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ON MAXIMAL PARALLELIZABLE REGIONS OF FLOWS OF THE PLANE

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Abstract: We consider maximal parallelizable regions of a flow of free mappings defined on the plane. We prove that each such a region is a union of equivalence classes of a relation. Moreover, we study the boundary of such regions by means the first prolongational limit sets.

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1. Introduction

Let $\{f^t : t \in \mathbf{R}\}$ be a flow of *free mappings* (i.e. for every $t \in \mathbf{R} \setminus \{0\}$ the function f^t is a homeomorphism of the plane onto itself without fixed points which preserves orientation). It follows from the Jordan Theorem that each orbit C of such a flow $\{f^t : t \in \mathbf{R}\}$ divides the plane into two simply connected regions. Note that each of them is invariant under f^t for $t \in \mathbf{R}$. Thus two different orbits C_p and C_q of points p and q, respectively, divide the plane into three simply connected invariant regions in such a way that one of which contains both C_p and C_q in its boundary. We will call this region by the *strip* between C_p and C_q and denote by D_{pq}.

An invariant region $M \subset \mathbf{R}^2$ is said to be *parallelizable* if there exists a homeomorphism φ mapping M onto $\mathbf{R}^2$ such that

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$$f^t(x) = \varphi^{-1}(\varphi(x) + (t, 0)) \quad \text{for } x \in M.$$

It is known that a region M is parallelizable if and only if there exists a homeomorphic image K of a straight line which is a closed set in M such that K has exactly one common point with every orbit of $\{f^t : t \in \mathbf{R}\}$ contained in M (see [2], p. 49 and e.g. [4]). Such a set K we will call a *section* in M.

Put

$J^+(q) := \{p \in \mathbf{R}^2 :$ there exist a sequence $(q_n)_{n\in\mathbf{N}}$ and a sequence $(t_n)_{n\in\mathbf{N}}$ such that $q_n \to q$, $t_n \to +\infty$, $f^{t_n}(q_n) \to p$ as $n \to +\infty\}$,

$J^-(q) := \{p \in \mathbf{R}^2 :$ there exist a sequence $(q_n)_{n\in\mathbf{N}}$ and a sequence $(t_n)_{n\in\mathbf{N}}$ such that $q_n \to q$, $t_n \to -\infty$, $f^{t_n}(q_n) \to p$ as $n \to +\infty$ $\}$.

The set $J(q) := J^+(q) \cup J^-(q)$ is called the *first prolongational limit set* of q. Let us observe that $p \in J(q)$ if and only if $q \in J(p)$ for any $p, q \in \mathbf{R}^2$. For a subset $H \subset \mathbf{R}^2$ we define

$$J(H) := \bigcup_{q\in H} J(q).$$

R.C. McCann has proved that a region M is a maximal parallelizable region (i.e. M is not contained properly in any parallelizable region) if and only if $J(M) = \operatorname{fr} M$ (see [5]).

We consider a relation in $\mathbf{R}^2$ defined in the following way:

$p \sim q$ if $p = q$ or p and q are endpoints of some arc$K$ for which $f^n(K) \to \infty$ as $n \to \pm\infty$.

By an arc$K$ with endpoints p and q we mean the image of a homeomorphism $c : [0, 1] \to c([0, 1])$ satisfying conditions $c(0) = p$, $c(1) = q$, where the topology on $c([0, 1])$ is induced by the topology of $\mathbf{R}^2$. It turns out that the relation defined above is an equivalence relation (see [3]) and has the same equivalence classes as the relation defined by S. Andrea in [1]. In [3] it has been proved that the interior of each equivalence class of the relation is parallelizable. Moreover, in [4] one can find the proof that each equivalence class is contained in a parallelizable region.

2. Properties of Parallelizable Regions

In this section we will prove some properties of parallelizable regions. In particular, these properties hold for maximal parallelizable regions.

Proposition 1. *Let M be a parallelizable region and let $p \in \operatorname{fr} M$. Then M is contained in one of the components of $\mathbf{R}^2 \setminus C_p$.*

Proof. Suppose, on the contrary, that there exist $q, r \in M$ which belong to different components of $\mathbf{R}^2 \setminus C_p$. Since M is arcwise connected, there exists an arc$L$ with endpoints q and r such that $L \subset M$. Then, by the Jordan Theorem, the arc$L$ has a common point with C_p. On the other hand, $C_p \cap M = \emptyset$, since M is an open invariant set and $p \in \operatorname{fr} M$. Thus we obtain a contradiction. □

Corollary 2. *Let M be a parallelizable region and let $p \in \operatorname{fr} M$. Then $\operatorname{cl} M \setminus C_p$ is contained in one of the components of $\mathbf{R}^2 \setminus C_p$.*

Proof. By Proposition 1 the region M is contained in one of the components of $\mathbf{R}^2 \setminus C_p$. Denote this component by H^+, and the other one by H^-. Fix a $q \in \operatorname{fr} M \setminus C_p$. Suppose that $q \in H^-$. Then there exists a ball $B(q, \rho)$ such that $B(q, \rho) \subset H^-$. Since $M \subset H^+$, we have $B(q, \rho) \cap M = \emptyset$, which contradicts the fact that $q \in \operatorname{fr} M$. □

Corollary 3. *Let M be a parallelizable region. Then $\operatorname{cl} M \setminus (C_p \cup C_q) \subset D_{pq}$ for all p, q such that $C_p \subset \operatorname{fr} M$, $C_q \subset \operatorname{fr} M$ and $C_p \neq C_q$.*

Proof. By Corollary 2, the set $\operatorname{cl} M \setminus (C_p \cup C_q)$ is contained in the component of $\mathbf{R}^2 \setminus C_p$ which contains C_q and in the component of $\mathbf{R}^2 \setminus C_q$ which contains C_p. Hence $\operatorname{cl} M \setminus (C_p \cup C_q) \subset D_{pq}$. □

3. Maximal Parallelizable Regions

Now we will prove a relation between maximal parallelizable regions of the flow $\{f^t : t \in \mathbf{R}\}$ and equivalence classes of the considered relation.

Theorem 4. *A maximal parallelizable region M of $\{f^t : t \in \mathbf{R}\}$ is a union of equivalence classes of the relation $\sim$.*

Proof. Let M be a maximal parallelizable region. Suppose, on the contrary, that there exists an equivalence class G_0 such that $p \in M$ and $q \notin M$ for some points $p, q \in G_0$. Denote by K a section of $\{f^t : t \in \mathbf{R}\}$ in M such that $p \in K$. Since $q \notin M$, we have $K \cap C_q = \emptyset$.

On account of the Whitney-Bebutov Theorem (see [2], p. 52) there exist an invariant neighbourhood V of q and a section L in V such that $q \in L$. We can take the neighbourhood V such that $V \cap C_p = \emptyset$. Put $K_0 := K \setminus D_{pq}$, $L_0 := L \setminus D_{pq}$. Since p and q belongs to the same equivalence class, there exists an arc$S$ with endpoints p and q which has exactly one common point with each orbit contained in D_{pq} such that $S \setminus \{p, q\} \subset D_{pq}$. It follows from the fact that equivalence class G_0 is contained in a parallelizable region H (see [4]). As S we can take the preimage of the segment with endpoints $\varphi(p)$, $\varphi(q)$, where φ is a homeomorphism mapping H onto $\mathbf{R}^2$ such that

$$f^t(x) = \varphi^{-1}(\varphi(x) + (t, 0)) \quad \text{for } x \in H.$$

Denote by M_1 the union of all orbits having a common point with $K_0 \cup S \cup L_0$. Then $K_0 \cup S \cup L_0$ is a section of M_1 and hence M_1 is a parallelizable region. This contradicts the fact that M is a maximal parallelizable region, since $M \subset M_1$ and $M \neq M_1$. □

4. Boundaries of Maximal Parallelizable Regions

In this section we will study boundaries of maximal parallelizable regions.

Proposition 5. *If $q \in \operatorname{int} G_0$ for an equivalence class G_0, then $q \notin J(\mathbf{R}^2)$.*

Proof. Let $q \in \operatorname{int} G_0$. Note that $q \notin J(\mathbf{R}^2)$ if and only if $J(q) = \emptyset$ (it follows from the fact that $p \in J(q)$ if and only if $q \in J(p)$ for every $p \in \mathbf{R}^2$). Fix $s \in \operatorname{int} G_0$ and $r \in \operatorname{int} G_0$ belonging to different components of $\mathbf{R}^2 \setminus C_q$. Then the strip D_{sr} between the orbits C_s and C_r is contained in G_0 (see [3]). In fact, it is contained in $\operatorname{int} G_0$, since for every $z \in D_{sr}$ we can find a ball centered at z which is contained in D_{sr}. Therefore $\operatorname{cl} D_{sr} \subset \operatorname{int} G_0$, since G_0 is invariant.

Take a ball $B(q, \varepsilon)$ such that $B(q, \varepsilon) \cap C_s = \emptyset$ and $B(q, \varepsilon) \cap C_r = \emptyset$. Then $B(q, \varepsilon) \subset D_{sr}$. Fix $(q_n)_{n \in \mathbf{N}}$ such that $q_n \to q$ as $n \to +\infty$. Then there exists an n_1 such that $q_n \in B(q, \varepsilon)$ for $n > n_1$.

Let A_0 be a compact set. Then $A := A_0 \cap \operatorname{cl} D_{sr}$ is also a compact set. Since $\operatorname{int} G_0$ is parallelizable (see [3]), there exists a $t_0 > 0$ for which $A \cap f^t(B(q, \varepsilon)) = \emptyset$ if $| t | > t_0$. Fix a sequence $(t_n)_{n \in \mathbf{N}}$ such that $t_n \to +\infty$ or $t_n \to -\infty$. Then $f^{t_n}(q_n) \notin A$ if $n > n_1$ and $| t_n | > t_0$. Take an n_2 such that $| t_n | > t_0$ for $n > n_2$ and put $n_0 := \max\{n_1, n_2\}$. Then $f^{t_n}(q_n) \notin A_0$ for $n > n_0$, since $f^{t_n}(q_n) \in D_{sr} \setminus A$ for $n > n_0$ and $(A_0 \setminus A) \cap D_{sr} = \emptyset$. Thus $f^{t_n}(q_n) \to \infty$ as $n \to \pm\infty$. □

Corollary 6. *Let M be a maximal parallelizable region and $q \in \operatorname{fr} M$. Then q belongs to the boundary of a class.*

Proof. Since $\operatorname{fr} M = J(M)$ for each maximal parallelizable region M (see [5]), we have $q \in J(M)$, and consequently $q \in J(\mathbf{R}^2)$. Hence by Proposition

5, the point q does not belong to the interior of any equivalence class. Thus q belongs to the boundary of a class. □

Remark 7. From Proposition 5 we get that if $q \in J(\mathbf{R}^2)$, then q belongs to the boundary of a class. The converse does not hold. If q belongs to the boundary of a class and every neighbourhood V of q has nonempty intersection with infinitely many classes, then q may not be an element of $J(\mathbf{R}^2)$.

Proposition 8. *Let M be a maximal parallelizable region and $p \in \operatorname{fr} M$. Let G_0 be the equivalence class which contains p. Assume that G_0 does not consist of just one orbit. Then $p \notin J(q)$ for each point q belonging to the component of $\mathbf{R}^2 \setminus C_p$ that does not contain M.*

Proof. Since $p \notin M$, we have by Theorem 4 that $M \cap G_0 = \emptyset$. Fix a point $s \in \operatorname{int} G_0$. Then the strip D_{ps} between the orbits C_s and C_p is contained in $\operatorname{int} G_0$ (see [3]). Therefore $G_0 \setminus C_p$ is contained in one of the components of $\mathbf{R}^2 \setminus C_p$. Denote by H_0 the component of $\mathbf{R}^2 \setminus C_p$ which contains $G_0 \setminus C_p$. On account of Proposition 1 the region M is contained in one of the components of $\mathbf{R}^2 \setminus C_p$. Let us note that H_0 is the component of $\mathbf{R}^2 \setminus C_p$ which does not contain M, since $p \in \operatorname{fr} M$ and $D_{ps} \cap M = \emptyset$.

Fix a point $q \in H_0$. If $q \in D_{ps} \cup C_s$, then $q \in \operatorname{int} G_0$. Hence by Proposition 5 we get that $J(q) = \emptyset$, and hence $p \notin J(q)$. Let $q \in H_0 \setminus (D_{ps} \cup C_s)$. Then p and q belongs to different components of $\mathbf{R}^2 \setminus C_s$. Denote by H_1 the component of $\mathbf{R}^2 \setminus C_s$ which contains q. From the definition of $J(q)$ we obtain that $J(q) \subset \operatorname{cl} H_1$, since each component of $\mathbf{R}^2 \setminus C_s$ is invariant. But $p \notin \operatorname{cl} H_1 = H_1 \cup C_s$. Thus $p \notin J(q)$. □

Remark 9. From Proposition 8 we get that if $p \in \operatorname{fr} M$, where M is a maximal parallelizable region, and C_p is properly contained in the class G_0 such that $p \in G_0$, then p cannot belong to the first prolongational limit set of any point from the component of $\mathbf{R}^2 \setminus C_p$ that does not contain M. However, p can belong to $J(q)$ for points q not only from M. Namely, by Corollary 2 the case where $q \in \operatorname{fr} M$ and $p \in J(q)$ cannot be exluded.

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