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NUMBER OF POINTS OVER $\mathbb{F}_q$ OF HYPERPLANE SECTIONS

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Abstract: Let $X \subset \mathbf{P}^n$ be a geometrically integral variety defined over $\mathbb{F}_q$. Set $S := X(\mathbb{F}_q)$. Let $PG(n,q)^*$ denote the set of all hyperplanse $H \subset \mathbf{P}^n$ defined over $\mathbb{F}_q$. For any $H \in PG(n,q)^*$ set $u(S,H) := \sharp(S \cap H)$. For any subset $A \subseteq PG(n,q)$ let $\mathfrak{G}(X,q;A)$ denote the set of all integers $u(S,H)$ when H varies in A. Here we study the sets $\mathfrak{G}(X,q;A)$ in a few examples in which X is a rational surface and the normalization of the geometrically integral $X \cap H$ are rational.

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1. Number of Points of Hyperplane Sections

Let $X \subset \mathbf{P}^n$ be a geometrically integral variety defined over $\mathbb{F}_q$. Set $S := X(\mathbb{F}_q)$. Let $PG(n,q)^*$ denote the set of all hyperplanse $H \subset \mathbf{P}^n$ defined over $\mathbb{F}_q$. For any $H \in PG(n,q)^*$ set $u(S,H) := \sharp(S \cap H)$. Let $\mathfrak{G}(S)$ or $\mathfrak{G}(X,q)$ denote the set of all integers $u(S,H)$ when H varies in $PG(n,q)^*$. For any subset $A \subseteq PG(n,q)$ let $\mathfrak{G}(S;A)$ or $\mathfrak{G}(X,q;A)$ denote the set of all integers $u(S,H)$ when H varies in A. The following example shows that even in very classical cases it is hopeless to try to know exactly the set $\mathfrak{G}(X,q)$ (see however Example 2), but in a few cases we will be able to describe $\mathfrak{G}(X,q;A)$ for geometrically interesting pairs (X,A).

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Example 1. Fix a prime power q and an integer $d \geq 3$ and let $X \subset \mathbf{P}^n$, $n := (d^2 + 3d)/2$, be the order d Veronese embedding of $\mathbf{P}^2$. To know $\mathfrak{G}(X, q)$ is equivalent to know the possible number of $\mathbb{F}_q$-points of all degree d plane curves. Even for $d = 3$ this is hopeless, except for very low q's, because it would require to know the number of $\mathbb{F}_q$-points of all elliptic curves. However, if $d = 3$ it is easy (and left to the reader) to describe the set $\mathfrak{G}(X, q; A)$, when A is the set of all hyperplanes tangent to X.

Example 2. Let $X \subset \mathbf{P}^n$, $n \geq 4$, be a geometrically integral minimal degree non-degenerate surface defined over $\mathbb{F}_q$. Equivalently, X is non-degenerate and $\deg(X) = n - 1$. The classification over $\bar{\mathbb{F}}_q$ is due to C. Segre. If X is singular, then it is a cone over a smooth rational normal curve of $\mathbf{P}^{n-1}$. First assume that X is a cone, say with vertex P. Since P is the unique singular point of X, it is defined over $\mathbb{F}_q$. All such cones are projectively equivalent over $\mathbb{F}_q$. Fix $H \in \mathfrak{G}(X, q)$. If $P \in H$, then $H \cap H$ is a union of lines through P (counted with some multiplicity). Call y the number of these lines (not counting the multiplicities). We have $u(S, H) = yq + 1$. We have $0 \leq y \leq n - 1$ and, for fixed X and q, all these integers y may be realized by some $H \in \mathfrak{G}(X, q)$ such that $P \in H$. Now assume that X is smooth. Over $\bar{\mathbb{F}}_q$ X is a ruled rational surface and its is uniquely determined, up to a projective transformation, by the choice of an integer e such that $0 \leq e \leq n - 3$ and $e \equiv n - 1 \pmod 2$. Over $\mathbb{F}_q$ the same classification holds, because (as we will see soon) the ruling of X is uniquely determined; here we use the assumption $n \geq 4$ (not needed in the cone case): a smooth quadric surface may be elliptic (no ruling) or hyperbolic (the rulings are defined over $\mathbb{F}_q$). Fix the integer e such that $0 \leq e \leq n - 3$ and $e \equiv n - 1 \pmod 2$ associated to X. $X \cong F_e$ is a $\mathbf{P}^1$-bundle $\pi : F_1 \to \mathbf{P}^1$ (defined over $\mathbb{F}_p$), $\mathrm{Pic}(X) \cong \mathbb{Z}^{\oplus 2}$ and we may take as a basis of $\mathrm{Pic}(X)$ a fiber f of π and a section h of π with negative self-intersection. We have $h^2 = -e$, $h \cdot f = 1$ and $f^2 = 0$. The linear system $|h + ((n-1-e)/2f)|$ is very ample and it is the linear system giving the embedding $X \subset \mathbf{P}^n$. We note that the ruling of F_e is unique if and only if $e = 0$, but if $e = 0$ the ruling π by lines on X is unique, because we assume $n \geq 4$. Fix $H \in \mathfrak{G}(X, q)$. If $H \cap X$ is integral, then $X \cap H \cong \mathbf{P}^1$ over $\mathbb{F}_q$ and hence $u(S, H) = q + 1$. Now assume that $X \cap H$ is not integral. There is an integer $\beta > 0$ and an integral curve $T \in |h + ((n - 1 - e - 2\beta)/2)f|$ such that T is integral and $X \cap H$ is scheme-theoretically the union of T and β fibers of π (counting the multiplicities). Furthermore, T is defined over $\mathbb{F}_q$. Hence $T \cong \mathbf{P}^1$ over $\mathbb{F}_q$. Thus $\sharp(T(\mathbb{F}_q)) = q + 1$. Call α, $0 \leq \alpha \leq \beta$, the number of fibers (not counting multiplicities) defined over $\mathbb{F}_q$ and appearing in $X \cap H$. Each such fiber intersect T at a unique point and this point is in $T(\mathbb{F}_q)$. Thus $u(S, H) = q + 1 + \alpha \cdot q$. Either $\beta = (n - 1 - e)/2$ or $(n - 1 - e - 2\beta)/2 \geq e$, i.e. $\beta \leq (n-1-3e)/2$ and all these β and α may be realized for a suitable $H \in \mathfrak{G}(X,q)$.

Example 3. Fix a prime-power q, and integer $d \geq 3$ and $Q \in \mathbf{P}^2(\mathbb{F}_q)$. Let $j_d : \mathbf{P}^2 \to \mathbf{P}^n$, $n = (d^2+3d)/2$,denote the order d Veronese embedding of the plane. Set $X := j_d(\mathbf{P}^2)$ and $P := j_d(Q)$. Both X and P are defined over $\mathbb{F}_q$. Let A be the set of all hyperplanes of $\mathbf{P}^n$ defined over $\mathbb{F}_q$, containing P and whose local equation at P vanishes with order at least $d-1$. Equivalently, the set $\{X \cap H\}_{H \in A}$ is the set of all degree d plane curves defined over $\mathbb{F}_q$ and with multiplicity at least $d-1$ at Q.

(a) Fix $H \in A$ such that the curve $X \cap H$ is geometrically integral. The linear projection from Q shows that the normalization map $\pi : C \to X \cap H$ is defined over $\mathbb{F}_q$ and that $C \cong \mathbf{P}^1$ over $\mathbb{F}_q$. Notice that $\sharp(\mathbf{P}^1(\mathbb{F}_q) = q+1$ and that $P \in (X \cap H)(\mathbb{F}_q)$. Thus $u(S,H) = q+2-t$, where $t := \sharp(\pi^{-1}(P) \cap C(\mathbb{F}_q)$.

Claim. *An integer t as above occurs (and hence $q+2-t \in \mathfrak{G}(X,q;A)$) if and only if $0 \leq t \leq \min\{d-2, q+1\}$ and $t \neq d-3$.*

Proof of the Claim. The "only if " part is obvious (just use the action of the absolute Galois group of $\mathbb{F}_q$ on the set of all lines in the tangent cone of $X \cap H$ at P). To check the "if" part we note that (if we see the curves $X \cap H$ as plane curves) all such integral curves $X \cap H$ are obtained in the following way, up to the action of an element of $PGL(2,q)$. Let $C \subset \mathbf{P}^d$ the degree d rational normal curve (seen as a curve over $\mathbb{F}_q$). Fix a degree $d-1$ effective divisor B of C defined over $\mathbb{F}_q$ and such that the reduction of its support contains exactly t points of $C(\mathbb{F}_q)$. Notice that $\dim(\langle B \rangle) = d-2$ and that the linear space $\langle B \rangle$ is defined over $\mathbb{F}_q$. Let $M \subset \langle B \rangle$ any $(d-3)$-dimensional linear subspace of $\langle B \rangle$ defined over $\mathbb{F}_q$. Take as $X \cap H$ the degree d plane curve which is the image of C by the linear projection from M. The point Q is the image of $\langle B \rangle$. This construction proves the "if" part of the Claim.

(b) $H \in A$ such that the curve $X \cap H$ is not geometrically integral. If $d = 3$ taking unions of lines through P we see that $\mathfrak{G}(X,q;A)$ contains the integers $1, 1+q, 1+3q$. Now assume $d \geq 4$. Take a geometrically integral solution with $q+2-t$ $\mathbb{F}_q$-points for the integer $d' := 3$ and $d-3$ lines through P, y of them (counting multiplicities) defined over $\mathbb{F}_q$. In this way we see that $\mathfrak{G}(X,q;A)$ contains all integers $q+2-t+yq$ with $1 \leq t \leq \min\{d-2, q+1\}$, $0 \leq y \leq \min\{d-3, q-1\}$ and $y \neq d-4$.

(c) From parts (a) and (b) we see that when $d \geq q+1$ the set $\mathfrak{G}(X,q;A)$ is the set of all integers x such that $1 \leq x \leq q^2+q+1$.

(d) Fix d and take $q \gg d$. We get $\sharp(\mathfrak{G}(X,q;A)) \geq d-3$ and that

$$\lim_{q \to +\infty} \max\{\mathfrak{G}(X,q;A)\}/\min\{\mathfrak{G}(X,q;A)\} = d.$$

Example 4. Fix a prime-power q, an integer $d \geq 3$ and a geometrically integral degree d surface X defined over $\mathbb{F}_q$ such that there is $P \in X(\mathbb{F}_q)$ which is a singular point of X with multiplicity $d-1$. Let A denote the set of all planes containing P and defined over $\mathbb{F}_q$. Call $\widetilde{X}$ and $\widetilde{A}$ the data of Example 3. It is easy to check that $\mathfrak{G}(X,q;A) \subseteq \mathfrak{G}(\widetilde{X},q;\widetilde{A})$ and that for every $x \in \mathfrak{G}(\widetilde{X},q;\widetilde{A})$ there is (X,A) such that $x \in \mathfrak{G}(X,q;A)$. However, it is easy to get example of pairs (X,A) (say without too many lines) such that $\mathfrak{G}(X,q';A) \subsetneqq \mathfrak{G}(\widetilde{X},q';\widetilde{A})$ for every q power q'.

Example 5. Fix a prime power q, integers $d \geq 4$, $s > 0$, $d_i > 0$, $1 \leq i \leq s$, such that $q \geq s-1$, and $\sum_{i=1}^{s} d_i(d_i-1) = d(d+3)$. Fix $Q_1, \dots, Q_s \in \mathbf{P}^2(\mathbb{F}_q)$, $Q_i \neq Q_j$ for $i \neq j$, such that no 3 of the points $Q_1, \dots, Q_s$ are collinear. Let $j_d : \mathbf{PP}^2 \to \mathbf{P}^n$, $n = (d^2+3d)/2$, the order d Veronese embedding of the plane. Set $X := j_d(\mathbf{P}^2)$ and $P_i := j_d(Q_i)$. The variety X and the points $P_1, \dots, P_s$ are defined over $\mathbb{F}_q$. Let A be the set of all hyperplanes of $\mathbf{P}^n$ defined over $\mathbb{F}_q$ and passing through each point P_i, $1 \leq i \leq s$, with multiplicity at least d_i. The case $s = 1$ of this example is just Example 3. Notice that $u(S,H) \geq s$ for all $H \in A$. Fix $H \in A$.

(a) Here we assume that $X \cap H$ is geometrically irreducible. Let $\pi : C \to X \cap H$ be the normalization. Since $\sum_{i=1}^{s} d_i(d_i-1) = d(d+3)$, the genus formula for plane curves gives $C \cong \mathbf{P}^1$ and hence $\sharp(C(\mathbb{F}_q) = q+1$. As in part (a) of Example 3 we get $u(S,H) = q+1+s-\sum i = 1^s t_i$, where $t_i := \sharp(\pi^{-1}(P_i) \cap C(\mathbb{F}_q))$. For all integers $e > 0$ set $E(e) := \{i : 1 \leq i \leq s; d_i - t_i = e\}$. Part (a) of Example 3 shows that the integers t_i, $1 \leq i \leq s$ are realized for some hyperplane H if the following conditions are satisfied:

(i) $0 \leq t_i \leq d_i$ and $t_i \neq d_i - 1$ for all i;

(ii) $e \cdot \sharp(E(e)) \leq q^e - q^{e-1}$ for all $e > 0$;

(iii) $\sum_{i=1}^{s} t_i \leq q+1$.

Conditions (i) and (iii) are obviously necessary. Since we assumed $s \leq q+1$ condition (ii) is always satisfied. Hence conditions (i) and (iii) are necessary and sufficient conditions (under the assumption that $X \cap H$ is integral).

(b) Here we assume that $X \cap H$ is not geometrically irreducible, but only raise a question. If $d_1 = d-1$, i.e. if $d_i = 1$ for all $i > s$, then $\max \mathfrak{G}(X,q;A) = 1 + dq$ if $d \leq q+1$ and $\max \mathfrak{G}(X,q;A) = q^2+q+1$ if $d \geq q+1$. Is for arbitrary $s, d_1, \dots, d_s$ the integer $\max \mathfrak{G}(X,q;A)$ computed by a hyperplane section which is a union of lines?

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