International Journal of Pure and Applied Mathematics
Volume 30 No. 2 2006, 171-177

THE CONVOLUTION PRODUCT OF $n-$DIMENSIONAL ULTRAHYPERBOLIC OPERATOR OF $(\frac{n}{2}-k-1)$-TH DERIVATIVE OF DIRAC'S DELTA IN HYPERCONE

Manuel A. Aguirre T.
Núcleo Consolidado Matemática Pura y
Aplicada (NuCOMPA)
Facultad de Ciencias Exactas
Universidad Nacional del Centro
Pinto 399, Tandil, 7000, ARGENTINA
e-mail: maguirre@exa.unicen.edu.ar

Abstract: In this paper we obtain a relation between the distribution $\delta^{(\frac{n}{2}-l-1)}(u)$ and the n-dimensional ultrahyperbolic operator iterated $s-times$. As a consequence we give a sense to convolution distributional product of $L^s\left\{\delta^{(\frac{n}{2}-k-1)}(u)\right\}*L^t\left\{\delta^{(\frac{n}{2}-l-1)}(u)\right\}$. Our convolution product result in a generalization of the convolution product $\delta^{(\frac{n}{2}-k-1)}(u)*\delta^{(\frac{n}{2}-l-1)}(u)$ due to M. Aguirre T. (c.f. [3]).

AMS Subject Classification: 46F10, 46F12
Key Words: theory of distributions, distributional convolution product

1. Introduction

Let $x=(x_1,x_2,...x_n)$ be a point of R^n. We shall write

$$x_1^2+...+x_\mu^2-x_{\mu+1}^2-...x_{\mu+\nu}^2=u, \tag{1}$$

$\mu+\nu=n$ (dimension of the space). By H_+ we designate the interior of the forward cone:

$$H_+=\{x\in R^n : x_1>0, u>0\}, \tag{2}$$

and by $\bar{H}_+$ designate its closure. Similarly, H_- designate the domain

Received: May 20, 2006

$$H_- = \{x \in R^n : x_1 < 0, u > 0\} \tag{3}$$

and $\bar{H}_-$ designate its closure.

The hypersurface $u = u(x) = 0$ is a hypercone with a singular point (the vertex) at the origin (see [7], p. 248).

Let $F(\lambda)$ be a function of the scalar variable λ, and let $\Phi(x)$ be a function endowed with the following propperties:

$$\begin{array}{ll} a) & \Phi(x) = F(u)\,, \\ b) & \operatorname{supp}\Phi(x) \subset \bar{H}_+\,, \\ c) & e^{\langle x,y\rangle}\Phi(x) \in L_1 \ \text{ if } \ y \in V_-, \end{array} \tag{4}$$

where

$$V_- = \left\{y \in R^n : y_1 > 0, y_1^2 + ... + y_\mu^2 - y_{\mu+1}^2 - ...y_{\mu+\nu}^2 > 0\right\}. \tag{5}$$

We call R the family of functions $\Phi(x)$ which satisfies conditions (4).

Similarly, we call A the family of functions which satisfies conditions

$$\begin{array}{ll} a') & \Phi(x) = F(u)\,, \\ b') & \operatorname{supp}\Phi(x) \subset \bar{H}_-\,, \\ c) & e^{\langle x,y\rangle}\Phi(x) \in L_1 \ \text{ if } \ y \in V_+, \end{array} \tag{6}$$

where

$$V_+ = \left\{y \in R^n : y_1 < 0, y_1^2 + ... + y_\mu^2 - y_{\mu+1}^2 - ...y_{\mu+\nu}^2 > 0\right\}. \tag{7}$$

We shall consider the following functions of the family R_α introduced by (see [8], p. 72):

$$R_\alpha(u) = \begin{cases} \frac{u^{\frac{\alpha-n}{2}}}{K_n(\alpha)}\,, & \text{if } \ x \in H_+\,, \\ 0\,, & \text{if } \ x \notin H_+\,. \end{cases} \tag{8}$$

Here α is a complex parameter and n the dimension of the space.

The constant $K_n(\alpha)$, is defined by

$$K_n(\alpha) = \frac{\pi^{\frac{n-1}{2}}\Gamma(\frac{2+\alpha-n}{2})\Gamma(\frac{1-\alpha}{2})\Gamma(\alpha)}{\Gamma(\frac{2+\alpha-\mu}{2})\Gamma(\frac{\mu-\alpha}{2})} \tag{9}$$

and μ is the number of positive terms of

$$u = x_1^2 + ... + x_\mu^2 - x_{\mu+1}^2 - ... - x_{\mu+\nu}^2, \tag{10}$$

$\mu + \nu = n$ (dimension of the space).

$R_\alpha(u)$, which is an ordinary function if Real$(\alpha) \geq n$, is a distribution of α.

By putting $\mu = 1$ in(8) and (9) and remembering the Legendre's duplication of $\Gamma(z)$

$$\Gamma(2z) = 2^{2z-1}\pi^{-\frac{1}{2}}\Gamma(z)\Gamma(z+\frac{1}{2}) \tag{11}$$

(see [6], Vol. I, p. 344, formula (15)) the formula(8) reduces to

$$M_\alpha(u) = \begin{cases} \frac{u^{\frac{\alpha-n}{2}}}{H_n(\alpha)}, & \text{if} \quad x \in H_+, \\ 0, & \text{if} \quad x \notin H_+, \end{cases} \tag{12}$$

where

$$u = x_1^2 - x_2^2 - ...x_n^2, \tag{13}$$

and

$$H_n(\alpha) = \pi^{\frac{n-2}{2}} 2^{\alpha-1}\Gamma(\frac{\alpha}{2})\Gamma(\frac{\alpha-n+2}{2}). \tag{14}$$

$M_\alpha(u)$ is, precisely, the hyperbolic kernel of Marcel Riesz (see [8], p. 31).

Lemma 1. *The ultrahyperbolic kernel of Marcel Riesz $R_\alpha^H(u)$ defined by(8) has the following propertie*

$$R_o(u) = (-1)^{n+1}\left\{\delta(x)\right\}, \tag{15}$$

$$R_{-2k}(u) = (-1)^{n+1}L^k\left\{\delta(x)\right\}, \tag{16}$$

$$L^k R_\alpha(u) = R_{\alpha-2k}(u) \tag{17}$$

and

$$L^k R_{2k}(u) = R_o(u) = (-1)^{n+1}\left\{\delta(x)\right\}. \tag{18}$$

The proof of Lemma 1 (formulae (15), (17), (18) and (16) for the case n odd), is given by S.E. Trione in [9] and the proofs of properties (15), (16), (17) and (18) for the case n even is given by Manuel A. Aguirre T. in [4].

In this paper we obtain a relation between the distribution $\delta^{(\frac{n}{2}-l-1)}(u)$ and the n-dimensional ultrahyperbolic operator iterated s-times. As a consequence, we give a sense to convolution distributional product of

$$L^s\left\{\delta^{(\frac{n}{2}-k-1)}(u)\right\} * L^t\left\{\delta^{(\frac{n}{2}-l-1)}(u)\right\}.$$

Our convolution product result in a generalization of the convolution product $\delta^{(\frac{n}{2}-k-1)}(u) * \delta^{(\frac{n}{2}-l-1)}(u)$ due to M. Aguirre T. (c.f. [3]).

Lemma 2. *Let s and l be non-negative integers and n an even positive integer then the following formula is valid*

$$L^s\left\{\delta^{(\frac{n}{2}-l-1)}(u)\right\}=\begin{cases}\frac{\Gamma(l)}{2^{-2s}\Gamma(l-s)}\delta^{(\frac{n}{2}-l-1+s)}(u) \text{ if } l>s\,,\\ (-1)^{\frac{\mu-1}{2}}\pi^{\frac{n-2}{2}}2^{2l-1}\Gamma(l)(-1)^{n+1}L^{s-l}\left\{\delta(x)\right\} \text{ if } l\leq s\end{cases} \quad (19)$$

under conditions i) μ and ν are both odd and ii) $\frac{n}{2}-l-1\geq 0$, where

$$\delta^{(\frac{n}{2}-k-1)}(u)=\frac{(\frac{n}{2}-k-1)!}{(-1)^{\frac{n}{2}-k-1}}\operatorname{Re}s_{\beta=-(\frac{n}{2}-k)}u^{\beta} \quad (20)$$

(see [1], p. 148, formulae (3.5) and (3.6), and [7], p. 352), $u=u(x)$ is defined by (10) and L^s is the n-dimensional ultrahyperbolic operator iterated s times defined by

$$L^s=\left\{\frac{\partial^2}{\partial x_1^2}+...+\frac{\partial^2}{\partial x_\mu^2}-\frac{\partial^2}{\partial x_{\mu+1}^2}-...-\frac{\partial^2}{\partial x_{\mu+\nu}^2}\right\}^s. \quad (21)$$

Proof. From [1], p. 149, formula(3.13) the following formula is valid

$$R_{2k}(u)=\frac{1}{(-1)^{\frac{\mu-1}{2}}\pi^{\frac{n-2}{2}}2^{2k-1}\Gamma(k)}\delta^{(\frac{n}{2}-k-1)}(u)\,, \quad (22)$$

if μ and ν are both odd ($\mu+\nu=n$ dimension of the space) and $k<\frac{n}{2}$, where $R_{2k}(u)$ is the ultrahyperbolic kernel of Marcel Riesz defined by (8). Now from (22) and considering the properties (17) we obtain

$$\begin{aligned}L^s\left\{\delta^{(\frac{n}{2}-l-1)}(u)\right\}&=(-1)^{\frac{\mu-1}{2}}\pi^{\frac{n-2}{2}}2^{2l-1}\Gamma(l)\left\{R_{2l-2s}(u)\right\}\\&=\frac{\Gamma(l)}{2^{-2s}\Gamma(l-s)}\delta^{(\frac{n}{2}-l-1+s)}(u)\,,\end{aligned} \quad (23)$$

if $l>s$ and $l<\frac{n}{2}$. On the other hand, using the properties (16) the following formula is valid

$$R_{2(l-s)}(u)=R_{-2(s-l)}(u)=(-1)^{n+1}L^{s-l}\left\{\delta(x)\right\}\,, \quad (24)$$

if $l\leq s$. Taking into account the formulae (24), (22) and (17) we have

$$L^s\left\{\delta^{(\frac{n}{2}-l-1)}(u)\right\}=(-1)^{\frac{\mu-1}{2}}\pi^{\frac{n-2}{2}}2^{2l-1}\Gamma(l)\left\{R_{2l-2s}(u)\right\}=$$

$$(-1)^{\frac{\mu-1}{2}}\pi^{\frac{n-2}{2}}2^{2l-1}\Gamma(l)\left\{R_{2l-2s}(u)\right\}=(-1)^{\frac{\mu-1}{2}}\pi^{\frac{n-2}{2}}2^{2l-1}\Gamma(l)\left\{R_{-2(s-l)}(u)\right\}$$
$$=(-1)^{\frac{\mu-1}{2}}\pi^{\frac{n-2}{2}}2^{2l-1}\Gamma(l)(-1)^{n+1}L^{s-l}\left\{\delta(x)\right\}, \quad (25)$$

if $l\leq s$ and $l<\frac{n}{2}$. From (23) and (25) we obtain the formula (19) and we conclude the proof of Lemma 2. □

Theorem 3. *Let s,t,k and l be non-negative integers and n an even positive integer such that $0\leq k+l-(s+t)\leq\frac{n}{2}-1$ then the following formula is valid*

$$L^{s}\left\{\delta^{(\frac{n}{2}-k-1)}(u)\right\}*L^{t}\left\{\delta^{(\frac{n}{2}-l-1)}(u)\right\}=A_{k,l}L^{s+t}\left\{\delta^{(\frac{n}{2}-(l+k)-1)}(u)\right\} \quad (26)$$

if $k>s$, $l>t$, where

$$A_{k,l,\mu}=\frac{\Gamma(k)\Gamma(l)(-1)^{\frac{\mu-1}{2}}\pi^{\frac{n-2}{2}}}{2\Gamma(k+l)}. \quad (27)$$

Proof. We know from [3] that the following formula is valid

$$\delta^{(\frac{n}{2}-k-1)}(u)*\delta^{(\frac{n}{2}-l-1)}(u)=a_{k,l}\delta^{(\frac{n}{2}-k-l-1)}(u), \quad (28)$$

under conditions i) μ and ν are both odd, and ii) $0\leq k+l\leq\frac{n}{2}-1$ (see [3], p. 36, formula (19)), where $\delta^{(\frac{n}{2}-k-1)}(u)$ is defined by (20) and

$$a_{k,l}=\frac{(-1)^{\frac{\mu-1}{2}}\pi^{\frac{n-2}{2}}\Gamma(k)\Gamma(l)}{2\Gamma(k+l)}. \quad (29)$$

From (19) for the case $k>s$ and $l>t$ and considering (27) we get

$$L^{s}\left\{\delta^{(\frac{n}{2}-k-1)}(u)\right\}*L^{t}\left\{\delta^{(\frac{n}{2}-l-1)}(u)\right\}=\frac{2^{2s}\Gamma(k)}{\Gamma(k-s)}.\frac{2^{2t}\Gamma(l)}{\Gamma(l-t)},$$

$$\delta^{(\frac{n}{2}-k-1+s)}(u)*\delta^{(\frac{n}{2}-l-1+t)}(u)$$
$$=\frac{2^{2s+2t}\Gamma(k)\Gamma(l)(-1)^{\frac{\mu-1}{2}}\pi^{\frac{n-2}{2}}}{\Gamma(k-s+l-t)2}\delta^{(\frac{n}{2}-(k+l)-1+s+t)}(u)$$
$$=\frac{\Gamma(k)\Gamma(l)(-1)^{\frac{\mu-1}{2}}\pi^{\frac{n-2}{2}}}{2\Gamma(k+l-1)}L^{s+t}\left\{\delta^{(\frac{n}{2}-(k+l)-1)}(u)\right\}, \quad (30)$$

if $0\leq k+l-s-t\leq\frac{n}{2}-1, k>s$ and $l>t$.

From (30) we obtain the formulae (26) and (27) and we conclude the proof of theorem. The formula (26) is a generalization of the convolution product of $\delta^{(\frac{n}{2}-k-1)}(u)*\delta^{(\frac{n}{2}-l-1)}(u)$ (see [3], p. 36, formula (19)). In fact, putting $s=t=0$ in (26) we obtain the formulae (28) and (29). □

Theorem 4. *Let s, t, k and l be non-negative integers and n an even positive integer such that $0 \leq k + l \leq \frac{n}{2} - 1$ then the following formula is valid*

$$L^s\left\{\delta^{(\frac{n}{2}-k-1)}(u)\right\} * L^t\left\{\delta^{(\frac{n}{2}-l-1)}(u)\right\} = B_{k,l}L^{s+t}\left\{\delta^{(\frac{n}{2}-(l+k)-1)}(u)\right\}, \quad (31)$$

if $k \leq s$ and $l \leq t$, where

$$B_{k,l} = (-1)^{n+1}A_{k,l,\mu} \quad (32)$$

and $A_{k,l,\mu}$ is defined by (27).

Proof. We know from [2], p. 346, formula (5.3) that the following is valid

$$L^s\{\delta(x)\} * L^t\{\delta(x)\} = L^{s+t}\{\delta(x)\}, \quad (33)$$

where L^r is the n-dimensional ultrahyperbolic operator iterated r times defined by (21). Now from (19) for the case $k \leq s$ and $l \leq t$ and considering the formula (33) we reach to

$$\begin{aligned} &L^s\left\{\delta^{(\frac{n}{2}-k-1)}(u)\right\} * L^t\left\{\delta^{(\frac{n}{2}-l-1)}(u)\right\} \\ &= \left((-1)^{\frac{\mu-1}{2}}\pi^{\frac{n-2}{2}}\right)^2 2^{2k-1}\Gamma(k)2^{2l-1}\Gamma(l)(-1)^{n+1}\left[L^{s-k}\{\delta(x)\} * L^{t-l}\{\delta(x)\}\right] \\ &= \left((-1)^{\frac{\mu-1}{2}}\pi^{\frac{n-2}{2}}\right)^2 2^{2k-1}\Gamma(k)2^{2l-1}\Gamma(l)(-1)^{n+1}L^{s+t-(k+l)}\{\delta(x)\} \\ &= \frac{(-1)^{\frac{\mu-1}{2}}\pi^{\frac{n-2}{2}}\Gamma(k)\Gamma(l)(-1)^{n+1}}{2\Gamma(k+l)}L^{s+t}\left\{\delta^{(\frac{n}{2}-(k+l)-1)}(u)\right\} \\ &\qquad \text{if } \ 0 \leq k+l \leq \frac{n}{2}-1, k \leq s \text{ and } l \leq t. \end{aligned} \quad (34)$$

From (34) we obtain the formulae (31) and (32) and we conclude the proof of theorem. ☐

Acknowledgments

Work is partially supported by “Comisión de Investigaciones Científicas y Tecnológicas de la Provincia de Buenos Aires CIC”, Argentina.

References

[1] Manuel A. Aguirre T., The distributional Hankel transform of marcel Riesz's ultrahyperbolic kernel, *Studies in Applied Mathematics*, **93** (1994), 133-162.

[2] Manuel A. Aguirre T., The dsitribution $\delta^{(k)}(P \pm i0 - m^2)$, *Journal of Computational and Applied Mathematics*, **88** (1997), 339-348.

[3] Manuel A. Aguirre T., Two special convolution products of $(\frac{n}{2} - k - 1)$-th derivatives of Dirac delta in hypercone, *Applied Mathematics E-Notes*, **1** (2001), 34-39.

[4] Manuel A. Aguirre T., Special convolution product of Marcel Riesz' ultra-hyperbolic kernel, To Appear.

[5] Manuel Aguirre T., Susana E. Trione, The distributional convolution products of Marcel Riesz'ultra-hyperbolic kernel, *Revista de la Unión Matemática Argentina*, **39** (1995).

[6] A. Erdelyi, Ed. *Higher Trascendental Functions*, Volume I and Volume II, McGraw-Hill, New York (1953).

[7] I.M. Gelfand, G.E. Shilov, *Generalized Function*, Volume I, Academic Press, New York (1964).

[8] Y. Nozaki, On Riemann-liouville integral of ultra-hiperbolic type, *Kodai Mathematical Seminar Reports*, **6**, No. 2 (1964), 69-87.

[9] S.E. Trione, On Marcel Riesz' ultrahyperbolic kernel, *Prepint* 116, Trabajos de Matemática, Instituto Argentina de Matemática-Conicet (1987).