994).

[14] J. Kingman, On queues in heavy traffic, *J. R. Statist. Soc.*, **24** (1961), 383-392.

[15] J. Kingman, The single server queue in heavy traffic, *Proc. Camb. Phil. Soc.*, **57** (1962), 902-904.

[16] C. Knessl, J.A. Morrison, Heavy-traffic analysis of the sojourn time in a three-mode Jackson network with overtaking, *Queueing Systems*, **8** (1991), 165-182.

[17] H. Kobyashi, Application of the diffusion approximation to queueing networks, *Journal of ACM*, **21** (1974), 316-328.

[18] S. Minkevičius, Weak convergence in multiphase queues, *Lietuvos Matematikos Rinkinys*, **26** (1986), 717-722, In Russian.

[19] S. Minkevičius, Law of the iterated logarithm for the extreme waiting time in multiphase queueing systems, *Lietuvos Matematikos Rinkinys*, **42** (2002), 470-485, In Russian.

[20] S. Minkevičius, Laws of the iterated logarithm for the sojourn time process in multiphase queueing systems, *Lietuvos Matematikos Rinkinys* (2006), In Russian, To Appear.

[21] M.I. Reiman, The heavy traffic diffusion approximation for sojourn times in Jackson networks, In: *Applied Probability-Computer Science: The Interface*, II. Progr. Comput. Sci., Birkhauser-Boston. MA, **3** (1982), 409-421.

[22] M.I. Reiman, Open queueing networks in heavy traffic, *Mathematics of Operations Research*, **9** (1984), 441-459.

[23] M.I. Reiman, B. Simon, J.St. Wilie, Simterpolation: A simulation based interpolation approximation for queueing systems, *Operations Research*, **40** (1992), 706-723.

[24] M.I. Reiman, L.M. Wein, Heavy traffic analysis of polling systems in tandem, *Operations Research*, **47** (1999), 706-723.

[25] T. Saati, K. Kerns, *Analytic Planning, Organization of Systems*, Mir. Moscow (1971), In Russian.

[26] R. Serfozo, Extreme values of queue length in $M/G/1$ and $GI/M/1$ systems, *Mathematics of Operations Research*, **13** (1988), 349-357.

[27] V. Strassen, An invariance principle for the law of the iterated logarithm, *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, **3** (1964), 211-226.

[28] W. Szczotka, F.P. Kelly, Asymptotic stationarity of queues in series and the heavy traffic approximation, *Annals of Probability*, **18** (1990), 1232-1248.

[29] W. Whitt, Some useful functions for functional limit theorems, *Mathematics of Operations Research*, **5** (1980), 67-85.

[30] Zhang Han Qin, Xu Guang Hui, Strong approximations for the open queueing network in heavy traffic, *Sci. China*, **A35** (1999), 521-535.