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HARMONIC MAJORANT OF A RADIAL SUBHARMONIC FUNCTION ON A STRIP AND THEIR APPLICATIONS

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Abstract: The regularity of value distribution for special subharmonic functions defined on a strip $D\times\mathbf{R}^l$ ($l \geq 2$) with a bounded domain D in $\mathbf{R}^m$ ($m \geq 1$) and satisfying the Phragmén-Lindelöf boundary condition will be systematically investigated.

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1. Introduction

Let D be a bounded domain in the m-dimensional Euclidean space $\mathbf{R}^m$ ($m \geq 1$). Denote the boundary and the closure of D in $\mathbf{R}^m$ by ∂D and $\overline{D}$, respectively. Let X denote a point $(x_1, x_2, \ldots, x_m)$ ($m \geq 2$) or x ($m = 1$). For the Dirichlet problem

$$\begin{aligned} &(\Delta_m + \lambda)F = 0 \quad \text{on } D\,, \\ &F = 0 \quad \text{on } \partial D\,, \end{aligned}$$

with the Laplace operator

$$\Delta_m = \begin{cases} \frac{d^2}{dx^2} & (m = 1)\,, \\ \sum_{i=1}^m \frac{\partial^2}{\partial x_i^2} & (m \geq 2)\,, \end{cases}$$

let λ_D and $f_D(X)$ denote the smallest positive eigenvalue and the positive

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normalized eigenfunction corresponding to λ_D, respectively, if they exist. If $D=(0,1)$ in $\mathbf{R}^1$, then we easily see that $\lambda_D=\pi^2$ and $f_D(x)=\sqrt{2}\sin\pi x$.

We introduce the system of the spherical coordinates (r,Θ), $\Theta=(\theta_1,\theta_2,\dots,\theta_{l-1})$

$$0\le r<\infty,\ -\frac{1}{2}\pi\le\theta_{l-1}<\frac{3}{2}\ (l\ge 2),\ 0\le\theta_j\le\pi\ (1\le j\le l-2,\ l\ge 3)$$

in $\mathbf{R}^l$ $(l\ge 2)$ which are related to the coordinates $(y_1,y_2,\dots,y_l)$ of a point $Y\in\mathbf{R}^l$ by

$$r=|Y|=\sqrt{y_1^2+y_2^2+\ldots+y_l^2},\ y_1=r\cos\theta_1,\ y_l=r(\Pi_{j=1}^{l-1}\sin\theta_j)\quad(l\ge 2),$$

$$y_k=r(\Pi_{j=1}^{k-1}\sin\theta_j)\cos\theta_{k-1}\quad(2\le k\le l-1,\ l\ge 3).$$

By $\mathbf{S}^{l-1}$, we denote the unit sphere $(l\ge 3)$ or the unit circle $(l=2)$ in $\mathbf{R}^l$. By s_l, we denote the surface area $(l\ge 3)$ or the length of the circumference $(l=2)$, of $\mathbf{S}^{l-1}$.

The set $D\times\mathbf{R}^l$ in $\mathbf{R}^{m+l}$ is called a cylinder $(l=1)$ or a strip $(l\ge 2)$. A strip is completely different from a cylinder in the following fact. When D has smooth boundary, infinite Martin boundary points of the cylinder are simply two points $+\infty$ and $-\infty$, but these set of the strip are complicately the set of all infinite points $M_\Theta,(1,\Theta)\in\mathbf{S}^{l-1}$, into which the set $\overline{D}\times\{Y\in\mathbf{R}^l;|Y|\ge n\}$ shrinks as $n\to\infty$ (see Brawn [6], Aikawa [2] and Gardiner [7]).

Let $g(X,Y)$ be a function defined on $D\times\mathbf{R}^l$ $(l\ge 2)$. We say that $g(X,Y)$ satisfies the Phragmén-Lindelöf boundary condition, if

$$\overline{\lim}_{(X,Y)\in D\times\mathbf{R}^l,(X,Y)\to(X_0,Y_0)}g(X,Y)\le 0$$

for every $(X_0,Y_0)\in\partial D\times\mathbf{R}^l$.

In [13], we gave a result of the Phragmén-Lindelöf type and extended it by giving a harmonic majorant for a subharmonic function defined on a cylinder and satisfying the Phragmén-Lindelöf boundary condition.

Hence in this paper we shall always be related to a function defined on a strip $D\times\mathbf{R}^l$ $(l\ge 2)$. When we try to make consideration similar to those in [13] for a subharmonic function defined on a strip and satisfying the Phragmén-Lindelöf boundary condition, we need a method of regarding the set all infinite Martin boundary points of the strip as if they were a single point. The following two results motivated the required method.

Aikawa [3, Theorem 1] proved the following Theorem A and extended a result of Brawn [6, Theorem 2] in which D is the interval $(0,1)$.

Theorem A. *Let D be a bounded domain in $\mathbf{R}^m$ $(m \geq 1)$ having smooth boundary ∂D. Let $u(X,Y)$ be a subharmonic function on the strip $D \times \mathbf{R}^l$ and satisfying the Phragmén-Lindelöf boundary condition. If*

$$\underline{\lim}_{r\to\infty} r^{(l-1)/2} \exp(-\sqrt{\lambda_D} r) \int_{\mathbf{S}^{l-1}} \int_D u^+(X,(r,\Theta)) f_D(X) dX dS(\Theta) = 0, \quad (1.1)$$

where $dS(\Theta)$ is the surface area element of $\mathbf{S^{l-1}}$ at $(1,\Theta)$, then

$$u(X,Y) \leq 0$$

on $D \times \mathbf{R}^l$.

Before Aikawa's paper [3] was published, Yoshida [12, Theorem 5] had proved the following Theorem B and extended a result of Brawn [6, Theorem 1] in which D is the interval $(0,1)$. Theorem B is much simpler than Theorem A, but Theorem A is better than Theorem B in the sense that (1.1) follows from (1.2).

Theorem B. *Let D be a bounded domain in $\mathbf{R}^m$ $(m \geq 1)$ having smooth boundary. Let $u(X,Y)$ be a subharmonic function on the strip $D \times \mathbf{R}^l$ and satisfying the Phragmén-Lindelöf boundary condition. If*

$$\underline{\lim}_{r\to\infty} r^{(l-1)/2} \exp(-\sqrt{\lambda_D} r) \sup_{X\in D, |Y|=r} u(X,Y) \leq 0, \quad (1.2),$$

then

$$u(X,Y) \leq 0$$

on $D \times \mathbf{R}^l$.

From (1.1) and (1.2), we pick up the following facts, which are easily proved (see Aikawa [3, Lemma 7] and also Armitage [4, Lemma]): *If $u(X,Y)$ is a subharmonic function defined on $D \times \mathbf{R}^l$, then*

$$C_u(X,Y) = s_l^{-1} \int_{\mathbf{s}^{l-1}} u(X,(r,\Theta)) dS(\Theta) \quad (0 \leq |Y| = r < \infty) \quad (1.3)$$

and

$$M_u(X,Y) = \max_{|Y|=r} u(X,Y) \quad (1.4)$$

are two subharmonic functions on $D \times \mathbf{R}^l$.

If we observe that the functions in (1.3) and (1.4) depend on X and $|Y|$, in fact, then we need to take the following definition into consideration. A function $g(X,Y)$ defined on a strip $D\times\mathbf{R}^l$ is said to be *radial* with respect to Y (simply, *radial*), if

$$g(X,Y)=g(X,|Y|)$$

for any $X\in D$ and any $Y\in\mathbf{R}^l$.

In this paper, in connection with a question about the regularity of the Nevanlinna norm

$$N(u)(r)=\int_D u(X,Y)f_D(X)dX \qquad (0<|Y|=r<\infty)$$

for a radial subharmonic function $u(X,Y)$ on $D\times\mathbf{R}^l$ satisfying the Phragmén-Lindelöf boundary condition , we shall prove some convexity property of $N(u)(r)$ and the existence of the limit

$$\mu(N(u))=\lim_{r\to\infty} r^{l/2-1}\{I_{l/2-1}(\sqrt{\lambda_D}r)\}^{-1}N(u)(r) \qquad (-\infty<\mu(N(u))\leq\infty),$$

where $I_{l/2-1}(t)$ is the third kind of Bessel function of order $l/2-1$ (Theorem 2 and Theorem 3). This will give a result concerning

$$N(C_u)(r)=s_l^{-1}\int_D(\int_{\mathbf{s}^{l-1}} u(X,(r,\Theta))dS(\Theta))f_D(X)dX$$

which is regarded the Nevanlinna norm of a subharmonic function $u(X,Y)$ on $D\times\mathbf{R}^l$ satisfying the Phragmén-Lindelöf boundary condition (Corollary 4).

Next we shall show that in the case $\mu(N(u^+))=0$ a Phragmén-Lindelöf type theorem for a radial subharmonic function on $D\times\mathbf{R}^l$ satisfying the Phragmén-Lindelöf boundary condition is obtained by applying it (Theorem 4). By applying Theorem 4 to special radial subharmonic functions, we shall obtain Theorem A, Theorem B and a new theorem of the Phragmén-Lindelöf type (Corollary 2 and Theorem 5).

Thirdly, we shall consider the case

$$\mu(N(u^+))\ <\ \infty \tag{1.5}$$

for a radial subharmonic function u on $D\times\mathbf{R}^l$ satisfying the Phragmén-Lindelöf boundary condition. Then we shall give a harmonic majorant of u (Theorem 6) which is the least harmonic majorant of u, if u is non-negative in addition (Corollary 3). By using Theorem 6, Theorem A and Theorem B will be generalized (Corollary 4 and Theorem 7).

Finally, in connection with a question about the regularity of the maximum modulus

$$L(u)(r)=\sup_{X\in D} u(X,Y) \qquad (|Y|=r)$$

of a radial subharmonic function $u(X,Y)$ on $D \times \mathbf{R}^l$ satisfying the Phragmén-Lindelöf boundary condition, we shall also prove the existence of the limit

$$\mu(L(u)) = \lim_{r\to\infty} r^{l/2-1} \frac{L(u)(r)}{I_{l/2-1}(\sqrt{\lambda_D} r)}$$

and some connection between $\mu(L(u))$ and $K(u)$ (Theorem 8), where

$$K(u) = \overline{\lim}_{r\to\infty} r^{l/2-1}\{I_{l/2-1}(\sqrt{\lambda_D} r)\}^{-1} \sup_{X\in D} \frac{u(X,Y)}{f_D(X)} \qquad (|Y| = r).$$

This result applied to a special case will give a result concerning

$$L(M_u)(r) = \sup_{X\in D, |Y|=r} u(X,Y)$$

which is regarded the maximum modulus of a subharmonic function $u(X,Y)$ on $D \times \mathbf{R}^l$ and satisfying the Phragmén-Lindelöf boundary condition (Theorem 9).

In order that the subsequent consideration may be simple, we put a strong assumption relative to D: if $m \geq 2$, then D is a $C^{2,\alpha}$-domain $(0 < \alpha < 1)$ in $\mathbf{R}^m$ surrounded by a finite number of mutually disjoint closed hypersurfaces (e.g. see Gilbarg and Trudinger [8, pp. 88-89] for the definition of $C^{2,\alpha}$-domains). Then $f_D(X)$ and any harmonic function in D vanishing on ∂D are twice continuously differentiable on $\overline{D}$ (see [8, Theorem 6.15]).

2. Statements of Our Results

Our results wll be stated by dividing into four parts.

2.1. The Fundamental Results

By $I_{l/2-1}(t)$ and $K_{l/2-1}(t)$, we denote two Bessel functions of the third kind of order $l/2-1$ as Watson denotes them (see [11, pp. 77-78]). Let ρ_1 and ρ_2 be two numbers satisfying $0 \leq \rho_1 < \rho_2 \leq \infty$. Let $\phi(t)$ be a real finite-valued function defined on an interval (ρ_1, ρ_2). For a positive number λ and for any given t_1, t_2 $(\rho_1 < t_1 < t_2 < \rho_2)$, the function $\Phi(t; \phi, \lambda, t_1, t_2)$ of t on (ρ_1, ρ_2) is defined by the determinant

$$\begin{vmatrix} \phi(t) & t^{1-l/2} I_{l/2-1}(\lambda t) & t^{1-l/2} K_{l/2-1}(\lambda t) \\ \phi(t_1) & t_1^{1-l/2} I_{l/2-1}(\lambda t_1) & t_1^{1-l/2} K_{l/2-1}(\lambda t_1) \\ \phi(t_2) & t_2^{1-l/2} I_{l/2-1}(\lambda t_2) & t_2^{1-l/2} K_{l/2-1}(\lambda t_2) \end{vmatrix}.$$

We put

$$T_l(\rho_1,\rho_2)=\{Y\in \mathbf{R}^l;\rho_1<|Y|<\rho_2\}$$

and

$$\overline{T_l(\rho_1,\rho_2)}=\{Y\in \mathbf{R}^l;\ \rho_1\le|Y|\le\rho_2\}\quad (0\le\rho_1<\rho_2<\infty).$$

Theorem 1. *Let ρ_1 and ρ_2 be two numbers satisfying $0\le\rho_1<\rho_2\le\infty$. Let $h(X,Y)$ be a radial harmonic function on $D\times T_l(\rho_1,\rho_2)$ and a continuous function on $\overline{D}\times\overline{T_l(r_1,r_2)}$ such that*

$$h(X,Y)=0$$

on $\partial D\times\overline{T_l(r_1,r_2)}$. Then there exist two constants C_1 and C_2 such that

$$N(h)(r)=C_1r^{1-l/2}K_{l/2-1}(\sqrt{\lambda_D}r)+C_2r^{1-l/2}I_{l/2-1}(\sqrt{\lambda_D}r)$$

for any r $(\rho_1<r<\rho_2)$. Hence

$$\Phi(r;N(h),\sqrt{\lambda_D},r_1,r_2)=0\quad(\rho_1<r<\rho_2)$$

for any r_1,r_2 $(\rho_1<r_1<r_2<\rho_2)$.

A function $\phi(t)$ is said to be $A_{\lambda,l}$-*convex* on $(0,\infty)$, if

$$\Phi(t;\phi,\lambda,t_1,t_2)\ge 0\quad (t_1\le t\le t_2)$$

for any $t_1,\ t_2$ $(0<t_1<t_2<\infty)$.

Theorem 2. *Let $u(X,Y)$ be a radial subharmonic function on a strip $D\times\mathbf{R}^l$ satisfying the Phragmén-Lindelöf boundary condition. Then $N(u)(r)$ is finite and $A_{\sqrt{\lambda_D},l}$-convex on $(0,\infty)$.*

Theorem 3. *If $u(X,Y)$ is a radial subharmonic function on $D\times\mathbf{R}^l$ satisfying the Phragmén-Lindelöf boundary condition, then*

$$r^{l/2-1}\{I_{l/2-1}(\sqrt{\lambda_D}r\}^{-1}N(u)(r)$$

is non-decreasing. In particular, $r^{l/2-1}\{I_{l/2-1}(\sqrt{\lambda_D}r)\}^{-1}N(u)(r)$ is a constant if and only if u is harmonic and vanishes on $\partial D\times\mathbf{R}^l$.

From Theorem 2 applied to $C_u(X,Y)$ with a subharmonic function u, we easily obtain the following Theorem 2, which generalizes a result of Brawn [6, Theorem 2] for $D=(0,1)$.

Corollary 1. *If $u(X,Y)$ is a subharmonic function on a strip $D\times\mathbf{R}^l$ satisfying the Phragmén-Lindelöf boundary condition, then*

$$r^{l/2-1}\{I_{l/2-1}(\sqrt{\lambda_D}r\}^{-1}N(C_u)(r)$$

is non-decreasing. In particular, $r^{l/2-1}\{I_{l/2-1}(\sqrt{\lambda_D}r\}^{-1}N(C_u)(r)$ is a constant if and only if u is harmonic and vanishes on $\partial D\times\mathbf{R}^l$.

2.2. Phragmén-Lindelöf Type Theorems

Theorems A and B are two special cases of the following Theorem 4.

Theorem 4. *Let $u(X,Y)$ be a radial subharmonic function on a strip $D\times\mathbf{R}^l$ satisfying the Phragmén-Lindelöf boundary condition. If*

$$\mu(N(u^+))=0, \tag{2.1}$$

then

$$u(X,Y)\leq 0$$

on $D\times\mathbf{R}^l$.

By applying Theorem 4 to the special functions (1.3) and (1.4), we obtain the following (I) and (II) in Corollary 2, which are different from the proofs of Theorem A in Aikawa [3] and Theorem B in Yoshida [12].

Corollary 2. *Let $u(X,Y)$ be a subharmonic function on a strip $D\times\mathbf{R}^l$ satisfying the Phragmén-Lindelöf boundary condition.*

(I) If

$$\underline{\lim}_{r\to\infty} r^{(l-1)/2}\exp(-\sqrt{\lambda_D}r)N(C_{u^+})(r)=0,$$

then

$$u(X,Y)\leq 0$$

on $D\times\mathbf{R}^l$.

(II) If

$$\underline{\lim}_{r\to\infty} r^{(l-1)/2}\exp(-\sqrt{\lambda_D}r)N(M_{u^+})(r)=0,$$

then

$$u(X,Y)\leq 0$$

on $D\times\mathbf{R}^l$.

Let $u(X,Y)$ be a subharmonic function defined on $D\times\mathbf{R}^l$. As the third example of radial subharmonic functions defined on $D\times\mathbf{R}^l$, we shall also consider

$$V_u(X,Y)=\frac{l}{s_l}\int_0^1 t^{l-1}\Big(\int_{\mathbf{S}^{l-1}} u(X,(rt,\Theta))dS(\Theta)\Big)dt \quad (0\leq |Y|=r)$$

which is the volume average over $\{Y'\in\mathbf{R}^l;\ |Y'|<r\}$ of a subharmonic function

$$u(X,Y')=u(X,(t,\Theta)) \quad (X\in D,\ Y'=(t,\Theta)\in\mathbf{R}^l)$$

defined on $D\times\mathbf{R}^l$.

The following Theorem 5 gives a theorem of the Phragmén-Lindelöf type different from Theorems A and B. Theorem 5 is better than Theorem A in the sense that (2.2) follows from (1.1).

Theorem 5. *Let* $u(X,Y)$ *be a subharmonic function on a strip* $D \times \mathbf{R}^l$ *satisfying the Phragmén-Lindelöf boundary condition. If*

$$\underline{\lim}_{r\to\infty} r^{(l-1)/2} \exp(-\sqrt{\lambda_D} r) N(V_{u^+})(r) = 0, \tag{2.2}$$

then

$$u(X,Y) \leq 0$$

on $D \times \mathbf{R}^l$.

Remark 1. Since $r^{l/2-1} I_{l/2-1}(\sqrt{\lambda_D} r)\}^{-1} N(C_{u^+})(r)$ is non-decreasing, for sufficiently large r

$$\begin{aligned} &\int_D \Big(\int_{\mathbf{s}^{l-1}} u^+(X,(\rho,\Theta)) dS(\Theta)\Big) f_D(X) dX \\ &\quad \leq \{\rho^{1-l/2} I_{l/2-1}(\sqrt{\lambda_D}\rho)\}\{r^{1-l/2} I_{l/2-1}(\sqrt{\lambda_D} r)\}^{-1} \\ &\qquad \times \int_D \Big(\int_{\mathbf{s}^{l-1}} u^+(X,(r,\Theta)) dS(\Theta)\Big) f_D(X) dX \end{aligned}$$

for any ρ, $\rho \leq r$. Here for these r there is a positive constant C_3 independent of r such that

$$\{\rho^{1-l/2} I_{l/2-1}(\sqrt{\lambda_D}\rho)\}\{r^{1-l/2} I_{l/2-1}(\sqrt{\lambda_D} r)\}^{-1} \leq C_3$$

for any ρ, $\rho \leq r$, because there is a positive limit

$$\lim_{\rho\to 0+} \rho^{1-l/2} I_{l/2-1}(\sqrt{\lambda_D}\rho) \tag{2.3}$$

exists and

$$\lim_{t\to+\infty} (2\pi t)^{1/2} e^{-t} I_{l/2-1}(t) = 1 \tag{2.4}$$

(see [5, Lemma A, (ii) and (iii)). These give that

$$N(V_{u^+})(r) \leq C_3 N(C_{u^+})(r)$$

for sufficiently large r. Hence (2.2) can be immediately deduced from (1.1).

2.3. Harmonic Majorants

The following Theorem 6 can be regarded as a generalization of Theorem 4.

Theorem 6. *Let u be a radial subharmonic function on $D \times \mathbf{R}^l$ satisfying the Phragmén-Lindelöf boundary condition. Then the limit $\mu(N(u))$ and $\mu(N(u^+))$ $(-\infty < \mu(N(u)) \leq \mu(N(u^+)) \leq \infty)$ exist. If (1.5) is satisfied, then*

$$u(X,Y) \leq \mu(N(u))r^{1-l/2}I_{l/2-1}(\sqrt{\lambda_D}r)f_D(X) \quad (|Y| = r) \tag{2.5}$$

on $D \times \mathbf{R}^l$.

Remark 2. Put

$$h(X,Y) = C_4 r^{1-l/2}I_{l/2-1}(\sqrt{\lambda_D}r)f_D(X)$$

with a real number C_4. Then h is a radial harmonic function on $D\times\mathbf{R}^l$ vanishing on $\partial D \times \mathbf{R}^l$ (for the proof of the case $D = (0,1)$, see Brawn [6, Lemma 1]) and

$$\mu(N(h)) = \lim_{r\to\infty} r^{l/2-1}\{I_{l/2-1}(\sqrt{\lambda_D}r)\}^{-1}N(h)(r) = C_4$$

Hence this h gives us the equality in (2.5).

For a non-negative radial subharmonic function, we have from Theorem 6 the following result.

Corollary 3. *Let $u(X,Y)$ be a non-negative radial subharmonic function on $D\times\mathbf{R}^l$ satisfying the Phragmén-Lindelöf boundary condition. If $\mu(N(u)) < \infty$, then*

$$h_u(X,Y) = \mu(N(u))r^{1-l/2}I_{l/2-1}(\sqrt{\lambda_D}r)f_D(X) \quad (|Y| = r)$$

is the least radial harmonic majorant of u.

By applying Theorem 6 to typical radial subharmonic functions $C_u(X,Y)$ and $M_u(X,Y)$, we have the following Corollary 4 and Theorem 7. The existence of the limit $\mu(N(C_u))$ is connected with a question about the regularity of the Nevanlinna norm of a subharmonic function satisfying the Phragmén-Lindelöf boundary condition, which was originally proved by Ahlfors [1] and Heins [9] for an analytic function on the typical strip $(0,\pi) \times \mathbf{R}^1$. Yoshida [13] generalized their results for a subharmonic function on the cylinder $D \times \mathbf{R}^1$.

Corollary 4. *Let $u(X,Y)$ be a subharmonic function on $D \times \mathbf{R}^l$ satisfying the Phragmén-Lindelöf boundary condition. If*

$$\underline{\lim}_{r\to\infty} r^{(l-1)/2}\exp(-\sqrt{\lambda_D}r)N(C_{u^+})(r) < \infty,$$

then the finite limit $\mu(N(C_u))$ exists and

$$C_u(X,Y) \leq \mu(N(C_u))r^{1-l/2}I_{l/2-1}(\sqrt{\lambda_D}r)f_D(X) \quad (|Y| = r)$$

on $D \times \mathbf{R}^l$.

If we apply Theorem 6 to $M_u(X,Y)$ in particular, we directly have a harmonic majorant of $u(X,Y)$.

Theorem 7. *Let* $u(X,Y)$ *be a subharmonic function on* $D \times \mathbf{R}^l$ *satisfying the Phragmén-Lindelöf boundary condition. If*

$$\underline{\lim}_{r\to\infty} r^{(l-1)/2} \exp(-\sqrt{\lambda_D} r) N(M_{u^+})(r) < \infty,$$

then the finite limit $\mu(N(M_u))$ *exists and*

$$u(X,Y) \le \mu(N(M_u)) r^{1-l/2} I_{l/2-1}(\sqrt{\lambda_D} r) f_D(X) \quad (|Y| = r) \tag{2.6}$$

on $D \times \mathbf{R}^l$.

Remark 3. As in Remark 2, that h gives us the equality in (2.6).

2.4. Regularity of Maximum Modulus

Theorem 8. *Let* $u(X,Y)$ *be a radial subharmonic function on* $D \times \mathbf{R}^l$ *satisfying the Phragmén-Lindelöf boundary condition. Then the limit* $\mu(L(u))$, $0 \le \mu(L(u)) \le \infty$, *exists and*

$$\mu(L(u)) = (K(u))^+ \max_{X\in D} f_D(X) = (\mu(N(u)))^+ \max_{X\in D} f_D(X). \tag{2.7}$$

The following Theorem 9 is also connected with a question about the regularity of maximum modulus

$$L(M_u)(r) = \max_{X\in D, |Y|=r} u(X,Y)$$

of a subharmonic function on $D \times \mathbf{R}^l$ satisfying the Phragmén-Lindelöf boundary condition, which was originally asked by Ahlfors [1] for an analytic function on the tipical strip $(0,\pi) \times \mathbf{R}^1$. Heins [9] answered to this question and Yoshida [13] also generalized Heins result for a subharmonic function on the cylinder $D \times \mathbf{R}^1$.

By applying Theorem 5 to $M_u(X,Y)$, we immediately have the following theorem.

Theorem 9. *Let* u *be a subharmonic function on* $D \times \mathbf{R}^l$ *satisfying the Phragmén-Lindelöf boundary condition. Then the limit*

$$\mu(L(M_u)) = \lim_{r\to\infty} r^{l/2-1} \{I_{l/2-1}(\sqrt{\lambda_D} r)\}^{-1} L(M_u)(r),$$

$$0 \le \mu(L(M_u)) \le \infty,$$

exists and

$$\mu(L(M_u)) = (K(M_u))^+ \max_{X\in D} f_D(X) = (\mu(N(u)))^+ \max_{X\in D} f_D(X),$$

where

$$K(M_u) = \overline{\lim}_{r\to\infty} r^{l/2-1}\{I_{l/2-1}(\sqrt{\lambda_D}r)\}^{-1} \sup_{X\in D,|Y|=r} \frac{u(X,Y)}{f_D(X).}$$

3. Proofs of Theorem 1 and Theorem 2

First of all we shall give some notations used through this paper. We put

$$B_l(r) = \{Y \in \mathbf{R}^l; |Y| < r\}, \quad \overline{B_l(r)} = \{Y \in \mathbf{R}^l; |Y| \leq r\},$$

and

$$S_l(r) = \{Y \in \mathbf{R}^l; |Y| = r\} \quad (0 < r < \infty).$$

Proof of Theorem 1. First we note that

$$\frac{\partial^2 h}{\partial r^2} = -\frac{l-1}{r}\frac{\partial h}{\partial r} - \Delta_m h\,. \tag{3.1}$$

Put

$$\phi(t) = t^{1-l/2} I_{l/2-1}(\sqrt{\lambda_D}t),$$

$$T_h(r) = s_l^{-1}\{\phi(r)\}^{-1}\int_D h(X,Y) f_D(X)dX \quad (|Y| = r)\,, \tag{3.2}$$

and

$$J_h(r) = r^{l-1}\{\phi(r)\}^2 \frac{d}{dr} T_h(r). \tag{3.3}$$

From Green's formula, we have

$$\int_D (\Delta_m h(X,Y)) f_D(X)dX = \int_D h(X,Y)(\Delta_m f_D(X))dX$$
$$(\rho_1 < |Y| = r < \rho_2).$$

Hence we obtain from (3.1) and (3.2) that

$$T_h''(r)\phi(r) = \lambda_D T_h(r)\phi(r) - 2T_h'(r)\phi'(r)$$

$$-T_h(r)\phi''(r) - \frac{l-1}{r}\{T_h'(r)\phi(r) + T_h(r)\phi'(r)\}.$$

With (3.3) this gives

$$r^{1-l}\frac{d}{dr}J_h(r) = -\phi(r)T_h(r)\{\phi''(r) + \frac{l-1}{r}\phi'(r) - \lambda_D\phi(r)\} \quad (\rho_1 < r < \rho_2).$$

Since

$$\phi''(r) + \frac{l-1}{r}\phi'(r) - \lambda_D\phi(r) = 0$$

(c.f. Armitage and Fugard [5, Lemma A, (vi)]), we have

$$r^{1-l}\frac{d}{dr}J_h(r) = 0,$$

which gives that $J_h(r)$ is constant on (ρ_1, ρ_2). Since

$$r^{-1}\{I_{l/2-1}(\sqrt{\lambda_D}r)\}^{-2} = -\frac{d}{dr}(\frac{K_{l/2-1}(\sqrt{\lambda_D}r)}{I_{l/2-1}(\sqrt{\lambda_D}r)})$$

(c.f. Armitage and Fugard [5, Lemma A, (vi)]), we obtain

$$r^{1-l/2}\{\phi(r)\}^{-1}N(h)(r) = C_1\frac{K_{l/2-1}(\sqrt{\lambda_D}r)}{I_{l/2-1}(\sqrt{\lambda_D}r)} + C_2 \quad (\rho_1 < r < \rho_2),$$

and hence

$$N(h)(r) = C_1 r^{1-l/2}K_{l/2-1}(\sqrt{\lambda_D}r) + C_2 r^{1-l/2}I_{l/2-1}(\sqrt{\lambda_D}r) \quad (\rho_1 < r < \rho_2),$$

where C_1 and C_2 are two constants independent of r, which is the conclusion of the first part.

Since $N(h)(r)$ takes the value $N(h)(t_i)$ at a point t_i $(i = 1, 2)$, the conclusion of the second pat follows immediately. □

Proof of Theorem 2. We shall first prove that

$$N(u)(r) > -\infty \quad (0 < r < \infty).$$

Take any number r $(0 < r < \infty)$ and define a boundary function Λ on the boundary $\partial(D \times B_l(r))$ of $D \times B_l(r)$ in $\mathbf{R}^{l+m}$ by

$$\Lambda(X,Y) = \begin{cases} u(X,Y) & \text{on } D \times S_l(r)\,, \\ 0 & \text{on } \partial D \times \overline{B_l(r)}. \end{cases}$$

This Λ is an upper semi-continuous function which is bounded above. Hence there exists a decreasing sequence $\{\Lambda_j\}$ of continuous functions on $\partial(D\times B_l(r))$ which converges to Λ as $j\to\infty$. Denote the solution of the Dirichlet problem with Λ_j on $D\times B_l(r)$ by H_j. Evidently we can assume that $H_j(X,Y)$ is radial on $D\times B_l(r)$ and

$$H_j(X,Y)\geq u(X,Y) \quad \text{on} \;\; D\times B_l(r).$$

Thus H_j decreases to a radial harmonic function h on $D\times B_l(r)$ such that h satisfies $u\leq h$ on $D\times B_l(r)$. In the same way as in Yoshida [12, pp.286-287], we can show that h has the vanishing boundary value on $\partial D\times B_l(r)$.

Let r_0 be a number satisfying $0<r_0<r$ and let $\{r_i\}$ $(r_0<r_i<r)$ be a sequences such that $r_i\to r$ $(i\to\infty)$. Then

$$\Phi(r_0; N(h), \sqrt{\lambda_D}, r^*, r_i)=0 \quad (i=1,2,3,\ldots)$$

for a number r^* $(0<r^*<r_0)$, i.e.

$$N(h)(r_i)=\{I(r_0)K(r^*)-K(r_0)I(r^*)\}^{-1} \tag{3.4}$$

$$\times[\{I(r_0)K(r_i)-I(r_0)K(r_i)\}N(h)(r^*)+\{I(r_i)K(r^*)-K(r_i)I(r^*)\}N(h)(r_0)]$$

from Theorem 1, where

$$I(t)=I_{l/2-1}(\sqrt{\lambda_D}t) \;\text{ and }\; K(t)=K_{l/2-1}(\sqrt{\lambda_D}t).$$

Observe that if we set

$$N(\Lambda_j)(r)=\int_D \Lambda_j(X,Y)f_D(X)dX \qquad (|Y|=r),$$

then

$$N(\Lambda_j)(r)\to N(u)(r) \qquad (j\to\infty). \tag{3.5}$$

Since

$$\overline{\lim}_{i\to\infty}N(h)(r_i)\leq \lim_{i\to\infty}N(H_j)(r_i)=N(\Lambda_j)(r)$$

for any j, we obtain from (3.4) that

$$N(\Lambda_j)(r)\geq \frac{\{I(r_0)K(r)-I(r_0)K(r)\}N(h)(r^*)+\{I(r)K(r^*)-K(r)I(r^*)\}N(h)(r_0)}{I(r_0)K(r^*)-K(r_0)I(r^*).}$$

This and (3.5) give that $N(u)(r)>-\infty$.

Next we shall show the main part of this theorem. Take any t_1 and any t_2 $(0 < t_1 < t_2 < \infty)$ and define a boundary function Λ on the boundary $\partial(D \times T_l(t_1, t_2))$ of $D \times T_l(t_1, t_2)$ by

$$\Lambda(X, Y) = \begin{cases} u(X, Y) & \text{on } D \times S_l(t_i) \quad (i = 1, 2)\,, \\ 0 & \text{on } \partial D \times \overline{T_l(t_1, t_2)}. \end{cases}$$

Take a decreasing sequence $\{\Lambda_j\}$ of continuous functions on $\partial(D \times T_l(t_1, t_2))$ which converges to Λ as $j \to \infty$. Denote the solution of the Dirichlet problem with Λ_j on $D \times T_l(t_1, t_2)$ by H_j. Then H_j decreses to a radial harmonic function h on $D \times T_l(t_1, t_2)$ such that h vanishes on $\partial D \times T_l(t_1, t_2)$. Since

$$N(u)(r) \le \lim_{j \to \infty} N(H_j)(r) \qquad (t_1 < r < t_2)$$

and

$$\lim_{j \to \infty} N(\Lambda_j)(t_i) = N(u)(t_i) \qquad (i = 1, 2),$$

where

$$N(\Lambda_j)(t_i) = \int_D \Lambda_j(X, Y) f_D(X) dX \qquad (|Y| = t_i) \quad (i = 1, 2),$$

we see that

$$\begin{aligned} &\Phi(r; N(u), \sqrt{\lambda_D}, t_1, t_2) \\ &\ge \lim_{j \to \infty} \begin{vmatrix} N(H_j)(r) & r^{1-l/2} I_{l/2-1}(\sqrt{\lambda_D} r) & r^{1-l/2} K_{l/2-1}(\sqrt{\lambda_D} r) \\ N(\Lambda_j)(t_1) & {t_1}^{1-l/2} I_{l/2-1}(\sqrt{\lambda_D} t_1) & {t_1}^{1-l/2} K_{l/2-1}(\sqrt{\lambda_D} t_1) \\ N(\Lambda_j)(t_2) & {t_2}^{1-l/2} I_{l/2-1}(\sqrt{\lambda_D} t_2) & {t_2}^{1-l/2} K_{l/2-1}(\sqrt{\lambda_D} t_2) \end{vmatrix} \\ &\qquad (t_1 < r < t_2). \end{aligned} \tag{3.6}$$

Take any r $(t_1 < r < t_2)$. Let $\{r_n^{(1)}\}$ and $\{r_n^{(2)}\}$ $(t_1 < r_n^{(1)} < r < r_n^{(2)} < t_2)$ be two sequences such that $r_n^{(i)} \to t_i$ $(n \to \infty)$ $(i = 1, 2)$. By applying Theorem 1 to h, we obtain that

$$\Phi(r; N(h), \sqrt{\lambda_D}, r_n^{(1)}, r_n^{(2)}) = 0$$

for any n. Since

$$N(H_j)(r_n^{(i)}) \to N(\Lambda_j)(t_i) \qquad (n \to \infty) \qquad (i = 1, 2)$$

for any j, we have from (3.6) that

$$\Phi(r; N(u), \sqrt{\lambda_D}, t_1, t_2) \ge 0.$$

This shows $A_{\sqrt{\lambda_D}, l}$-convexity of N_u on $(0, \infty)$. $\square$

4. Proofs of Theorem 3 and Theorem 4

We denote

$$\frac{K_{l/2-1}(t)}{I_{l/2-1}(t)}$$

by $L_{l/2-1}(t)$. We remark that $L_{l/2-1}(t)$ is strictly decreasing on $(0,\infty)$ (see Armitage and Fugard [5, Lemma A, (iv)]). In this section, we sometimes denote $I_{l/2-1}(\lambda t), K_{l/2-1}(\lambda t)$ and $L_{l/2-1}(\lambda t)$ with a positive constant λ by $I(\lambda t), K(\lambda t)$ and $L(\lambda t)$, respectively.

Lemma 4.1. *The following (I), (II) and (III) are equivalent:*

(I) $\phi(t)$ is $A_{\lambda,l}$-convex on $(0,\infty)$,

(II) $\dfrac{t^{l/2-1}\phi(t)}{I_{l/2-1}(\lambda t)}$ is a convex function of $L_{l/2-1}(\lambda t)$ on $(0,\infty)$,

(III) $\dfrac{t^{l/2-1}\phi(t)}{K_{l/2-1}(\lambda t)}$ is a convex function of $L^{-1}_{l/2-1}(\lambda t)$ on $(0,\infty)$.

Proof. Since

$$\begin{vmatrix} \phi(t) & t^{1-l/2}I(\lambda t) & t^{1-l/2}K(\lambda t) \\ \phi(t_1) & {t_1}^{1-l/2}I(\lambda t_1) & {t_1}^{1-l/2}K(\lambda t_1) \\ \phi(t_2) & {t_2}^{1-l/2}I(\lambda t_2) & {t_2}^{1-l/2}K(\lambda t_2) \end{vmatrix}$$

$$= -(tt_1t_2)^{1-l/2}I(\lambda t)I(\lambda t_1)I(\lambda t_2)\begin{vmatrix} \dfrac{t^{l/2-1}\phi(t)}{I(\lambda t)} & L(\lambda t) & 1 \\ \dfrac{{t_1}^{l/2-1}\phi(t_1)}{I(\lambda t_1)} & L(\lambda t_1) & 1 \\ \dfrac{{t_2}^{l/2-1}\phi(t_2)}{I(\lambda t_2)} & L(\lambda t_2) & 1 \end{vmatrix} \geq 0\,,$$

$(t_1 \leq t \leq t_2)$, it immediately follows that (I) is equivalent to (II). The equivalence between (I) and (III) can be proved in the same way. □

Lemma 4.2. *Let λ be a positive constant and $\phi(t)$ be $A_{\lambda,l}$-convex on $(0,\infty)$.*

(I) Both limits

$$\mu(\phi) = \lim_{t\to\infty}\frac{t^{l/2-1}\phi(t)}{I_{l/2-1}(\lambda t)}, \quad -\infty < \mu(\phi) \leq \infty$$

and

$$\eta(\phi) = \lim_{t\to 0}\frac{t^{l/2-1}\phi(t)}{K_{l/2-1}(\lambda t)}, \quad -\infty < \eta(\phi) \leq \infty$$

exist.

(II) If $\eta(\phi) \le 0$, then $\dfrac{t^{l/2-1}\phi(t)}{I_{l/2-1}(\lambda t)}$ is non-decreasing.

Proof. From Lemma 4.1, it follows that

$$\frac{{t_2}^{l/2-1}\phi(t_2)/K(\lambda t_2) - {t_1}^{l/2-1}\phi(t_1)/K(\lambda t_1)}{1/L(\lambda t_2) - 1/L(\lambda t_1)}, \tag{4.1}$$

respectively

$$\frac{{t_2}^{l/2-1}\phi(t_2)/I(\lambda t_2) - {t_1}^{l/2-1}\phi(t_1)/I(\lambda t_1)}{L(\lambda t_2) - L(\lambda t_1)} \tag{4.2}$$

is a non-decreasing (resp. a non-increasing) function of t_2 (resp. t_1) ($t_1 \ne t_2$), when t_1 in (4.1) (resp. t_2 in (4.2)) is fixed. Since

$$\lim_{t\to\infty} \frac{1}{L_{l/2-1}(\lambda t)} = \infty \qquad (\text{resp. } \lim_{t\to 0} L_{l/2-1}(\lambda t) = \infty)$$

(see Armitage and Fugard [5, Lemma A, (iii)]), we know the existence of $\mu(\phi)$ (resp. $\eta(\phi)$). If $\eta(\phi) \le 0$, then (4.2) is non-positive for any $t_1 (\ne t_2)$. Hence $\frac{t^{l/2-1}\phi(t)}{I_{l/2-1}(\lambda t)}$ is non-decreasing. □

Proof of Theorem 3. By Theorem 2, $N(u)(r)$ is $A_{\sqrt{\lambda_D},l}$-convex on $(0,\infty)$. Since $u(X,Y)$ is bounded above on $D \times \{Y \in \mathbf{R}^l : 0 < |Y| < \delta\}$ ($\delta > 0$), we see

$$\eta(N(u)) = \lim_{r\to 0+} \{K_{l/2-1}(\sqrt{\lambda_D} r)\}^{-1} t^{l/2-1} N(u)(r) \le 0,$$

because

$$\begin{cases} -(\log t)^{-1} K_0(t) & (l = 2), \\ t^{l/2-1} K_{l/2-1}(t) & (l \ge 3) \end{cases} \tag{4.3}$$

has a positive finite limit as $t \to 0+$ (see [5, p.80, (12)-(15)]). Thus the first part of the conclusion follows from (II) of Lemma 4.2.

Now we shall prove the seond part of Theorem 3. First, suppose that u is harmonic and vanishes on $\partial D \times \mathbf{R}^{\mathbf{l}}$. Then from Theorem 1 with $\rho_1 = 0$ and $\rho_2 = \infty$ we have

$$N(u)(r) = C_1 r^{1-l/2} K_{l/2-1}(\sqrt{\lambda_D} r) + C_2 r^{1-l/2} I_{l/2-1}(\sqrt{\lambda_D} r) \quad (0 < r < \infty).$$

Since $|u|$ is bounded on $D \times B_l(\delta)$ ($\delta > 0$), we know $C_1 = 0$ from (2.3) and (4.3). Hence

$$r^{l/2-1} \{I_{l/2-1}(\sqrt{\lambda_D} r)\}^{-1} N(u)(r) = C_2 \quad (0 < r < \infty),$$

which shows 'if' part.

To prove 'only if' pat, take any positive intger n. By the same way as in Theorem 2, we have a radial harmnic functin h_n in $D \times B_l(n)$ vanishing on $\partial D \times B_l(n)$ such that h_n satisfies

$$u \leq h_n \tag{4.4}$$

in $D \times B_l(n)$. Since $N(h_n)(n) = N(u)(n)$, we have

$$N(h_n)(r) = N(u)(r) \quad (0 < r \leq n) \tag{4.5}$$

from Theorm 1. Hence from (4.4) and (4.5) we have $h_n(X,Y) = u(X,Y)$ a.e. on D for any $Y, |Y| \leq n$. Thus we see that $h_n = u$ on $D \times B_l(n)$ and hence u is a harmonic function on $D \times B_l(n)$ vanishing on $\partial D \times B_l(n)$. As $n \to \infty$, we have the conclusin of 'only if' part. □

Proof of Theorem 4. Since $r^{l/2-1}\frac{N(u^+)(r)}{I_{l/2-1}(\sqrt{\lambda_D}r)}$ is non-decreasing from Theorem 3, (2.1) gives $N(u^+)(r) \equiv 0$ and hence for any $Y,\ |Y| > 0,\ \ u^+(X,Y) = 0$ a.e. on D. Theorem 4 immediately follows from the volume mean-value property of $u^+(X,Y)$. □

5. Proofs of Theorem 6 and Corollary 3

We denote the Green function on $D \times \mathbf{R}^l$ by $G(P,Q), P \in D \times \mathbf{R}^l$, $Q \in D \times \mathbf{R}^l$. When we remember that the set of infinite Martin boundary points of $D \times \mathbf{R}^l$ is equivalent to the unit sphere $\mathbf{S}^{l-1}$, we also denote the Martin function on $\partial D \times \mathbf{R}^l$ by $K(P,Q), P \in D \times \mathbf{R}^l$, $Q \in (\partial D \times \mathbf{R}^l) \cup S^{l-1}$. For a positive measure λ on $D \times \mathbf{R}^l$ (resp. $\partial D \times \mathbf{R}^l$), denote the value of the integral

$$\int_{D \times \mathbf{R}^l} G(P,Q) d\lambda(Q) \quad (\text{resp.} \int_{\partial D \times \mathbf{R}^l} K(P,Q) d\lambda(Q))$$

at a point $P = (X,Y) \in D \times \mathbf{R}^l$ by $G_\lambda(P)$ or $G_\lambda(X,Y)$ (resp. $K_\lambda(P)$ or $K_\lambda(X,Y)$).

The following lemma immediately follows from Aikawa [2, Lemma 3].

Lemma 5.1. *Given $r_1 > 0$ there are two positive constants C_5, $C_5 > r_1$, and C_6 depending on r_1 such that if $r_2 \geq C_5$, then $G(P,Q)$ (resp. $K(P,Q)$)$< C_6$ for any $P \in D \times T_l(r_2, \infty)$ and any $Q \in D \times \overline{B_l(r_1)}$ (resp. $Q \in \partial D \times \overline{B_l(r_1)}$).*

Lemma 5.2. *Let $u(X,Y)$ be a radial subharmonic function on $D \times \mathbf{R}^l$ satisfying the Phragmén-Lindelöf boundary condition. If (1.5) is satisfied, then there exists a constant C_7 such that*

$$u(X,Y) \leq C_7 r^{1-l/2} I_{l/2-1}(\sqrt{\lambda_D} r) f_D(X) \quad (|Y| = r)$$

on $D \times \mathbf{R}^l$.

Proof. Take any positve integer n and defne a boundary fnction Λ_n on the boundary $\partial(D \times B_l(n))$ of $D \times B_l(n)$ in $\mathbf{R}^{l+m}$ by

$$\Lambda_n = \begin{cases} u^+ & \text{on } D \times S_l(n), \\ 0 & \text{on } \partial D \times \overline{B_l(n)}. \end{cases}$$

This Λ_n is an upper semi-continous function which is bounded above. Hence there exists a decreasing sequence $\{(\Lambda_n)_j\}$ of continuous functions on $\partial(D \times B_l(n))$ which converges to Λ_n as $j \to \infty$. Denote the solution of the Dirichlet problem with $\{(\Lambda_n)_j\}$ on $D \times B_l(n)$ by $(H_n)_j$. Evidently we can assume that $(H_n)_j$ is radial on $D \times B_l(n)$ and $(H_n)_j \geq u^+$ on $D \times B_l(n)$. Thus $(H_n)_j$ decreases to a radial harmonic function h_n on $D \times B_l(n)$ as $j \to \infty$ such that h_n satisfies $u^+ \leq h_n$ on $D \times B_l(n)$. In the same way as in Yoshida [12, pp. 286-287], we can show that h_n has the vanishing boundary value on $\partial D \times B_l(n)$. Since $D \times B_l(n)$ is a bounded regulr domain, h_n is the least harmonic majorant of u^+ in $D \times B_l(n)$. Hence h_n increases with n and

$$\lim_{n\to\infty} h_n = h$$

is a harmonic function or otherwise identically $+\infty$ in $D \times \mathbf{R}^l$ such that

$$u^+ \leq h \tag{5.1}$$

on $D \times \mathbf{R}^l$. By Theorem 1

$$\frac{r^{l/2-1} N(h_n)(r)}{I_{l/2-1}(\sqrt{\lambda_D} r)} = \frac{r^{l/2-1} N(u^+)(n)}{I_{l/2-1}(\sqrt{\lambda_D} n)} \qquad (0 < r < n)$$

and hence for any fixed r $(0 < r < \infty)$

$$\frac{r^{l/2-1} N(h)(r)}{I_{l/2-1}(\sqrt{\lambda_D} r)} = \lim_{n\to\infty} \frac{n^{l/2-1} N(u^+)(n)}{I_{l/2-1}(\sqrt{\lambda_D} n)} = \mu(N(u^+)) < +\infty.$$

This shows that h is not identically $+\infty$ and hence h is a radial harmonic function on $D \times \mathbf{R}^l$. To see that h vanishes on $\partial D \times \mathbf{R}^l$, apply Theorem 3 to $-h$. Thus h is a positive radial harmonic function on $D \times \mathbf{R}^l$ and vanishing on $\partial D \times \mathbf{R}^l$.

Since the Martin boundary of $D \times \mathbf{R}^l$ is $(\partial D \times \mathbf{R}^l) \cup \mathbf{S}^{l-1}$, h is the sum of two integrals

$$h(X,Y) = \int_{\mathbf{S}^{l-1}} K((X,Y), M_\Theta) d\nu(\Theta) + K\nu(X,Y) \tag{5.2}$$

by the Martin representation theorem (see Helms [9, Theorem 12.17]), where ν is a positive measure on $(\partial D \times \mathbf{R}^l) \cup \mathbf{S}^{l-1}$. Since h is radial, the measure ν is uniformly distributed on $\mathbf{S}^{l-1}$ and equal to the surface measure $dS(\Theta)$ of $\mathbf{S}^{l-1}$ multiplied by a constant. Hence if we note that

$$\int_{\mathbf{S}^{l-1}} K((X,Y), M_\Theta) dS(\Theta) = Cr^{1-l/2} I_{l/2-1}(\sqrt{\lambda_D} r) f_D(X) \quad (|Y| = r),$$

where C is a positive constant (see Aikawa [3, p. 124]), then we have

$$\int_{\mathbf{S}^{l-1}} K((X,Y), M_\Theta) d\nu(\Theta) = C_7 r^{1-l/2} I_{l/2-1}(\sqrt{\lambda_D} r) f_D(X) \quad (|Y| = r). \tag{5.3}$$

Now we shall show that $K\nu \equiv 0$ on $D \times \mathbf{R}^l$. Then (5.1), (5.2) and (5.3) give the conclusion of this lemma.

Suppose that $\nu(\partial D \times \mathbf{R}^l) > 0$. Then there is $r_1 > 0$ such that $\nu(\partial D \times \overline{B_l(r_1)}) > 0$. Put

$$h_1(P) = \int_{\partial D \times B_l(r)} K(P,Q) d\nu(Q) \quad (P \in D \times \mathbf{R}^l).$$

Since $h_1 \leq K\nu \leq h$, h_1 vanishes on $\partial D \times \mathbf{R}^l$. From Lemma 5.1 we observe that h_1 is bounded on $D \times T_l(r_2, \infty)$ for some $r_2 > 0$ and hence bounded on $D \times \mathbf{R}^l$. Therefore (2.4) and Theorem 4 lead to $h_1 \equiv 0$ on $D \times \mathbf{R}^l$. This is a contradiction. □

Proof of Theorem 6. Let u be a radial subharmonic function on $D \times \mathbf{R}^l$satisfying the Phragmén-Lindelöf boundary condition. With the constant C_7 in (5.3), put

$$v(X,Y) = C_7 r^{1-l/2} I_{l/2-1}(\sqrt{\lambda_D} r) f_D(X) - u(X,Y) \quad (|Y| = r).$$

Then by Lemma 5.2 v is a positive superharmonic function on $D \times \mathbf{R}^l$. By the Riesz decomposition theorem (see e.g. Helms [9, Theorem 6.18]) it is presented as the sum

$$v(X,Y) = G\lambda(X,Y) + h_v(X,Y)$$

of the Green potential $G\lambda(X,Y)$ with a positive measure λ on $D \times \mathbf{R}^l$ and a positive harmonic function $h_v(X,Y)$ on $D \times \mathbf{R}^l$ which is also the sum of two integrals

$$h_v(X,Y) = \int_{s^{l-1}} K((X,Y), M_\Theta) d\nu(\Theta) + K\nu(X,Y)$$

by the Martin representation theorem (see Helms [9, Theorem 12.17]), where $M_\Theta, \Theta \in \mathbf{S}^{l-1}$, is an infinite Martin boundary point and ν is a positive measure on $(\partial D \times \mathbf{R}^l) \cup \mathbf{S}^{l-1}$. Since $v(X,Y)$ is radial, $h_v(X,Y)$ is also radial. Hence the measure ν is uniformly distributed on $\mathbf{S}^{l-1}$ and equal to the surface measure of $\mathbf{S}^{l-1}$ multiplied by a constant. Hence we have

$$\int_{\mathbf{S}^{l-1}} K((X,Y), M_\Theta) d\nu(\Theta) = C_8 r^{1-l/2} I_{l/2-1}(\sqrt{\lambda_D} r) f_D(X) \quad (|Y| = r),$$

where C_8 is a positive constant. Thus we obtain

$$u(X,Y) = (C - C_8) r^{1-l/2} I_{l/2-1}(\sqrt{\lambda_D} r) f_D(X) - G\lambda(X,Y) - K\nu(X,Y) \qquad (|Y| = r). \tag{5.4}$$

Observing that $G\lambda(X,Y)$ and $K\nu(X,Y)$ are also radial, we see

$$N(u)(r) = (C - C_8) r^{1-l/2} I_{l/2-1}(\sqrt{\lambda_D} r) - N(G\lambda)(r) - N(K\nu)(r) \qquad (0 < r < \infty).$$

If we can prove that

$$\mu(N(G\lambda)) = \mu(N(K\nu)) = 0, \tag{5.5}$$

then $\mu(N(u)) = C - C_8$. Since

$$G\lambda(X,Y) \geq 0 \ \text{ and } \ K\nu(X,Y) \geq 0$$

on $D \times \mathbf{R}^l$, we immediately obtain from (5.4) that

$$u(X,Y) \leq \mu(N(u)) r^{1-l/2} I_{l/2-1}(\sqrt{\lambda_D} r) f_D(X)$$

on $D \times \mathbf{R}^l$, which is the conclusion of this theorem. □

We shall show (5.5). Let $\lambda^*(\rho)$ (resp. $\nu^*(\rho)$) $(0 < \rho < \infty)$ be the measure on $D \times \mathbf{R}^l$ (resp. $\partial D \times \mathbf{R}^l$) defined by

$$\lambda^*(\rho)(E) = \lambda(E \cap (D \times T_l(\rho, \infty)) \ \ (\text{resp. } \nu^*(E \cap (\partial D \times T_l(\rho, \infty))))$$

for every Borel subset E of $D \times \mathbf{R}^l$ (resp. $\partial D \times \mathbf{R}^l$). By applying Theorem 2 to $-G\lambda(X,Y)$ (resp. $-K\nu(X,Y)$) on $D \times \mathbf{R}^l$ (resp. $\partial D \times \mathbf{R}^l$), we have

$$N(G\lambda)(r) < \infty \ \ (\text{resp. } N(K\nu)(r) < \infty) \quad (0 < r < \infty).$$

Take a positive number r_1 from (2.4) such that

$$r_1^{1-l/2} I_{l/2-1}(\sqrt{\lambda_D} r_1) > 1$$

and take any positive number ε. Since

$$N(G\lambda)(r_1) \ \ (\text{resp. } N(K\nu)(r_1)) \ < \infty$$

we can find a sufficiently large r_2 such that

$$N(G\lambda^*(r_2))(r_1) \ \ (\text{resp. } N(K\nu^*(r_2))(r_1)) \ < \frac{\varepsilon}{2.}$$

Then

$$r_1^{l/2-1} N(G\lambda^*(r_2))(r_1)\{I_{l/2-1}(\sqrt{\lambda_D} r_1)\}^{-1} \tag{5.6}$$
$$(\text{resp. } r_1^{l/2-1} N(K\nu^*(r_2))(r_1)\{I_{l/2-1}(\sqrt{\lambda_D} r_1)\}^{-1}) \ < \frac{\varepsilon}{2.}$$

Since $-G\lambda^*(r_2)$ (resp. $-K\nu^*(r_2)$) is a non-positive subharmonic function on $D \times \mathbf{R}^l$, $N(-G\lambda^*(r_2))(r)$ (resp. $N(-K\nu^*(r_2))(r)$) is $A_{\sqrt{\lambda_D},l}$-convex on $(0, \infty)$ by Theorem 2. From Lemma 4.2, a limit

$$\eta(N(-G\lambda^*(r_2))) \ \ (\text{resp. } \eta(N(-K\lambda^*(r_2)))) \ \leq 0$$

exists and

$$r^{l/2-1}\{I_{l/2-1}(\sqrt{\lambda_D} r)\}^{-1} N(-G\lambda^*(r_2))(r)$$
$$(\text{resp. } r^{l/2-1}\{I_{l/2-1}(\sqrt{\lambda_D} r)\}^{-1} N(-K\nu^*(r_2))(r))$$

is non-decreasing. Hence we see from (5.6) that

$$r^{l/2-1}\{I_{l/2-1}(\sqrt{\lambda_D} r)\}^{-1} N(G\lambda^*(r_2))(r) \tag{5.7}$$
$$(\text{resp. } r^{l/2-1}\{I_{l/2-1}(\sqrt{\lambda_D} r)\}^{-1} N(K\nu^*(r_2))(r)) \ < \frac{\varepsilon}{2} \quad (r \geq r_1).$$

Let $\lambda^{**}(\rho)$ (resp. $\nu^{**}(\rho)$) $(0 < \rho < \infty)$ be the measure on $D \times \mathbf{R}^l$ (resp. $\partial D \times \mathbf{R}^l$) defined by

$$\lambda^{**}(\rho)(E) = \lambda(E \cap (D \times \overline{B_l(\rho)})) \ \ (\text{resp. } \nu^{**}(\rho)(E) = \nu(E \cap (\partial D \times \overline{B_l(\rho)}))$$

for every Borel subset E of $D \times \mathbf{R}^l$ (resp. $\partial D \times \mathbf{R}^l$). If we take a sufficiently large r_3, then we see from Lemma 5.1 that

$$N(G\lambda^{**}(r_2))(r) \ \ (\text{resp. } N(K\nu^{**}(r_2))(r)) \ \leq C_9 \quad (r \geq r_3),$$

where C_9 is a constant. Hence there is a number $r_4, r_4 > r_3$ from (2.4) such that

$$\frac{r^{l/2-1} N(G\lambda^{**}(r_2))(r)}{I_{l/2-1}(\sqrt{\lambda_D} r)} \ \ (\text{resp. } \frac{r^{l/2-1} N(K\nu^{**}(r_2))(r)}{I_{l/2-1}(\sqrt{\lambda_D} r)}) \ < \frac{\varepsilon}{2} \quad (r \geq r_4). \tag{5.8}$$

Thus (5.7) and (5.8) give

$$\frac{r^{l/2-1}N(G\lambda)(r)}{I_{l/2-1}(\sqrt{\lambda_D}r)} \quad \left(\text{resp. } \frac{r^{l/2-1}N(K\nu)(r)}{I_{l/2-1}(\sqrt{\lambda_D}r)}\right) < \varepsilon \qquad (r \geq \max(r_1, r_4)),$$

which shows $\mu(N(G\lambda))$ (resp. $\mu(N(K\nu))$) $= 0$.

Proof of Corollary 3. Let $h(X,Y)$ be any radial harmonic majorant of $u(X,Y)$ on $D \times \mathbf{R}^l$. Put

$$h_1(X,Y) = h_u(X,Y) - h(X,Y).$$

Then h_1 is also a radial harmonic function satisfying the Phragmén-Lindelöf boundary condition. Since $\mu(N(u)) \leq \mu(N(h))$, and $\mu(N(h_u)) = \mu(N(u))$, we have

$$\mu(N(h_1)) = \mu(N(h_u)) - \mu(N(h)) \leq \mu(N(u)) - \mu(N(u)) = 0.$$

Hence Theorem 2 gives that $h_1 \leq 0$. Now we can conclude that

$$h_u(X,Y) \leq h(X,Y)$$

on $D \times \mathbf{R}^l$, which shows that $h_u(X,Y)$ is the least harmonic majorant of $u(X,Y)$. □

6. Proof of Theorem 8

Let $v(X,Y)$ be a radial subharmonic function on $D \times \mathbf{R}^l$. For a positive number δ, we put

$$E_v(r;\delta) = \{X \in D; v(X,Y) \leq -\delta r^{1-l/2} I_{l/2-1}(\sqrt{\lambda_D}r)\} \quad (0 < |Y| = r < \infty)$$

and

$$\zeta_v(\delta) = \overline{\lim}_{r\to\infty} \int_{E_v(r;\delta)} f_D(X)dX.$$

Lemma 6.1. *Let $v(X,Y)$ be a radial subharmonic function on $D \times \mathbf{R}^l$ satisfying the Phragmén-Lindelöf boundary condition. If there is a positive number R such that*

$$v(X,Y) \leq 0 \quad \text{on} \quad D \times T_l(R,\infty),$$

then

$$v(X,Y) \leq -\delta\zeta_v(\delta) r^{1-l/2} I_{l/2-1}(\sqrt{\lambda_D}r) f_D(X) \quad (|Y| = r)$$

on $D \times \mathbf{R}^l$ for any $\delta > 0$.

Proof. Take any positive number δ and a sequence $\{r_n\}, r_n \to \infty$, such that

$$\lim_{n\to\infty} \int_{E_v(r_n;\delta)} f_D(X)dX = \zeta_v(\delta).$$

Since

$$N(v)(r_n) \le \int_{E_v(r_n;\delta)} v(X,Y) f_D(X)dX$$

$$\le -\delta {r_n}^{1-l/2} I_{l/2-1}(\sqrt{\lambda_D} r_n) \int_{E_v(r_n;\delta)} f_D(X)dX \qquad (|Y| = r_n > R),$$

we have

$$\mu(N(v)) \le -\delta\zeta_v(\delta).$$

Theorem 6 immediately gives the conclusion. □

Proof of Theorem 8. If $\mu(N(u^+)) = \infty$, then $(\mu(N(u)))^+ = \infty$ and $\mu(L(u))$ is evidently equal to ∞. Since

$$(K(u))^+ = r^{l/2-1}\{I_{l/2-1}(\sqrt{\lambda_D} r)\}^{-1} \sup_{X\in D} \frac{u^+(X,Y)}{f_D(X)}$$

$$\ge \{r^{l/2-1} I_{l/2-1}(\sqrt{\lambda_D} r)\}^{-1} \frac{L(u)(r)}{\max_{X\in D} f_D(X)} \qquad (|Y| = r),$$

we also have $(K(u))^+ = \infty$. Hence (2.7) holds.

Suppose that

$$\mu(N(u^+)) < \infty.$$

Then

$$u(X,Y) \le \mu(N(u)) r^{1-l/2} I_{l/2-1}(\sqrt{\lambda_D} r) f_D(X) \quad (|Y| = r)$$

on $D \times \mathbf{R}^l$ from Theorem 6. Hence $K(u) \le \mu(N(u)) < \infty$. On the other hand, $K(u) \ge \mu(N(u)) > -\infty$, because

$$\frac{r^{l/2-1} N(u)(r)}{I_{l/2-1}(\sqrt{\lambda_D} r)} \le r^{l/2-1}\{I_{l/2-1}(\sqrt{\lambda_D} r)\}^{-1} \sup_{X\in D} \frac{u(X,Y)}{f_D(X).}$$

Thus we see $-\infty < K(u) < \infty$ and

$$K(u) = \mu(N(u)). \tag{6.1}$$

First we shall consider the case where $0 \le K(u) < \infty$. For any $\varepsilon > 0$ and a number $R_1 = R_1(\varepsilon)$,

$$u(X,Y) \le (K_u + \varepsilon) r^{1-l/2} I_{l/2-1}(\sqrt{\lambda_D} r) f_D(X) \tag{6.2}$$

on $D \times T_l(R_1, \infty)$ and hence

$$\overline{\lim}_{r\to\infty} r^{l/2-1}\{I_{l/2-1}(\sqrt{\lambda_D}r)\}^{-1}L(u)(r) \le K(u)\max_{X\in D} f_D(X). \qquad (6.3)$$

Put

$$\tau(u) = \underline{\lim}_{r\to\infty}\frac{r^{l/2-1}L(u)(r)}{I_{l/2-1}(\sqrt{\lambda_D}r)}$$

and assume that

$$\tau(u) < K(u)\max_{X\in D} f_D(X).$$

Then we can find a number $\delta_1 > 0$ and a set $S_u, S_u \subset D$, such that

$$\int_{S_u} f_D(X)dX > 0$$

and

$$K(u)f_D(X) - \tau(u) \ge 2\delta_1 \qquad (6.4)$$

on S_u. Put

$$v_1(X,Y) = u(X,Y) - (K_u + \varepsilon)r^{1-l/2}I_{l/2-1}(\sqrt{\lambda_D}r)f_D(X). \qquad (6.5)$$

We see from (6.2) that

$$v_1(X,Y) \le 0$$

on $D \times T_l(R_1, \infty)$. Hence we have from Lemma 6.1 that

$$u(X,Y) \le \{K(u) + \varepsilon - \delta_1\zeta_{v_1}(\delta_1)\}r^{1-l/2}I_{l/2-1}(\sqrt{\lambda_D}r)f_D(X)$$

on $D \times \mathbf{R}^l$, which gives

$$K(u) \le K(u) - \delta_1\zeta_{v_1}(\delta_1).$$

If we can show that

$$\zeta_{v_1}(\delta_1) > 0, \qquad (6.6)$$

then we have a contradiction. Thus we obtain

$$\tau(u) \ge K(u)\max_{X\in D} f_D(X).$$

Hence (6.1), (6.3) and this give the existence of $\mu(L(u))$ and (2.7).

To prove (6.6), take a sequence $\{r_k\}$, $r_k \to \infty$ $(k \to \infty)$, such that

$$r_k^{l/2-1}\frac{u(X,Y)}{I_{l/2-1}(\sqrt{\lambda_D}r_k)} \le \tau(u) + \delta_1 \qquad (6.7)$$

for every Y, $|Y| = r_k$ $(k = 1, 2, 3, \ldots)$. Then from (6.4), (6.5) and (6.7)

$$r^{l/2-1}\frac{v_1(X,Y)}{I_{l/2-1}(\sqrt{\lambda_D}r)} \le \tau(u) + \delta_1 - K(u)f_D(X) \le -\delta_1$$

for every Y, $|Y| = r_k$ $(k = 1, 2, 3, \ldots)$ and $X \in S_u$. Hence we see that

$$E_{v_1}(r_k; \delta_1) \supset S_u \quad (k = 1, 2, 3, \ldots)$$

and

$$\zeta_{v_1}(\delta_1) \ge \int_{S_u} f_D(X)dX > 0.$$

Secondly we consider the case where

$$-\infty < K(u) < 0. \tag{6.8}$$

Take a positive number ε satisfying $K(u) + \varepsilon < 0$ and a number $R_2 = R_2(\varepsilon)$ such

$$r^{l/2-1}\frac{u(X,Y)}{I_{l/2-1}(\sqrt{\lambda_D}r)} \le K(u) + \varepsilon)f_D(X) \tag{6.9}$$

on $D \times T_l(R_2, +\infty)$, which shows

$$\overline{\lim}_{r\to\infty}\frac{r^{l/2-1}L(u)(r)}{I_{l/2-1}(\sqrt{\lambda_D}r)} \le 0. \tag{6.10}$$

Suppose that $\tau(u) < 0$. Then there are a sequence $\{r_k^*\}, r_k^* \to \infty$, and a positive number δ_2 such that

$$\frac{L(u)(r_k^*)}{I_{l/2-1}(\sqrt{\lambda_D}r_k^*)} \le -2\delta_2 \quad (k = 1, 2, 3, \ldots). \tag{6.11}$$

Put

$$v_2(X,Y) = u(X,Y) - (K(u) + \varepsilon)r^{1-l/2}I_{l/2-1}(\sqrt{\lambda_D}r)f_D(X). \tag{6.12}$$

Then

$$v_2(X,Y) \le 0$$

on $D \times T_l(R_2, \infty)$ from (6.9). Applying Lemma 6.1 again, we see that

$$u(X,Y) \le \{K(u) + \varepsilon - \delta_2\zeta_{v_2}(\delta_2)\}r^{1-l/2}I_{l/2-1}(\sqrt{\lambda_D}r)f_D(X) \quad (|Y| = r)$$

on $D \times \mathbf{R}^l$, and

$$K(u) \leq K(u) - \delta_2 \zeta_{v_2}(\delta_2).$$

If we can show that

$$\zeta_{v_2}(\delta_2) > 0, \tag{6.13}$$

then we have a contradiction. Hence we obtain $\tau(u) \geq 0$. With (6.1), (6.8) and (6.10) this gives the existence of $\mu(L(u))$ and

$$\mu(L(u)) = 0 = (K(u))^+ \max_{X \in D} f_D(X) = (\mu(u))^+ \max_{X \in D} f_D(X),$$

which also is (2.7).

To prove (6.13), put

$$T_u = \{X \in D; -K(u) f_D(X) \leq \delta_2\}.$$

It is evident that

$$\int_{T_u} f_D(X) dX > 0.$$

We see from (6.11) and (6.12) that

$$r_k^{*l/2-1} \frac{v_2(X,Y)}{I_{l/2-1}(\sqrt{\lambda_D} r_k^*)}$$

$$\leq r_k^{*l/2-1} \frac{u(X,Y)}{I_{l/2-1}(\sqrt{\lambda_D} r_k^*)} - K(u) f_D(X) \leq -2\delta_2 + \delta_2 = -\delta_2$$

for every $Y, |Y| = r_k^*$ $(k = 1, 2, 3, \ldots)$ and $X \in T_u$, which shows

$$E_{v_2}(r_k^*; \delta_2) \supset T_u \qquad (k = 1, 2, 3, \ldots).$$

Hence we have

$$\zeta_{v_2}(\delta_2) \geq \int_{T_u} f_D(X) dX > 0.$$

Thus we complete the proof of Theorem 8. □

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