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QUASIDISKS AND THE MAX PROPERTY

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Abstract: Let D be a Jordan proper subdomain of R^2 whose boundary contains at least three points, $D^* = \overline{R}^2 \setminus \overline{D}$, the exterior of D. We say that D has the max property if there exists a constant $c \geq 1$ such that each pair of points $x_1, x_2 \in D \setminus \{\infty\}$ can be joined by an arc γ in D for which

$$|x - y| \leq c \max_{j=1,2} |x_j - y|$$

for each $x \in \gamma$ and each $y \in \partial D$. In this paper, the authors prove that D is a quasidisk if and only if both D and D^* have the max property.

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1. Introduction

We shall assume throughout this paper that D is a Jordan proper subdomain of $\overline{R}^2$ whose boundary containing at least three points, $D^* = \overline{R}^2 \setminus \overline{D}$. For convenience we shall adopt the notation and terminology as in paper [1]. For $x \in R^2$ and $0 < r < \infty$, let $B^2(x, r) = \{z \in R^2 : |z - x| < r\}$, $\overline{B}^2(x, r)$ be the closure of $B^2(x, r)$, $S^1(x, r) = \partial B^2(x, r)$, $B^2(r) = B^2(0, r)$ and $B^2 = B^2(1)$.

We say that D has the max property if there exists a constant $c \geq 1$ such that each pair of points $x_1, x_2 \in D \setminus \{\infty\}$ can be joined by an arc γ in D for which

$$|x - y| \leq c \max_{j=1,2} |x_j - y| \tag{1.1}$$

for each $x \in \gamma$ and each $y \in \partial D$.

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The max property is an important concept for a domain D, it has been used extensively in the research fields of combinatorial optimization [2], approximation algorithm [3], sensitivity analysis [4], Hilbert space [5] and nonlinear fractional theory [6], etc.

D is called a quasidisk if there exists a $K-$quasiconformal mapping $(K \geq 1)$ $f : \overline{R}^2 \to \overline{R}^2$ such that D is the image of the unit disk B^2 under f.

It is well-known that quasidisks play a very important role in quasiconformal mappings [7], complex dynamics [8], Fuchsian groups [9] and Teichmüller space theory [10], etc. A lot of interesting geometric and analytic properties for quasidisks are obtained by many authors.

The main purpose of the present paper is to use the max property to depict the geometric characteristics of quasidisks, we obtain the following result.

Theorem. *D is a quasidisk if and only if both D and D^* have the max property.*

For convenience we need to define and introduce the following important concepts, they will be used in the next sections.

Let $c \geq 1$ be a constant. (1) If for any $x_0 \in R^2$ and $0 < r < +\infty$, each pair of points $x, y \in D \cap \overline{B}^2(x_0, r)$ can be joined by an arc γ in $D \cap \overline{B}^2(x_0, cr)$, then we call D is a $c-$inner linearly locally connected domain, denoted by $D \in c - ILC$; (2) If for any $x_0 \in R^2$ and $0 < r < +\infty$, each pair of finite points $x, y \in D \setminus B^2(x_0, r)$ can be joined by an arc γ in $D \setminus B^2(x_0, r/c)$, then we call D is a $c-$outer linearly locally connected domain, denoted by $D \in c - OLC$.

D is called a linearly locally connected domain if $D \in c - ILC$ and $D \in c - OLC$ at the same time for some $c \geq 1$.

Let $b > 0$ be a constant, D is called a $b-$cigar domain if each pair of points $x_1, x_2 \in D \setminus \{\infty\}$ can be joined by an arc $\gamma \subseteq D$ for which

$$\min_{j=1,2} \operatorname{dia}(\gamma(x_j, x)) \leq bd(x, \partial D) \qquad \text{for all} \qquad x \in \gamma, \tag{1.2}$$

where $\gamma(x_j, x)$ is the part of γ between x_j and x, and $d(x, \partial D)$ is the Euclidean distance from x to ∂D.

We say that D is a cigar domain if D is a $b-$cigar domain for some $b > 0$.

The following result was obtained by F.W. Gehring and B.G. Osgood [11] in 1982.

Theorem A. *D is a quasidisk if and only if D is a linearly locally connected domain.*

2. The Max Property of The Unit Disk B^2

In this section, we shall prove the following results

Lemma 2.1. *The unit disk B^2 has the max property.*

Proof. For each pair of points $x_1, x_2 \in B^2$, let γ be the hyperbolic geodesic in B^2 with endpoints x_1 and x_2, for any $x \in \gamma$ and $y \in S^1$, denote

$$t_j = \frac{|x_j - y|}{|x - y|}, \qquad L = \max_{j=1,2} t_j, \quad j = 1, 2. \tag{2.1}$$

If constructing Möbius transformation f as follows:

$$f(z) = \frac{x - x_2}{x - x_1} \frac{z - x_1}{z - x_2},$$

then $f(x_1) = 0$, $f(x) = 1$ and $f(x_2) = \infty$. Taking $y' = f(y)$, $z' = f(\infty)$, from the basic properties of Möbius transformation we get

$$t_j = \frac{|f(x_j) - f(y)|}{|f(x) - f(y)|} \frac{|f(x) - f(\infty)|}{|f(x_j) - f(\infty)|}. \tag{2.2}$$

Taking $j = 1, 2$, (2.2) implies

$$\begin{cases} |y'||1 - z'| = t_1 |z'||1 - y'|, \\ |1 - z'| = t_2 |1 - y'|. \end{cases} \tag{2.3}$$

From the properties of Möbius transformation we know that $f(\gamma)$ is a hyperbolic geodesic in $f(B^2)$ and $f(B^2) = \{z : \text{Re} z > t, \ t \le 0\}$, hence $y', z' \in \{z : \ \text{Re} z \le 0\}$. This implies

$$1 + |y'|^2 \le |1 - y'|^2 \le 2(1 + |y'|^2) \tag{2.4}$$

and

$$1 + |z'|^2 \le |1 - z'|^2 \le 2(1 + |z'|^2). \tag{2.5}$$

Combining (2.1), (2.3), (2.4) and (2.5) we get

$$\begin{aligned} (1 + |y'|^2)(1 + |z'|^2) &\le (1 + |y'|^2)|1 - z'|^2 \\ &= (t_2^2 + t_1^2 |z'|^2)|1 - y'|^2 \le 2L^2(1 + |z'|^2)(1 + |y'|^2). \end{aligned} \tag{2.6}$$

(2.6) implies

$$L \ge \frac{1}{\sqrt{2}}. \tag{2.7}$$

(2.1) and (2.7) yield

$$|x - y| \le \sqrt{2} \max_{j=1,2} |x_j - y|.$$

Hence B^2 has the max property. □

3. The Max Property and Quasiconformal Mappings

The main aim of this section is to prove that the max property is invariance under quasiconformal mappings. Lemma 3.3 is our main result in this section. To prove Lemma 3.3 we need to introduce two lemmas firstly.

Lemma 3.1. (see [11]) *If Γ is a family of closed curves γ each of which separates z_1, w_1 from z_2, w_2 and if*

$$s(z_1, w_1) \geq l, \qquad s(z_2, w_2) \geq l,$$

then

$$\mathcal{M}(\Gamma) \leq \frac{\pi}{l^2}, \tag{3.1}$$

where $\mathcal{M}(\Gamma)$ is the modulus of Γ, $s(z_1, z_2) = \inf\limits_{\beta} \int_\beta \frac{2|dz|}{1+|z|^2}$ and the infinum is taken over all locally rectifiable arcs β which joins z_1 and z_2 in $\overline{R}^2$.

Lemma 3.2. (see [11]) *If Γ is a family of curves γ and if for each t with $a < t < b$ the circle $|z - z_1| = t$ contains a $\gamma \in \Gamma$, then*

$$\mathcal{M}(\Gamma) \geq \frac{1}{2\pi} \log \frac{b}{a}. \tag{3.2}$$

Lemma 3.3. *Suppose that $f : R^2 \to R^2$ is a $K-$quasiconformal mapping, if D has the max property, then $D' = f(D)$ has the max property, too.*

Proof. For each pair of points $x_1', x_2' \in D' \setminus \{\infty\}$, taking $x_1 = f^{-1}(x_1')$, $x_2 = f^{-1}(x_2')$, then $x_1, x_2 \in D \setminus \{\infty\}$, since D has the max property, hence there exists a constant $\lambda \geq 1$ such that x_1 and x_2 can be joined by an arc γ in D for which

$$|x - y| \leq \lambda \max_{j=1,2} |x_j - y| \tag{3.3}$$

for each $x \in \gamma$ and $y \in \partial D$.

Let $\gamma' = f(\gamma)$, then $\gamma' \subseteq D'$ is an arc which joining x_1' and x_2'. For any $x' \in \gamma'$ and $y' \in \partial D'$, taking $x = f^{-1}(x')$ and $y = f^{-1}(y')$, then $x \in \gamma$, $y \in \partial D$ and (3.3) holds.

Without loss of generality, we may assume

$$|x - y| \leq \lambda |x_2 - y|. \tag{3.4}$$

If $|x_2' - y'| < |x' - y'|$, let Γ' be the family of circles $|z - y'| = t$ $(|x_2' - y| < t < |x' - y'|)$, then Lemma 3.2 implies

$$\mathcal{M}(\Gamma') \geq \frac{1}{2\pi} \log \frac{|x' - y'|}{|x_2' - y'|}. \tag{3.5}$$

On the other hand, since each $\beta' \in \Gamma'$ separates y', x_2' from x', ∞, hence each $\beta \in \Gamma = f^{-1}(\Gamma')$ separates y, x_2 from x, ∞. Let $g(z) = \frac{z-y}{x_2-y}$, then $g(y) = 0$, $g(x_2) = 1$, $g(\infty) = \infty$, $\mathcal{M}(\Gamma) = \mathcal{M}(g(\Gamma))$ and each $\alpha \in g(\Gamma)$ separates 0, 1 from $g(x)$, ∞ with

$$s(g(y), g(x_2)) = s(0, 1) = \frac{\pi}{2} \tag{3.6}$$

and

$$s(g(x), g(\infty)) = 2(\frac{\pi}{2} - \arctan |g(x)|) = 2\left(\frac{\pi}{2} - \arctan \left|\frac{x-y}{x_2-y}\right|\right). \tag{3.7}$$

(3.6) and $\lambda \geq 1$ imply

$$s(g(y), g(x_2)) \geq 2(\frac{\pi}{2} - \arctan \lambda). \tag{3.8}$$

(3.4) and (3.7) yield

$$s(g(x), g(\infty)) \geq 2(\frac{\pi}{2} - \arctan \lambda). \tag{3.9}$$

Combining (3.8), (3.9) and Lemma 3.1 we get

$$\mathcal{M}(\Gamma) = \mathcal{M}(g(\Gamma)) \leq \frac{\pi}{(\pi - 2\arctan \lambda)^2}. \tag{3.10}$$

From the basic properties of $K-$quasiconformal mappings [1] we have

$$\mathcal{M}(\Gamma') \leq K\mathcal{M}(\Gamma). \tag{3.11}$$

Combining (3.5), (3.10) and (3.11) we obtain

$$|x' - y'| \leq e^{\frac{2k\pi^2}{(\pi-2\arctan \lambda)^2}} |x_2' - y|. \tag{3.12}$$

If $|x_2' - y'| \geq |x' - y'|$, then it is obvious that (3.12) is true. □

From the above argument we know that

$$|x' - y'| \leq \lambda' \max_{j=1,2} |x_j' - y|,$$

where $\lambda' = e^{\frac{2k\pi^2}{(\pi-2\arctan \lambda)^2}} |x_j' - y|$, and hence D' has the max property.

Remark 3.1. If we take $f(x) = \frac{x}{|x|^2}$ with $f(0) = \infty$ and $f(\infty) = 0$, then f is a Möbius transformation of $\overline{R}^2$. Making use of Lemma 2.1 and the same method in Lemma 3.3 we can obtain that $(B^2)^*$ has the max property with $\lambda' = \frac{2\pi^2}{(\pi-2\arctan\sqrt{2})^2}$.

4. Cigar Domains and Linearly Locally Connected Domains

In this section, we shall establish the relation between cigar domains and linear locally connected domains.

Lemma 4.1. *If D is a $c-$cigar domain, then $D \in (2c+2) - OLC$.*

Proof. Take $b = 2c+2$. If $D \notin b-OLC$, then there exist $y_0 \in R^2$, $0 < r < \infty$ and $x_1, x_2 \in D \setminus B^2(y_0, r)$, such that x_1 and x_2 can not be joined by any arc in $D \setminus B^2(y_0, r/b)$.

Since D is a $c-$cigar domain, there exists an arc $\gamma \subseteq D$ such that γ joining x_1 and x_2 with

$$\min_{j=1,2} \text{dia}\,(\gamma(x_j, x)) \leq cd(x, \partial D) \tag{4.1}$$

for all $x \in \gamma$.

It is obvious that $\gamma \cap S^1(y_0, r/b) \neq \emptyset$. If taking $y \in \gamma \cap S^1(y_0, r/b)$, then (4.1) implies

$$d(y, \partial D) \geq \frac{1}{c} \min_{j=1,2} \text{dia}\,(\gamma(x_j, y)) \geq \frac{1}{c} \min_{j=1,2} |x_j - y| \geq \frac{1}{c}(1 - \frac{1}{b})r. \tag{4.2}$$

But

$$d(y_0, \partial D) \leq \frac{r}{b}. \tag{4.3}$$

The above (4.2), (4.3) and the triangular inequality yield

$$\frac{1}{c}(1 - \frac{1}{b})r \leq d(y, \partial D) \leq |y - y_0| + d(y_0, \partial D) \leq \frac{2}{b}r,$$

$$b \leq 2c + 1. \tag{4.4}$$

(4.4) contradicts with $b = 2c+2$. Hence $D \in (2c+2) - OLC$. □

Lemma 4.2. *If D^* is a c_0-cigar domain, then $D \in (16c_0 + 21)-ILC$.*

Proof. Take $\delta = 8c_0 + 10$. For any $u \in R^2$, $s > 0$, and $z_1, z_2 \in D \cap \overline{B}^2(u, s)$, $z_1 \neq z_2$. Denote $z = \frac{1}{2}(z_1 + z_2)$ and $r = |z_1 - z_2|$. Next we first prove that z_1, z_2 must be in the same component of $\overline{B}^2(z, \frac{1}{2}\delta r) \setminus D^*$.

If z_1, z_2 belong to different components of $\overline{B}^2(z, \frac{1}{2}\delta r) \setminus D^*$, then z_1, z_2 must be in the different components of $\overline{B}^2(z, \frac{1}{2}r) \setminus D^*$. Let β be the line segment which joins z_1 and z_2, then β contains a subcurve $\alpha \subseteq D^*$ such that α divide D^* into D_1 and D_2, and $\text{dia}\,(D_j) \geq \frac{1}{2}r(\delta - 1)$, $j = 1, 2$. This yields

$$\min_{j=1,2} \text{dia}\,(D_j) \geq \frac{1}{2}r(\delta - 1). \tag{4.5}$$

For any $x \in \alpha$, if $D_1 \nsubseteq B^2(x,(2c_0+2)\mathrm{dia}\,(\alpha))$ and $D_2 \nsubseteq B^2(x,(2c_0+2)\mathrm{dia}\,(\alpha))$, then take

$$x_j \in D_j \setminus \overline{B}^2(x,(2c_0+2)\mathrm{dia}\,(\alpha)), \qquad j=1,2.$$

Since D^* is a c_0-cigar domain, there exists an arc $\gamma \subseteq D^*$ joining x_1 and x_2 with

$$\min_{j=1,2} \mathrm{dia}\,(\gamma(x_j,w)) \le c_0 d(w,\partial D^*) \tag{4.6}$$

for all $w \in \gamma$.

Take $y \in \gamma \cap S^1(x,\mathrm{dia}\,(\alpha))$, then we can get

$$\begin{cases} |y-x_j| \ge (2c_0+1)\mathrm{dia}\,(\alpha), \\ \min\limits_{j=1,2} \mathrm{dia}\,(\gamma(x_j,y)) \le c_0 d(y,\partial D^*), \end{cases} \qquad j=1,2. \tag{4.7}$$

This implies

$$d(y,\partial D^*) \ge \frac{2c_0+1}{c_0}\mathrm{dia}\,(\alpha). \tag{4.8}$$

But

$$d(x,\partial D^*) \le \mathrm{dia}\,(\alpha). \tag{4.9}$$

(4.8), (4.9) and the triangular inequality yield

$$\frac{2c_0+1}{c_0}\mathrm{dia}\,(\alpha) \le d(y,\partial D^*) \le |y-x| + d(x,\partial D^*) \le 2\mathrm{dia}\,(\alpha),$$

so

$$\mathrm{dia}\,(\alpha) < 0,$$

this is obviously impossible. Hence $D_1 \subseteq \overline{B}^2(x,(2c_0+2)\mathrm{dia}\,(\alpha))$ or $D_2 \subseteq \overline{B}^2(x,(2c_0+2)\mathrm{dia}\,(\alpha))$, and we can obtain

$$\min_{j=1,2} \mathrm{dia}\,(D_j) \le 2(2c_0+2)\mathrm{dia}\,(\alpha). \tag{4.10}$$

(4.5), (4.10) and $\mathrm{dia}\,(\alpha) \le r$ imply

$$\delta \le 8c_0+9,$$

this contradicts with $\delta = 8c_0+10$. Hence z_1, z_2 must be in the same component of $\overline{B}^2(z,\frac{1}{2}\delta r) \setminus D^*$, and there exists an arc $\gamma \subseteq D$ joining z_1 and z_2 with

$$\mathrm{dia}\,(\gamma) \le \delta r = \delta|z_1-z_2| \le 2\delta s. \tag{4.11}$$

The above (4.11) implies

$$\begin{aligned}\gamma &\subseteq D\cap \overline{B}^2(u, s+\mathrm{dia}\,(\gamma)) \subseteq D\cap \overline{B}^2(u,(2\delta+1)s)\\ &= D\cap \overline{B}^2(u,(16c_0+21)s).\end{aligned}$$

Hence $D\in(16c_0+21)-ILC$, this completes the proof of Lemma 4.2. □

5. Max Property and Cigar Domains

In this section, we shall prove that the exterior of a domain with max property must be a cigar domain.

Lemma 5.1. *If D has the max property, then D^* is a cigar domain.*

Proof. For any finite points $z_1, z_2 \in D^*$, let γ be the hyperbolic geodesic which joining z_1 and z_2 in D^*, for any $z\in\gamma\backslash\{z_1,z_2\}$, suppose that $f: B^2\to D^*$ is a conformal mapping with $f(0)=z$ and

$$f^{-1}(\gamma)\subseteq R=\{z : z\in R^2,\ \mathrm{Im}=0\},$$

where f^{-1} is inverse of f. Then there exist

$$x_1\in\{z : z\in S^1,\ \mathrm{Im}z>0\}\quad\text{and}\quad x_2\in\{z : z\in S^1,\ \mathrm{Im}z<0\}$$

by [12], Corollary 10.3, such that $\alpha_j=f([0,x_j))$ is rectifiable with

$$l(\alpha_j)<a_0d(z,\partial D^*),\quad j=1,2,$$

where a_0 is an absolute constant, and $[0,x_j)$ is the half open segment which joins the origin O and x_j, $j=1,2$.

Let $y_j=f(x_j)$, $\alpha=\alpha_1\cup\alpha_2$, then $y_j\in\partial D$ and

$$l(\alpha)\le l(\alpha_1)+l(\alpha_2)<2a_0d(z,\partial D^*)\,. \tag{5.1}$$

For above $y_1,y_2\in\partial D^*=\partial D$, there exist a constant $b\ge1$ and a simple curve $\beta\subseteq D$ which joins y_1 and y_2 such that

$$|y-y'|\le b\max_{j=1,2}|y'-y_j|\quad\text{for all}\quad y\in\beta \text{ and } y'\in\partial D \tag{5.2}$$

by D has the max property.

If taking $y'=y_1$ in (5.2), then we get

$$|y-y_1|\le b|y_1-y_2| \tag{5.3}$$

for any $y \in \beta$.

(5.3) and the triangular inequality imply

$$\mathrm{dia}\,(\beta) \leq 2b|y_1 - y_2| \leq 2bl(\alpha). \tag{5.4}$$

If we denote by D_0 the bounded domain with boundary $\alpha \cup \beta$, then one of the points z_1 and z_2 must be in D_0. Without loss of generality, we may assume that $z_1 \in D_0$, then we can obtain

$$\begin{aligned}\mathrm{dia}\,(\gamma(z_1, z)) \leq \mathrm{dia}\,(D_0) = \mathrm{dia}\,(\partial D_0) \leq \mathrm{dia}\,(\alpha) + \mathrm{dia}\,(\beta) \\ \leq l(\alpha) + \mathrm{dia}\,(\beta) \leq 2a_0(2b+1)d(z, \partial D^*)\end{aligned} \tag{5.5}$$

by (5.1) and (5.4). This yields

$$\min_{j=1,2} \mathrm{dia}\,(\gamma(z_j, z)) \leq 2a_0(b+1)d(z, \partial D^*), \tag{5.6}$$

and (5.6) implies that D^* is a cigar domain. □

6. The Proof of Theorem

In this section, we shall prove the main result of the present paper by the above lemmas.

Theorem. *D is a quasidisk if and only if both D and D^* have the max property.*

Proof. (⇒) If D is a quasidisk, then there exists a quasiconformal mapping $f : R^2 \to R^2$ such that $f(B^2) = D$ and $f((B^2)^*) = D^*$ or $f(B^2) = D^*$ and $f((B^2)^*) = D$. In each case, Lemma 2.1, Remark 3.1 and Lemma 3.3 imply both D and D^* have the max property.

(⇐) (1) Since D has the max property, hence D^* is a cigar domain by Lemma 5.1, this and Lemma 4.2 imply $D \in ILC$.

(2) since D^* has the max property, hence D is a cigar domain by Lemma 5.1, this and Lemma 4.1 imply $D \in OLC$.

From the above (1), (2) and Theorem A we know that D is a quasidisk. □

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