International Journal of Pure and Applied Mathematics
Volume 30 No. 2 2006, 247-257

ON THE SOLUTION OF AN MHD FLOW PAST AN OSCILLATING PLANE WALL: ANALYTICAL SOLUTION VS. NUMERICAL SOLUTION

Ch. Mamaloukas [§]

Department of Informatics
Athens University of Economics and Business
76, Patision Str., Athens, 10434, GREECE
email: mamkris@aueb.gr

Abstract: In this paper, the numerical solution of an incompressible, viscous, electrically conducting, fluid stream past an oscillating plane wall, is being compared with the analytical solutions presented by Poria et al [6]. In the beginning the basic equation, governing the motion of such a flow is expressed in non-dimensional form. Then we attempt to find possible analytical solutions for the velocity field and compute them separately with appropriate boundary and initial conditions. It is shown that this equation admits of two solutions that satisfy the respective sets of boundary and initial conditions. Then we try to solve the same equation expressed in non-dimensional form numerically.

The numerical calculations have been performed by a finite difference method using the well-known Crank-Nicholson implicit scheme. The results, showing the development of the velocity field in time for different values of a magnetic parameter, are presented graphically and discussed.

AMS Subject Classification: 65C20
Key Words: MHD fluid flow, oscillating plane wall, Crank-Nicholson implicit scheme

Received: June 15, 2006

[§]Correspondence address: 13, K. Theotoki, Str., A. Patisia, Athens, 11141, GREECE

1. Introduction

The unsteady flow of a viscous fluid, which sets up from rest when a plane wall oscillates with a prescribed velocity, was first studied by Stokes according to Bansal [1] and is termed as Stokes's second problem (Schlichting [7]). In the literature either it is called Stoke's first problem (Zeng and Weinbaum [11]) or simply Rayleigh's problem (Bansal [1]).

Kerczek and Davis [9] performed the linear stability analysis of the Stoke's layers on the oscillating surface.

The transient solution for the flow due to the oscillating plate has been given by Panton [5]. He has assumed that for large times steady-state flow is set-up with the same frequency as the velocity of the plane boundary. In order to obtain the starting solution, since the problem is linear, a transient solution must be added to the steady-state solution. Panton [5] has presented a closed-form solution to the transient component of Stoke's problem using the steady state component as the initial profile; however, he has not given the expression for the starting velocity field and the expression of the transient solution for the cosine oscillation of the plane boundary. Recently, Erdogan [2] obtained an analytical solution describing the flow at small and large times after the start of the boundary by the Laplace transform method. He has considered the flow of a viscous fluid produced by a plane boundary moving in its own plane with a sinusoidal variation of velocity. He obtained the steady solution and also the transient solution by subtracting the steady state solution from the starting solution. Mamaloukas [3] has reported some effects of a transverse magnetic field on the flow of a viscous conducting fluid, produced by a plane wall, oscillating harmonically in its own plane. Finally, Poria et al [6] studied the unsteady undirectional motion of an incompressible viscous fluid, produced by sinusoidal oscillation of a rigid plane wall and subjected to a uniform magnetic field acting perpendicularly to the flow direction. There was found two analytical solutions of this motion which satisfy the respective sets of boundary and initial solutions.

The problem of harmonically oscillating stream flow of a viscous fluid over an oscillating plate is not only of fundamental theoretical interest but also of practical importance as it occurs in many applied problems. It arises in acoustic streaming around an oscillating body. Another example is an unsteady boundary layer with fluctuation in the free-stream velocity [8].

In the present analysis, our aim is first to determine the analytic solutions of the problem and to compare them with the numerical calculations that have been performed by a finite difference method, using the well-known Crank-

Nicholson implicit scheme.

2. Formulation of the Problem

Let us consider the flow of an incompressible viscous fluid produced by the oscillation of a rigid plane wall ($y = 0$) and subjected to a uniform transverse magnetic field B_0. In the second Stokes problem, such a velocity field satisfies the boundary and initial conditions $u\left(0,t\right) = U_0 \cos\left(\omega t\right), u\left(\infty,t\right) = 0$ or $u\left(0,t\right) = U_0 \sin\left(\omega t\right), u\left(\infty,t\right) = 0$. Choosing the x-axis as the plane of the wall and assuming that the fluid stays in the region $y \geq 0$, the momentum equation governing the motion of the fluid can be written as (White [10])

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} - \frac{\sigma B_0^2}{\rho} u \,, \tag{2.1}$$

where σ is the electrical conductivity and B_0 is the applied magnetic field perpendicular to the flow direction.

We non-dimensionalize the coordinate y, time t and velocity u as

$$\overline{u} = \frac{u}{U_0}, \quad \tau = \omega t, \quad \eta = y \left(\frac{\omega}{\nu}\right)^{\frac{1}{2}} . \tag{2.2}$$

In view of (2) the equation (1) is transformed to

$$\frac{\partial \overline{u}}{\partial \tau} = \nu \frac{\partial^2 \overline{u}}{\partial \eta^2} - M \overline{u} \,, \tag{2.3}$$

where $M = \frac{\sigma B_0^2}{\rho \omega}$ is the magnetic parameter, σ is the electrical conductivity and B_0 is the strength of the applied magnetic field.

The two sets of boundary and initial conditions now are transformed respectively to

$$\overline{u}\left(0,\tau\right) = \cos\tau, \quad \overline{u}\left(\infty,t\right) = 0\,, \quad \text{or} \quad \overline{u}\left(0,\tau\right) = \sin\tau, \quad \overline{u}\left(\infty,\tau\right) = 0\,. \tag{2.4}$$

3. Analytical Solution

To solve equation (3) analytically, we assume (Poria et al [6]) the form of $\overline{u}$ as

$$\overline{u} = f\left(\eta\right) e^{i\tau} \,. \tag{3.1}$$

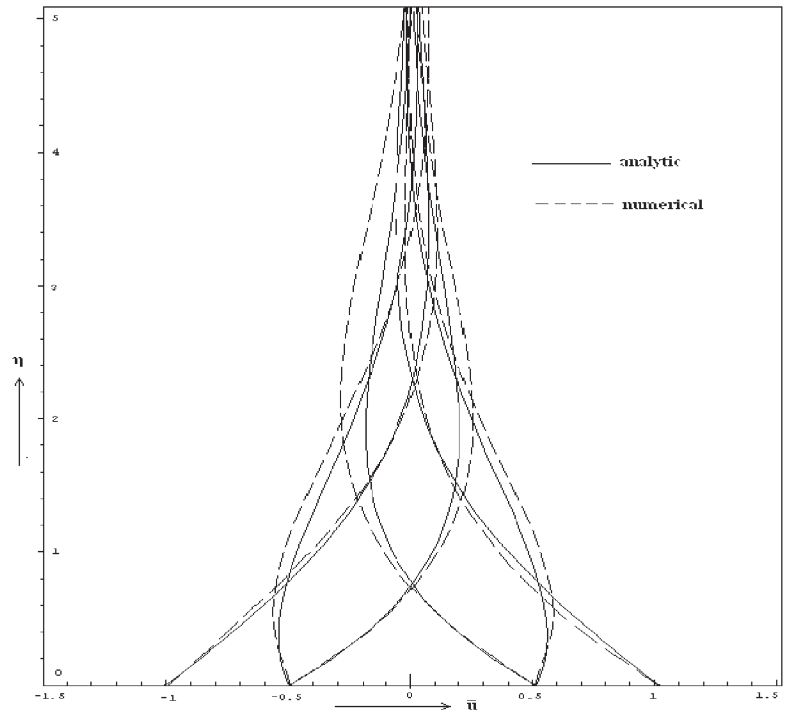

Figure 1: Variation of velocity with distance from the wall with time increments $\frac{\pi}{3}$ for the cosine oscillation of the plane wall and $M = 0$, obtained by analytical solution and numerical solution

Substituting (5) in (3), we obtain

$$f''(\eta) - \alpha^2 f(\eta) = 0; \quad \alpha^2 = i + M\,, \tag{3.2}$$

where $f'' = \frac{\partial^2 f}{\partial \eta^2}$.

We now put $\alpha^2 = X_1 + iY_1$ and especially $X_1 = M,\ Y_1 = 1$ and $\alpha = X_2 + iY_2$. It can be easily found that

$$X_2 = \pm\left[\frac{X_1 \pm \left(X_1^2 + Y_1^2\right)^{\frac{1}{2}}}{2}\right]^{\frac{1}{2}}, \quad Y_2 = \pm\left[\frac{-X_1 \pm \left(X_1^2 + Y_1^2\right)^{\frac{1}{2}}}{2}\right]^{\frac{1}{2}}, \tag{3.3}$$

and $\quad Y_1 = 2X_2Y_2\,.$

Now, since X_2, Y_2 are real, $\left(X_1^2 + Y_1^2\right)^{\frac{1}{2}} > X_1,\ Y_1 = 2X_2Y_2 > 0$ and the

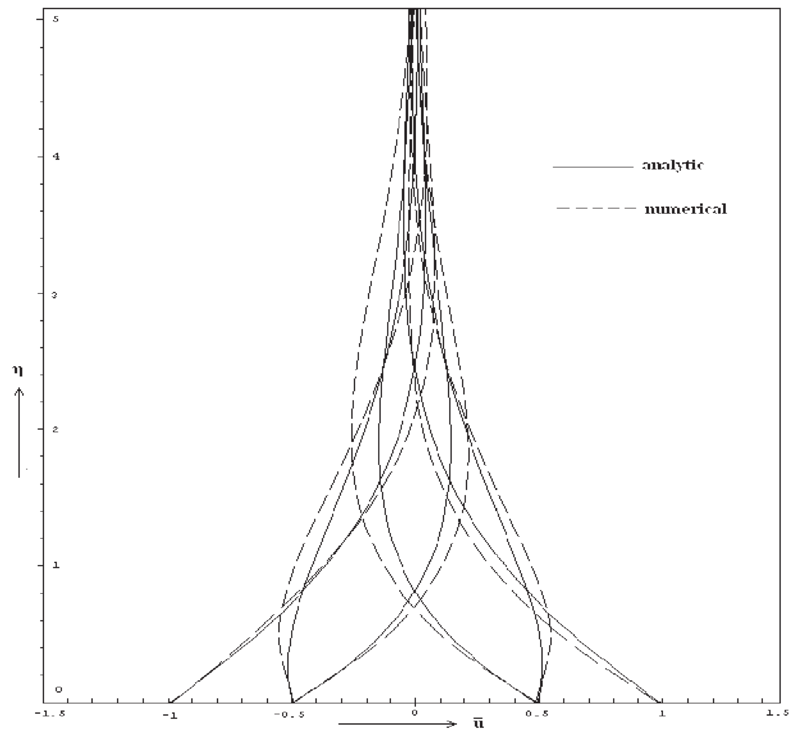

Figure 2: Variation of velocity with distance from the wall with time increments $\frac{\pi}{3}$ for the cosine oscillation of the plane wall and $M = 0.2$, obtained by analytical solution and numerical solution

solution should vanish as $y \to \infty$, we take

$$X_2 = \left[\frac{X_1 + \left(X_1^2 + Y_1^2\right)^{\frac{1}{2}}}{2}\right]^{\frac{1}{2}}, \; Y_2 = \left[\frac{-X_1 + \left(X_1^2 + Y_1^2\right)^{\frac{1}{2}}}{2}\right]^{\frac{1}{2}},$$

or

$$X_2 = \left[\frac{M + \left(1 + M^2\right)^{\frac{1}{2}}}{2}\right]^{\frac{1}{2}}, \; Y_2 = \left[\frac{-M + \left(1 + M^2\right)^{\frac{1}{2}}}{2}\right]^{\frac{1}{2}}. \tag{3.4}$$

We choose the form for $f(\eta)$ as $f(\eta) = e^{-\alpha\eta}$. Accordingly,

$$\overline{u} = e^{-\alpha\eta} e^{i\tau} = e^{-(X_2 + iY_2)\eta} \left(\cos\tau + i\sin\tau\right),$$

or

$$\overline{u} = e^{-X_2\eta} \left[\cos\left(\tau - Y_2\eta\right) + i\sin\left(\tau - Y_2\eta\right)\right]. \tag{3.5}$$

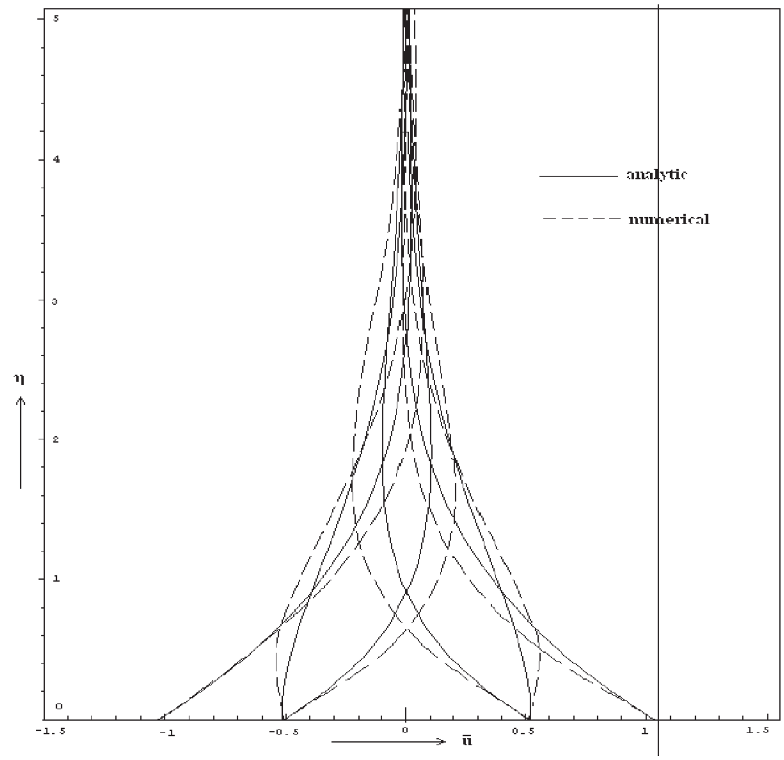

Figure 3: Variation of velocity with distance from the wall with time increments $\frac{\pi}{3}$ for the cosine oscillation of the plane wall and $M = 0.5$, obtained by analytical solution and numerical solution

Thus, (9) is the general solution of equation (3). From (9), taking (8) into account, we obtain two particular solutions for the velocity field corresponding, respectively to the boundary and initial conditions (4) as

$$\overline{u}_1 = e^{-\left[\frac{M+\left(1+M^2\right)^{\frac{1}{2}}}{2}\right]^{\frac{1}{2}}\eta} \cos\left(\tau - \left[\frac{-M+\left(1+M^2\right)^{\frac{1}{2}}}{2}\right]^{\frac{1}{2}}\eta\right) \tag{3.6}$$

and

$$\overline{u}_1 = e^{-\left[\frac{M+\left(1+M^2\right)^{\frac{1}{2}}}{2}\right]^{\frac{1}{2}}\eta} \sin\left(\tau - \left[\frac{-M+\left(1+M^2\right)^{\frac{1}{2}}}{2}\right]^{\frac{1}{2}}\eta\right) . \tag{3.7}$$

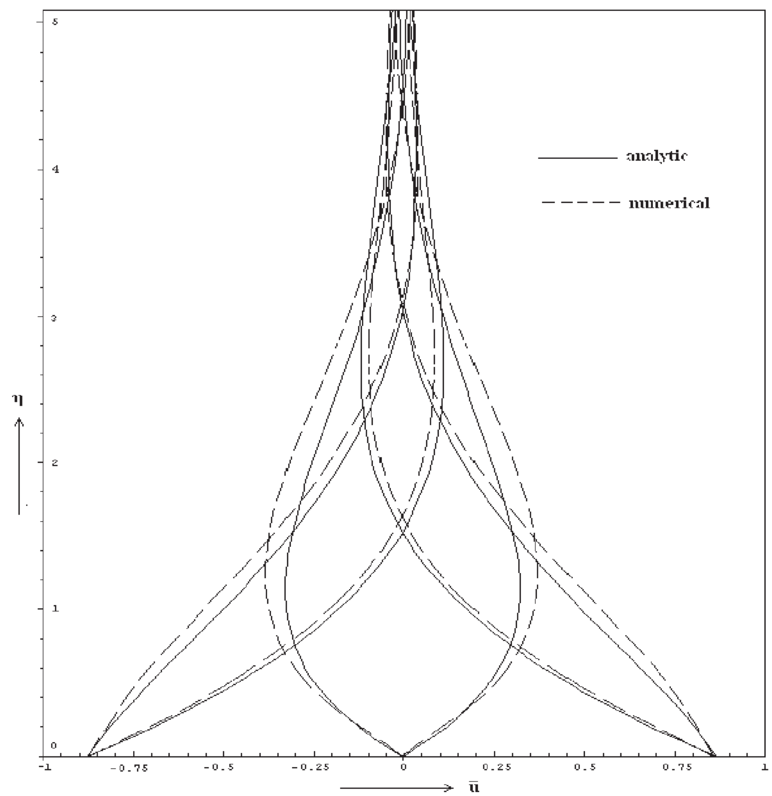

Figure 4: Variation of velocity with distance from the wall with time increments $\frac{\pi}{3}$ for the sine oscillation of the plane wall and $M = 0$, obtained by analytical solution and numerical solution

We now proceed to the numerical solution.

4. Numerical Computations

The equation (3) with boundary and initial conditions, given by (4) are solved separately by a finite difference technique.

We use Crank-Nicholson implicit scheme as appropriate for the parabolic type of equation (3). In this scheme, the time derivative term $\frac{\partial \overline{u}}{\partial \tau}$ is represented by forward - differences as given by

$$\frac{\partial \overline{u}}{\partial \tau} = \overline{u}_\tau\big|_{i,j} = \frac{\overline{u}_{i,j+1} - \overline{u}_{i,j}}{k} + 0\,(R) \tag{4.1}$$

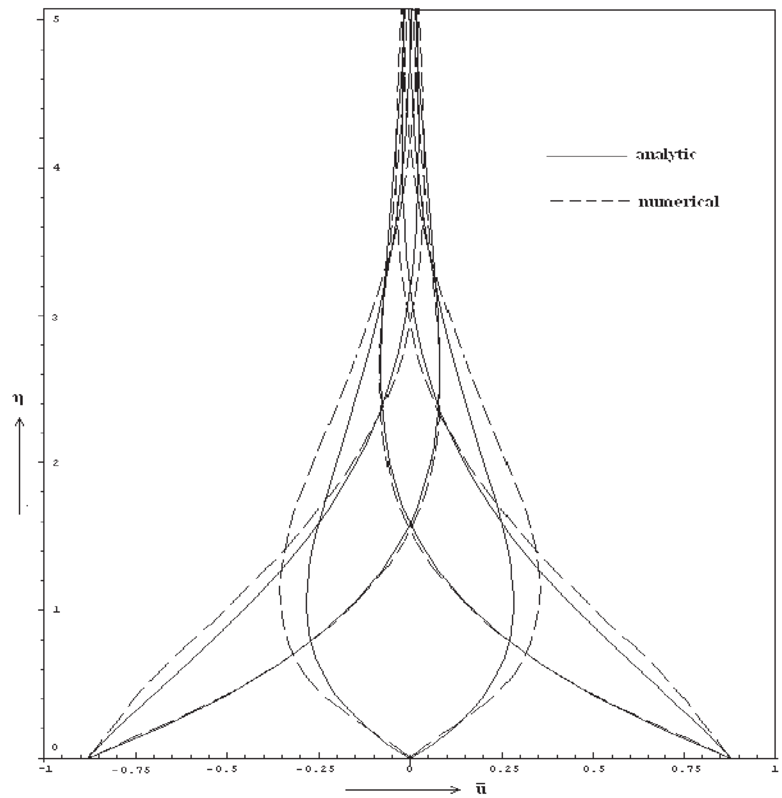

Figure 5: Variation of velocity with distance from the wall with time increments $\frac{\pi}{3}$ for the sine oscillation of the plane wall and $M = 0.2$, obtained by analytical solution and numerical solution

while the space double derivative is represented by the average central differences at the present and new time step (Niyogi [5], Mamaloukas [3]) is given by:

$$\frac{\partial^2 \overline{u}}{\partial \eta^2} = \overline{u}_{\eta\eta}\big|_{i,j} = \frac{1}{2} \times \left[\frac{\overline{u}_{i+1,j+1} - 2\overline{u}_{i,j+1} + \overline{u}_{i-1,j+1}}{h^2} + \frac{\overline{u}_{i+1,j} - 2\overline{u}_{i,j} + \overline{u}_{i-1,j}}{h^2} \right] + 0^2 (R) \; . \quad (4.2)$$

Accordingly, we write equation (3) in the form:

$$\frac{\overline{u}_{i,j+1} - \overline{u}_{i,j}}{\Delta \tau} = \frac{1}{2} \left[\frac{\overline{u}_{i+1,j+1} - 2\overline{u}_{i,j+1} + \overline{u}_{i-1,j+1} + \overline{u}_{i+1,j} - 2\overline{u}_{i,j} + \overline{u}_{i-1,j}}{(\Delta \eta)^2} \right]$$

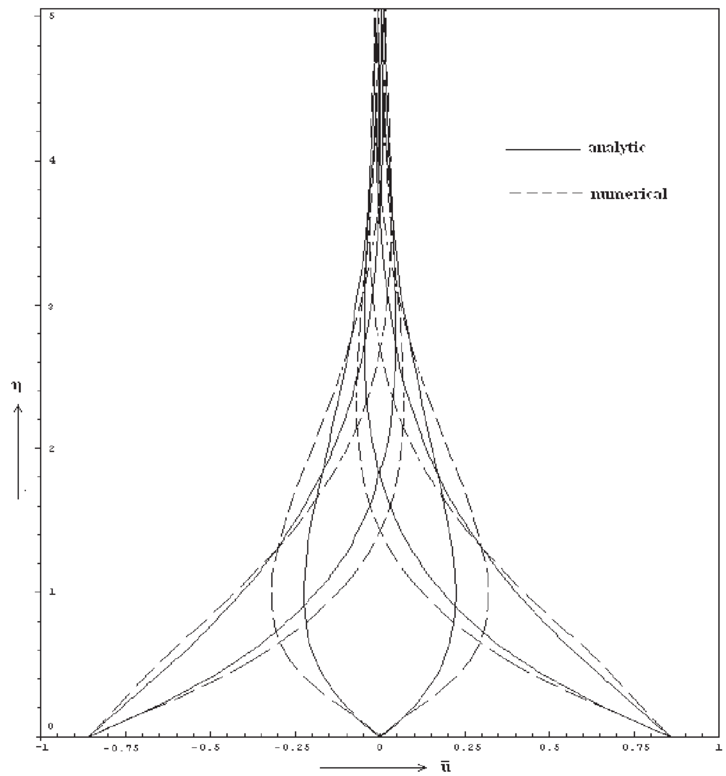

Figure 6: Variation of velocity with distance from the wall with time increments $\frac{\pi}{3}$ for the sine oscillation of the plane wall and $M = 0.5$, obtained by analytical solution and numerical solution

$$-\frac{M}{2}\left(\overline{u}_{i,j+1}-\overline{u}_{i,j}\right), \quad (4.3)$$

or,

$$-r\overline{u}_{i-1,j+1}+(1+2r+MrP)\,\overline{u}_{i,j+1}-r\overline{u}_{i+1,j+1} = r\left(\overline{u}_{i-1,j}-\overline{u}_{i+1,j}\right)+(1-MrP-2r)\,\overline{u}_{i,j}, \quad (4.4)$$

where

$$r=\frac{\Delta\tau}{2\left(\Delta\eta\right)^2}, \quad P=\left(\Delta\eta\right)^2 . \quad (4.5)$$

The system of algebraic equations in tri-diagonal form that follows from (15) is solved by Thomas algorithm for each time level. In this problem 25×25 grid-points have been considered for numerical computation is obtained at each grid-points at each time level.

5. Results and Discussions

Solutions for the velocity $\overline{u}(\eta,\tau)$ as obtained by analytical solution (10) and (11) and by solving the finite-difference equation (15) with the specified sets of initial and boundary conditions (4) and for various values of times τ and the magnetic parameter M are represented by graphs over a cycle, respectively in Figures 1, 2, 3 and 4, 5, 6.

Figures 1, 2, 3 exhibit the time developments of the velocity field due to cosine oscillation (10) at $\eta = 0$ of the infinite plate and the variations of the velocity profiles in time with the cosine oscillation, taking the values of M as $M = 0,\ 0.2$ and $M = 0.5$. It is seen that the harmonic waves created over a cycle, in the fluid lag behind in phase and damp out as η increases. Furthermore it is clear from these figures that the velocity field is suppressed with the imposition of magnetic field. Such suppression increases with the increase of the value of M.

Similar features are noticed in Figures 4, 5, 6 for the case with sine oscillation (11) for the same set of values of M as $M = 0,\ 0.2$ and $M = 0.5$. It is observed that the unsteady behavior of the velocity field of the sine oscillations differs much from the same with the cosine oscillations. The reason may be attributed to the fact that in the case of the sine oscillations the fluid is at rest at time $\tau = 0$.

Comparing the velocity profiles obtained by the analytical and the numerical solution we see that the more magnetic parameter M increases the less the matching results are satisfactory in the near field while the matching results in the far field are satisfactory in any case.

The effects of the magnetic field on the velocity field, for the case of sine oscillation in both the right and left half cycles are qualitatively similar to that of with cosine oscillation. Only, the amount of suppression of the velocity field in the right half cycle and the amount of enhancement of the velocity field in the left half cycle are both large in the case with sine oscillation. Such enhancement of the velocity field is found to increase with the increase of M.

It is to be mentioned that in the present treatment the values of the magnetic parameter is chosen to be small enough so that the induced magnetic field may be negligible.

References

[1] J.L. Bansal, *Viscous Fluid Dynamics*, Oxford and IBH, Publishing Co, Calcutta (1977).

[2] M. Erdogan, A note on an unsteady flow of a viscous fluid due to an oscillating plane wall, *Int. J. Non-Linear Mech.*, **35** (2000), 1-6.

[3] C. Mamaloukas, Numerical solution of the flow of a viscous conducting fluid produced by an oscillating plane wall subjected to a transverse magnetic field, In: *BSG Proceedings*, **10** (2004), 87-94.

[4] P. Niyogi, *Lecture Notes on Finite-Difference for Partial Differential Equation with Applications to CFD*, IIT Kharagpur, W.B, India (1998).

[5] R. Panton, The Transients for Stokes's oscillating plane: A solution in terms of tabulated functions, *J. Fluid Mech.*, **31** (1968), 819-825.

[6] S. Poria, C. Mamaloukas, et al, Some effects of a transverse magnetic field on the flow of a viscous conducting fluid produced by an oscillating plane wall, *Journal of Technical Physics*, **42**, No 4 (2001), 397-405.

[7] H. Schlichting, *Boundary Layer Theory*, 6-th Edition, McGraw. Hill, New York (1968).

[8] N. Tokuda, On the impulsive motion of a flat plate in a viscous fluid, *J. Fluid Mech.*, **33** (1968), 657-672.

[9] C. Von Kerczek, S.H. Davis, Linear stability theory of oscillating Stokes layers, *J. Fluid Mech.*, **62** (1974), 753-773.

[10] F.M. White, *Viscous Fluid Flow*, McGraw-Hill, New York (1991).

[11] Y. Zeng, S. Weinbaum, Stokes problem for moving half-planes, *J. Fluid Mech.*, **287** (1995), 59-74.