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LAGRANGE INTERPOLATION ON THE PROJECTIVE LINE AND ON PROJECTIVE CURVES: VERY SPECIAL INTERPOLATION SUBSETS

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Abstract: Let X be an integral projective curve, $L \in \mathrm{Pic}(X)$ and $V \subseteq H^0(X,L)$ a linear subspace. Here we study the existence of $P_1,\dots,P_{v-1} \in X_{reg}$, $v := \dim(V)$ such that for every $Q \in X_{reg}\backslash\{P_1,\dots,P_{v-1}\}$ non non-zero $f \in V$ vanishes at Q and at each P_i, $1 \le i \le v-1$.

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1. Interpolation on $\mathbf{P}^1$

Fix an algebraically closed base field $\mathbb{K}$, an integer $d > 0$ and a non-zero $\mathbb{K}$-vector space of homogeneous degree d polynomial in the variables x_0, x_1. Set $v := \dim(V)$. Hence $0 < v \le d+1$. If $v = d+1$, then every $S \subset \mathbf{P}^1$ such that $\sharp(S) = d+1$ is a unique Lagrange interpolation set for $V = H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(d))$. From now on we assume $v \le d$. A subset $S \subset \mathbf{P}^1$ with $\sharp(S) = v-1$ will be said V-good or good for V if for all $P \in \mathbf{P}^1\backslash S$ the set $S \cup \{P\}$ is a unique interpolation set for V, i.e. 0 is the only homogeneous polynomial $f \in V$ such that $f(Q) = 0$ for all $Q \in S \cup \{P\}$. Since $\mathbb{K}$ is algebraically closed, every non-constant polynomial has at least one zero. Hence no one-dimensional V has a V-good set. Thus from now on we will assume $v \ge 2$ and hence $d \ge 2$. Fix d, v, V. For any $A \subset \mathbf{P}^1$ set $V(-A) := \{f \in V : f(P) = 0 \text{ for all } P \in A\}$.

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If $\sharp(A) = v-1$ and $\dim(V(-A)) \geq 2$, then A is not V-good. At least if $\mathrm{char}(\mathbb{K}) = 0$ for any V (assuming $v < d$) there exists at least one A such that $\sharp(A) = v-1$ and $\dim(V(-A)) \geq 2$. Hence such a set A is not V-good for this very stupid reason.

Theorem 1. *Fix integers $d \geq v \geq 2$ and $S \subset \mathbf{P}^1$ such that $\sharp(S) = v-1$. Then the set $\Phi(S)$ of all v-dimensional linear subspaces $V = H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(d))$ is not empty. Fix an ordering $P_1, \dots, P_{v-1}$ of the elements of S. Set $\Psi(d,v) := \{(a_1, \dots, a_{v-1}) \in \mathbb{N}^{v-1} : a_i > 0$ for all i and $a_1 + \dots + a_{v-1} = d\}$. Up to a non-zero multiplicative constant there is a unique $f_{a_1,\dots,a_{v-1};P_1,\dots,P_{v-1}} \in H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(d)) \backslash \{0\}$ such that $f_{a_1,\dots,a_{v-1};P_1,\dots,P_{v-1}}$ vanishes with order a_i at each P_i. Let $\mathfrak{G}(v, f_{a_1,\dots,a_{v-1};P_1,\dots,P_{v-1}})$ denote the set of all v-dimensional linear subspaces $V = H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(d))$ containing $f_{a_1,\dots,a_{v-1};P_1,\dots,P_{v-1}}$. Then $\Phi(S) \subseteq \bigcup_{(a_1,\dots,a_{v-1}) \in \Psi(d,v)} \mathfrak{G}(v, f_{a_1,\dots,a_{v-1};P_1,\dots,P_{v-1}})$ and there is a non-empty $(v-1)(d+2-v)$-dimensional open subset $U_{f_{a_1,\dots,a_{v-1};P_1,\dots,P_{v-1}}}$ of $(v, f_{a_1,\dots,a_{v-1};P_1,\dots,P_{v-1}})$ such that $\Phi(S) = \bigcup_{(a_1,\dots,a_{v-1}) \in \Psi(d,v)} U_{f_{a_1,\dots,a_{v-1};P_1,\dots,P_{v-1}}}$. Furthermore, we may choose this open subsets with the additional property that for all $((a_1, \dots, a_{v-1}), (b_1, \dots, b_{v-1})) \in \Psi(d,v) \times \Psi(d,v)$ either*

$$U_{f_{a_1,\dots,a_{v-1};P_1,\dots,P_{v-1}}} = U_{f_{b_1,\dots,b_{v-1};P_1,\dots,P_{v-1}}},$$

or

$$U_{f_{a_1,\dots,a_{v-1};P_1,\dots,P_{v-1}}} \cap U_{f_{b_1,\dots,b_{v-1};P_1,\dots,P_{v-1}}} = \emptyset.$$

Proof. We just saw that if $V \in \Phi(S)$, then $\dim(V(-S)) = 1$. Fix $(a_1, \dots, a_{v-1}) \in \Psi(d,v)$ and take the corresponding polynomial $f_{a_1,\dots,a_{v-1};P_1,\dots,P_{v-1}}$. There is a non-zero open subset $\Omega(P_1, \dots, P_{v-1})$ of the Grassmannian of all $(v-1)$-dimensional linear subspaces of $H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(d))$ such that $W \in \Omega(P_1, \dots, P_{v-1})$ if and only $\dim(W(-A)) = 0$. Set

$$U_{f_{a_1,\dots,a_{v-1};P_1,\dots,P_{v-1}}} := \{\mathbb{K} f_{b_1,\dots,b_{v-1};P_1,\dots,P_{v-1}} + W\}_{W \in \Omega(P_1,\dots,P_{v-1})} \cong \Omega(P_1, \dots, P_{v-1}).$$

All other assertions follows from this construction. □

Now we consider the same topic for a curve with higher genus. We allow the case of a singular curve X, but we only consider subsets of X_{reg}. Fix an integer $v > 0$. Let X be an integral projective curve, $L \in \mathrm{Pic}(X)$ and $V \subseteq H^0(X, L)$ a linear subspace. Set $v := \dim(V)$. A set $S \subset X_{reg}$ such that $\sharp(S) = v-1$ will be said V-good (resp. almost V-good) if $V(-(S \cup \{P\})) = \{0\}$ for all $P \in X \backslash S$ (resp. $P \in X_{reg} \backslash S$).

Example 1. Let X be an integral projective curve such that $g := p_a(X) > 0$. Fix integers d, v and a_i, $1 \le i \le v-1$, such that $d \ge 2g$ and $2 \le v \le d-g$, $a_i > 0$ for all i and $a_1 + \cdots + a_{v-1} = d$. Fix $P_1, \ldots, P_{v-1}$ in X_{reg} such that $P_i \neq P_j$ for all $i \neq j$. Set $L := \mathcal{O}_X(a_1P_1 + \cdots + a_{v-1}P_{v-1}) \in \mathrm{Pic}^d(X)$. Since $d \ge 2g$, $h^0(X, L) = d + 1 - g > v$ and $h^0(X, \mathcal{I}_Q \otimes L) = h^0(X, L) - 1$ for all $Q \in X$. The effective divisor $a_1P_1 + \cdots + a_{v-1}P_{v-1}$ induces (up to a non-zero multiplicative constant) a unique $f \in H^0(X, L)\backslash\{0\}$. Let $V \subset H^0(X, L)$ be any v-dimensional linear subspace containing f. $P_1, \ldots, P_{v-1}$ is V-good if and only if $\mathbb{K}f = V(-P_1 - \cdots - P_{v-1})$, i.e. if and only if $\dim(V(-P_1 - \cdots - P_{v-1})) = 1$. Notice that if $h^0(X, L(-P_1 - \cdots - P_{v-1})) = h^0(X, L) - v + 1$, i.e. if $h^1(X, L(-P_1 - \cdots - P_{v-1})) = 0$, then this condition is satisfied when we take as V a general v-dimensional linear subspace containing f. Obviously, $h^1(X, L(-P_1 - \cdots - P_{v-1})) = 0$ if $d \ge 2g + v - 2$.

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